

Pragati

INTEGRAL CALCULUS WITH APPLICATIONS

A.K. Hazra

PRAGATI PRAKASHAN, MEERUT

INTEGRAL CALCULUS

with

APPLICATIONS

RB YG

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Preface

The text is an outgrowth of author's 30 years' teaching. In the first lecture on this topic, the author used to advise each batch of students to start preparing a table of integrals with types, procedures and results and fill it up day by day as the learning of integrals proceed on. Ultimately a table like one given herewith would be prepared. Knowledge of procedures is better than memorizing the results. This is still an advice to the learners. Because learning of integral calculus is not so easy as differential calculus and the drawbacks are stated in sec. 1-5 of chapter 1 to make the learners serious and sincere in learning integral calculus from the very beginning.

A good number of types of integrals are collected and described with procedural details. All attempts have been made to make the material lucid & easily graspable to learners.

The author does not claim its originality rather than same extension of work particularly in chapters 3, 17 and illustration of the materials throughout the book. Formulae for the calculation of the coefficients of partial fraction using derivatives are derived and introduced in chapter 3. A part of this approach for linear factors only is available in "A text book of Integral Calculus" of Ganesh Prasad, 1910, published by Longmans, Green and Co. Ltd. This approach is convenient for computing a part of the integral without computing the whole. Chapter 17, introduces a general rule for the reduction of some elliptic integrals to standard form in sec. 17-4 (D).

The book is divided into FOUR parts : Part I — consists of first 14 chapters on single integrals, Part II comprises of 2 chapters on Jacobian and multiple integrals, Part III introduces Elliptic Integrals and Elliptic functions in a single chapter and Part IV contains the last 5 chapters on elementary ordinary differential equations (ODE). Each part contains the respective applications in geometry, physics and engineering.

A review of trigonometric hyperbolic functions in chapter 1 and 3 dimensional surfaces in chapter 16 has been added for ready references. Procedures for curve tracing in both cartesian and polar co-ordinates are embodied in chapter 11, which are a bare necessity for determining limits of integration in applied problems. Solved problems mainly on areas give almost detail description of curve tracing and determining limits to facilitate the understanding of the beginners.

In Part III on ODE, chapter 22 is a systematic procedure for the reduction of order of ODE which are available scatteredly and in different contexts.

In chapter 8, Riemann integrability has been described and illustrated with examples which is rarely available in other texts.

No knowledge of integral calculus is assumed and only pre-requisite is some knowledge of differentiation.

The author is indebted to the authorities on the subject and the great and renowned authors of this subject. Heartiest thanks to publishers M/s Pragati Prakashan for sincere co-operation and publishing the book in the shortest possible time.

Any suggestion for the improvement of the book is cordially invited.

Finally, the author extends his sincere thanks to (i) his three dependants namely his wife Mrs. Jahan Ara Begum, two sons Dr. Renzil Ahsan (Tutul) and Zeaul Ahsan (Mithili) for their co-operation in preparing the manuscript and (ii) Mr. Shah Alam for type some parts of the manuscript.

— A. K. Haya

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— A. K. Haysa

Dedicated to the sweet memories of the bereaved souls of

my Mother	: Bibi Fatema
Youngest Sister	: Chabi
Daughter	: Shampa (1967-1971)
Eldest Brother	: Khailur Rahaman Hazra
4th Brother	: Matiur Rahaman Hazra
Father	: Nazem Ali Hazra
Maternal Uncle	: Ali Akber
2nd Brother	: Abdul Aziz Hazra
2nd Brother's Wife	: Bibi Marium

$$\begin{aligned}
 & \text{P 40-40} \int (x-3)^3 \sqrt{6x-x^2} \, dx \\
 & \int -2(x-3)^2 \, dz \\
 & \int -2^2(x^2-6x+9) \, dz \\
 & = -2^2(-z^2+9) \, dz \quad \text{f f}
 \end{aligned}$$

$$\begin{aligned}
 & 6x-x^2 = z^2 \\
 & 2(3-x) = z \quad dz \\
 & (x-3) \, dx = -z \, dz
 \end{aligned}$$

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Table of Integrals with Procedures

Sl. Nos.	Types of Integrals	Procedures	Results
1.	$\int x^n dx, n \neq -1$	Variable with const. power. Definition of integration as the reverse process of differentiation.	$\frac{x^{n+1}}{n+1} + C, n \neq -1$
2.	$\int \frac{1}{x} dx$	Do	$\log x + C$
3.	$\int \frac{f'(x)}{f(x)} dx$	Numerator is the diff. coeff. of the denominator, substitution $f(x)$, using 2.	$\log f(x) + C$
4.	$\int e^x dx$	Definition of integration as the reverse process of differentiation	$e^x + C$
5.	$\int a^x dx$	Do, (const.)	$\frac{a^x}{\log a} + C$
6.	$\int \sin x dx$	Do	$-\cos x + C$
7.	$\int \cos x dx$	Do	$\sin x + C$
8.	$\int \tan x dx$	By 3 above	$\log \sec x + C$
9.	$\int \sec^2 x dx$	Def. of int. as above	$\tan x + C$
10.	$\int \operatorname{cosec}^2 x dx$	Do	$-\cot x + C$
11.	$\int \sec x \tan x dx$	Do	$\sec x + C$
12.	$\int \operatorname{cosec} x \cot x dx$	Do	$-\operatorname{cosec} x + C$
13.	$\int \operatorname{cosec} x dx$	Express in terms of sine, then express in terms of half angle, convert to tangent, use 3 above	$\log \tan \frac{x}{2} + C$
14.	$\int \sec x dx$	Do	$\log \tan \left(\frac{x}{2} + \frac{\pi}{4} \right) + C$
15.	$\int \cot x dx$	By 3 above	$\log \sin x + C$
16.	$\int \sinh x dx$	Def. of int. as above	$\cosh x + C$
17.	$\int \cosh x dx$	Do	$\sinh x + C$

18.	$\int \tanh x \, dx$	Def. of int. as above	$\log \cosh x + C$
19.	$\int \operatorname{sech}^2 x \, dx$	Do	$\tanh x + C$
20.	$\int \operatorname{cosech}^2 x \, dx$	Do	$-\coth x + C$
21.	$\int \operatorname{sech} x \, dx$	$\frac{1}{\cosh x} = \frac{\cosh x}{1 + \sinh^2 x}$, Put $\sinh x = v$	$\tan^{-1}(\sinh x) + C$
22.	$\int \operatorname{cosech} x \, dx$	$\frac{1}{\sinh x} = \frac{\sinh x}{\cosh^2 x - 1}$, Put $\cosh x = u$, integrate and simplify.	$\log \left(\tanh \frac{x}{2} \right) + C$
23.	$\int \coth x \, dx$	Use 3.	$\log \sinh x + C$
24.	$\int \operatorname{sech} x \tanh x \, dx$	Def. of int.	$-\operatorname{sech} x + C$
25.	$\int \operatorname{cosech} x \coth x \, dx$	Do	$-\operatorname{cosech} x + C$
26.	$\int \frac{dx}{x^2 + a^2}$	Let $x = a \tan \theta$.	$\frac{1}{a} \tan^{-1} \frac{x}{a} + c$
27.	$\int \frac{dx}{x^2 - a^2}$	P.F., use 3.	$\frac{1}{2} \log \frac{x-a}{x+a} + c$
28.	$\int \frac{dx}{a^2 - x^2}$	Do	$\frac{1}{2} \log \frac{a+x}{a-x} + c$
29.	$\int \frac{dx}{ax^2 + bx + c}$	Express $ax^2 + bx + c$ as sum or difference of two squares as 26-28	either of the form 25, 26 or 27.
30.	$\int \frac{dx}{\sqrt{(x^2 + a^2)}}$	Let $x = a \tan \theta$ or $x = a \sinh \theta$	$\log(x + \sqrt{x^2 + a^2}) + c$ or $\sinh^{-1} \frac{x}{a} + c$
31.	$\int \frac{dx}{\sqrt{(x^2 - a^2)}}$	Let $x = a \sec \theta$ or $a \cosh \theta$	$\log(x + \sqrt{x^2 - a^2}) + c$ or $\cosh^{-1} \frac{x}{a} + c$
32.	$\int \frac{dx}{\sqrt{(a^2 - x^2)}}$	Let $x = a \sin \theta$	$\sin^{-1} \frac{x}{a} + c$
33.	$\int \frac{dx}{\sqrt{(ax^2 + bx + c)}}$	Reduce to one of 30, 31, 32	one of the form of 30-32
34.	$\int \sqrt{(x^2 + a^2)} \, dx$	Integrate by parts, second function = 1, use 30 or let $x = a \tan \theta$	$\frac{1}{2} \left[x \sqrt{x^2 + a^2} + a^2 \log(x + \sqrt{x^2 + a^2}) \right] + c$ or $\frac{1}{2} \left[x \sqrt{x^2 + a^2} + a^2 \sinh^{-1} \frac{x}{a} \right] + c$

35.	$\int \sqrt{x^2 - a^2} dx$	Integrate by parts as in 34 and use 31 or let $x = a \sec \theta$	$\frac{1}{2} \left[x \sqrt{x^2 - a^2} - a^2 \cosh^{-1} \frac{x}{a} \right] + c$
36.	$\int \sqrt{a^2 - x^2} dx$	Integrate by parts as in 34 and use 32 or let $x = a \sin \theta$	$\frac{1}{2} \left[x \sqrt{a^2 - x^2} + a^2 \sin^{-1} \frac{x}{a} \right] + c$
37.	$\int \sqrt{ax^2 + bx + c} dx$	Reduce to one of 34-36	one of the form of 34-36.
38.	$\int \frac{px + q}{ax^2 + bx + c} dx$	Balance numerator as the diff. coeff. of the denominator. Break the integral into two. First will be of type 3 and 2nd one will be 29 and hence any one of 26-28	$\frac{p}{2a} \log(ax^2 + bx + c) +$ one of the 26-28
39.	$\int \frac{px + q}{\sqrt{(ax^2 + bx + c)}} dx$	Balance numerator using diff. coeff. of $ax^2 + bx + c$. Break the integral into two. First one is integrable by substitution, 2nd is of type 33.	$\frac{p}{2} \sqrt{ax^2 + bx + c} +$ one of the forms of 30-32
40.	$\int \frac{(px + q)}{\sqrt{ax^2 + bx + c}} dx$	Balance $(px + q)$ as the diff. coeff. of $(ax^2 + bx + c)$. Integral breaks into two—1st integrable by substitution and 2nd of form 37 and hence to 34-36.	$\frac{p}{3a} (ax^2 + bx + c)^{3/2} +$ one of the forms 34-36.
41.	$\int \frac{x^{2m+1}}{(a+bx)^n} dx$	$m > 0$, put $(a+bx) = u$, use binomial expansion, integrate by division	come out automatically.
42.	$\int \frac{dx}{x^m (a+bx)^n}$	$m+n > 1$, +ve integer, put $u = \frac{a+bx}{x}$ proceed as in 41	Do
43.	$\int \frac{x^{2m+1}}{(\sqrt{a+bx^2})^n} dx$	m +ve integer, $a+bx^2 = u^2$, binomial expansion	Do
44.	$\int \frac{dx}{x^n \sqrt{a+bx^2}}$	n +ve integer, put $x = \frac{1}{u}$, reduces to 43.	Do
45.	$\int \{ \sin px \cos qx \text{ or } \cos px \cos qx \text{ or } \sin px \sin qx \} dx$	Use addition formulae to express the product as a sum. Integral reduces 6 and 7.	Do
46.	$\int \frac{a \cos x + b \sin x}{c \cos x + d \sin x} dx$	Numerator as sum of denominator and a diff. of coeff. of denominator	$lx + m \log(\cos x + d \sin x)$ l, m to be determined
47.	$\int \sin^n x \cos^n x dx$	(i) either m or n +ve integer put $\sin x$ or $\cos x = u$ according as n or m odd. (ii) m, n both +ve integer, express into multiple angles.	Come out automatically

		(iii) $(m+n)$ is -ve even integer, convert $\tan x$ and one $\sec^2 x$, put $\tan x = u$	
48.	$\int \tan^m x \sec^n x dx$ or $\int \cot^m x \operatorname{cosec}^n x dx$	(i) n is +ve integer, separate one $\sec^2 x$ or $\operatorname{cosec}^2 x$, then use $u = \tan x$ or $\cot x$ as the case may be. Use 49. (ii) m, n both +ve odd, separate one $\tan x \sec x$ or $\cot x \operatorname{cosec} x$, put $\sec x$ or $\operatorname{cosec} x = v$ (iii) m +ve even, n +ve odd, express to powers of $\sec x$ or $\operatorname{cosec} x$. Use 50.	Come out automatically
49.	$\int \tan^n x dx$, $\int \cot^n x dx, n \neq 1$	Separate one $\tan^2 x$ or one $\cot^2 x$. Express $\tan^2 x = \sec^2 x - 1$ or $\cot^2 x = \operatorname{cosec}^2 x - 1$. Integrals break into two—one integral of the same form with reduced power. Proceed on reducing power	Do
50.	$\int \sec^n x dx, \operatorname{cosec}^n x dx$	n +ve integer separating one $\sec^2 x$ or $\operatorname{cosec}^2 x$, put $\tan x = u$ or $\cot x = u$.	Do
51.	$\int \frac{dx}{a \cos x + b \sin x + c}$ and $\int \frac{dx}{a \cosh x + b \sinh x + c}$	(i) a, b, c all non zero; (ii) $b = 0$, (iii) $a = 0$. For all six integrals, express all in half angles with pure const. term. Convert to tangent, put $\tan \frac{x}{2} = u$ or $\tanh \frac{x}{2} = u$. Integral reduces to 29 and hence to 26–28 (any one).	Do
52.	$\int \frac{\sin^n x}{a + b \cos x} dx$ and $\int \frac{\cos^n x}{a + b \sin x} dx$	(i) n is even, $\sin^n x$ or $\cos^n x$ to powers of $\cos x$ or $\sin x$, divide & integrate. (ii) n is odd, put $a + b \cos x = u$ or $a + b \sin x = u$. Use binomial expansion & integrate.	Do
53.	$\int \frac{dx}{a \cos x + b \sin x}$	Use const. substitution $a = r \cos \theta$, $b = r \sin \theta$ or vice-versa to reduce it to 13 or 14.	Do
54.	$\int \frac{a \cos x + b \sin x + c}{d \cos x + e \sin x + f} dx$, $\int \frac{a \cosh x + b \sinh x + c}{d \cosh x + e \sinh x + f} dx$	Express numerator = l (denominator) + m (diff. coeff. of denominator) + n . Equate coeff. of $\cos, \sin$ and const. Solve l, m, n .	$lx + m \log (\text{denominator}) + nx$ form 51 with a, b, c all non-zero.
55.	$\int \frac{ax^2 + c}{a^2x^4 + bx^2 + c^2} dx$	Divide by x^2 , write denominator as sum of two squares, on substitution for $\left(ax \pm \frac{c}{x}\right)$, the integral reduces to any one of 26–28.	Come out automatically

56.	$\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}},$ $\int \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}$	To come out of the surd, $x-\alpha$ or $x=\alpha \cos^2 \theta + \beta \sin^2 \theta$ may be substituted and then integral reduces to 32.	Come out automatically
57.	$\int \frac{dx}{(ax+b)\sqrt{cx+d}}$	Put $cx+d=u^2$ for same reason as 56.	Do
58.	$\int \frac{\sum_{r=0}^n a_r x^r}{\sqrt{ax^2+bx+c}} dx$	Write integral $\left(\sum_{r=0}^{n-1} A_{n-r-1} x^r \right) \sqrt{ax^2+bx+c} + A_n \int \frac{dx}{\sqrt{ax^2+bx+c}}$ Diff. & equate coeffs. of like powers of x solve for consts. The last integral is of type 33.	Do
59.	$\int \frac{dx}{(px+q)^r \sqrt{ax^2+bx+c}}$	Put $px+q=\frac{1}{u}$. Use 58.	Do
60.	$\int \frac{dx}{(ax^2+b)\sqrt{cx^2+d}}$	Put $x=\frac{1}{u}$ and $c+du^2=p^2$ to come out of surd. Use any one of 26-28	Do
61.	$\int \frac{dx}{x(a+bx^n)},$ $\int \frac{dx}{x\sqrt{a+bx^n}}$	Put $x^n=\frac{1}{u}$, 1st one easily integrable and 2nd reduces to any one of 30-32.	1st $=\frac{1}{na} \log \frac{x^n}{a+bx^n} + c$ 2nd will come out depending on a, b .
62.	$\int \frac{dx}{(a+cx^2)^{3/2}}$	Put $x=\frac{1}{u}$, next put $ax^2+c=p$	$\frac{x}{a(a+cx^2)^{1/2}} + c$
63.	$\int \frac{dx}{(ax^2+bx+c)\sqrt{ex+f}}$	Put $ex+f=u^2$, P.F. after factorization.	Come out automatically
64.	$\int \frac{dx/(Ax^2+Bx+C)}{\sqrt{ax^2+bx+c}}$	Put $x=\frac{pt+q}{t+1}$, assuming none of the quadratic expression has real zeros and uncommon imaginary zero, where p and q are two distinct roots of $\{Ab-Ba\}m^2 - 2\{Ca-Ac\}m + \{Bc-Cb\}=0$. Integral reduces to the form $\int \frac{(t+1)(p-q)}{(Pt^2+Qt+R)\sqrt{Ut^2+Vt+W}} dt$. Now put $Ut^2+Vt+W=u^2$ to remove surd and reduces to 29.	Do
65.	$\int R(x, x^{p_1/q_1}, x^{p_2/q_2}, \dots, x^{p_n/q_n}) dx,$ R denotes rational function.	Put $x=t^m$, m is the L.C.M. of $q_1, q_2, \dots, q_n$. Surds will be eliminated.	Do

66.	$\int R \left(x, \left(\frac{ax+b}{cx+d} \right)^{p_1/q_1}, \dots, \left(\frac{ax+b}{cx+d} \right)^{p_n/q_n} \right) dx$, R denotes rational function	Put $x = t^m$, m is the L.C.M. of $q_1, q_2, \dots, q_n$. Surds will be eliminated.	Come out automatically
67.	$\int R(x, \sqrt{ax+b}, \sqrt{a_1x+b_1} + b_1) dx$, R for rational function.	Put $x = ly + m$ where $\sqrt{ax+b} = \sqrt{al}\sqrt{y+2}$ and $\sqrt{a_1x+b_1} = \sqrt{a_1l}\sqrt{ay+2}$. Then $t = \frac{a_1b-ab_1}{4aa_1}$ and $m = \frac{a_1b+ab_1}{2aa_1}$ and the integral reduces to $\int R_1(y_1\sqrt{y+2}, \sqrt{y-2}) dy$. Now put $y = t^2 + \frac{1}{t^2}$. Use P.F. if necessary.	Do
68.	$\int R(x, (ax+b)^{1/n}) dx$, $n > 0$ integer	Put $ax+b = t^n$. P.F. may be required to integrate.	Do
69.	$\int R(x, \sqrt{ax^2+bx+c}) dx$	(1) Put $\sqrt{ax^2+bx+c} = t \pm x\sqrt{a}$ if $a > 0$ (2) " " " " " " " " " " " " if $c > 0$ (3) " " " " " " " " " " " " $ax^2+bx+c = a(x-\alpha)(x-\beta)$	Do
70.	$I_{m,n} = \int \sin^m x \cos^n x dx$	(1) separate one $\cos x$ and integrate by parts ($\sin^m x \cos x$) as second function, a reduction formula will be obtained (2) separate one $\sin x$ and integrate by parts ($\cos^n x \sin x$) as second function, a reduction formula will be obtained.	(1) $I_{m,n} = \frac{\cos^{n-1} x \sin^{m+1} x}{m+n} + \frac{n-1}{m+n} I_{m,n-2}$ reduces power of $\cos x$ by 2. (2) $I_{m,n} = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} I_{m-2,n}$ reduces power of $\sin x$ by 2, $m+n \neq 0$ in both.
71.	(i) $I_m = \int \sin^m x dx$ (ii) $I_n = \int \cos^n x dx$	(i) put $n=0$ in 70 (2) (ii) put $m=0$ in 70 (1)	(i) $I_m = -\frac{1}{m} \sin^{m-1} x \cos x + \frac{m-1}{m} I_{m-2}$ (ii) $I_n = \frac{1}{n} \sin x \cos^{n-1} x + \frac{n-1}{n} I_{n-2}$

72.	$I_{m,n} = \int \operatorname{cosec}^m x \sec^n x dx$	m, n both -ve in 70. In 70 (1) replace n by $(n+2)$. Then replace m by $-m$ and n by $-n$. Interchange of sides. Similarly from (2)	$(1) \frac{\operatorname{cosec}^{m-1} x \sec^{n-1} x}{n-1} + \frac{m+n-2}{n-1} I_{m,n-2}$ reduces power of $\sec x$ by 2, $(2) \frac{-\operatorname{cosec}^{m-1} x \sec^{n-1} x}{m-1} + \frac{m+n-2}{m-1} I_{m-2,n}$
73.	(i) $I_n = \int \sec^n x dx$ (ii) $I_m = \int \operatorname{cosec}^m x dx$	(i) put $m=0$ in 72 (1). (ii) put $n=0$ in 72 (2).	$(i) \frac{\tan x \sec^{n-2} x}{n-1} + \frac{n-2}{n-1} I_{n-2}$ $(ii) \frac{-\cot x \operatorname{cosec}^{m-2} x}{m-1} + \frac{m-2}{m-1} I_{m-2}$
74.	$I_{m,p} = \int x^m (a+bx^n)^p dx$	(i) separate x^{n-1} from x^m and integrate by parts taking $\{(a+bx^n)^p \cdot n bx^{n-1}\}$ as second function, solve for $I_{m,p}$ obtain a reduction formula reducing power of x in x^m by n . (ii) separate one $(a+bx^n)$, decompose into two integrals, replace m by $(m+n)$ and p by $(p-1)$ in (i) and use it in the decomposition form; gives a reduction formula reducing power of $(a+bx^n)$ by unity.	$(i) \frac{x^{m-n+1} (a+bx^n)^{p+1}}{(np+m+1)b} - \frac{(m-n+1)a}{(np+m+1)b} I_{m-n,p}$ $(ii) \frac{x^{m+1} (a+bx^n)^p}{np+m+1} + \frac{anp}{np+m+1} I_{m,p-1}$ $np+m+1 \neq 0$ in both.
75.	$I = \int u(x) v(x) dx$	(i) Logarithmic, & inverse trigonometric must be chosen as first function. (ii) For algebraic, smaller powered termed should be chosen first function (iii) In the product of exp. and sine or cosine any one can be chosen first function but once chosen maintain it for subsequent times also (iv) For logarithmic and inverse trigonometric without product is to be integrated by parts taking unity as second function.	$I = u(x) \int v(x) dx$ $\uparrow$ first function $\uparrow$ second function function function $- \int \left[\frac{d}{dx} u(x) v(x) dx \right] dx$
76.	$\int e^{ax} \{af(x) + f'(x)\} dx$	When exp. function appears as a multiple, try to express other part as a sum of a function and its derivative.	$e^{ax} f(x) + c.$

PART I

Indefinite and Definite Integrals with Applications

CH. 1—14

METHOD OF INTEGRATION AND ELEMENTARY INTEGRALS

1.1 DEFINITION

Integration originates from two different concepts. So the methods of integration are two — (i) integration is the reverse or inverse process of differentiation, which is the subject matter of indefinite integration and (ii) integration is a process of summation which is the subject matter of definite integration. The latter type results into a unique one and the former type is an easy one. The second type is useful for engineering and other physical applications of real life problems. The definition and method of integration of the first category will be discussed in this chapter and few others in subsequent chapters.

DEFINITION

If $F(x)$ is the derivative of $f(x)$ for general real values of x i.e. $\frac{d}{dx} F(x) = f(x)$, then $F(x)$ is called the indefinite integral or antiderivative of $f(x)$ and is denoted by $F(x) = \int f(x) dx$. (1)

In this definition three points are to be noted — (i) *general value* of x , (ii) *indefinite* and (iii) *real*. Let us discuss these three things sequentially: *Firstly*, general value does not mean all values of x i.e. the point of discontinuity, if any, is to be *disregarded*. For example, if $\frac{d}{dx} \left(-\frac{1}{x-a} \right) = \frac{1}{(x-a)^2}$ then $\int \frac{1}{(x-a)^2} dx = -\frac{1}{x-a}$ inspite of the point of discontinuity at $x = a$. *Secondly*, the word indefinite implies that the integral is *not unique*. Because, for some constant C , $\frac{d}{dx} [F(x) + C] = f(x)$ and in that case $\int f(x) dx = F(x) + C$. This C can have infinite number of values. In this sense this type of integral is *indefinite* and an arbitrary constant is to be added. This arbitrary constant is known as *Constant of Integration*. The theorem 1 (below) is concerned with these indefiniteness and constant of integration. *Thirdly*, x is *always real*. Because there is another important branch of mathematics—complex analysis for imaginary or complex x . So, the *convention* is that the integral of a real $f(x)$ is also real.

In the above definition the symbol $\int (\dots) dx$ replaces the symbol $\frac{d}{dx} (\dots)$. Hence, integration is the inverse or reverse process of differentiation. Again if both sides of $F(x) = \int f(x) dx$ is differentiated with respect to x , then by definition $\frac{d}{dx} (F(x)) = \frac{d}{dx} (\int f(x) dx) = f(x)$. Hence differentiation and integration are inverse to one another.

Theorem 1 : If $f(x)$ is the derivative of both $F(x)$ and $G(x)$ i.e. $F'(x) = f(x)$ and $G'(x) = f(x)$ then $F(x) \sim G(x) = \text{const.}$

Proof. By hypothesis, $F'(x) \sim G'(x) = f(x) \sim f(x) = 0$.

That is, $\frac{d}{dx} [F(x) \sim G(x)] = 0$

or, integrating with respect to (w.r.t.) x ,

$F(x) \sim G(x) = \text{const.} = C$ (say)

$\therefore F(x) = G(x) + C$.

It is evident from Theorem 1 that if $G(x)$ is the integral of $g(x)$ then $[G(x) + C]$ is also an integral of $g(x)$. This const. C is the constant of integration as stated above and it is convention to add an arbitrary constant in case of indefinite integration.

1.2 SYMBOLS

As the symbol $\frac{d}{dx} \equiv D$ is used to indicate differentiation w.r.t x , so to indicate integration w.r.t x the symbol $\int (\dots) dx$ is used. The differential dx indicates the *variable of integration*, x . From definition 1.1, the symbols D^{-1} or $\frac{1}{D}$ also indicates integration w.r.t. x . Hence the expression $\int f(x) dx$ is called the *integral of $f(x)$ w.r.t. x* and $f(x)$ is called the *integrand*. If there exists a function $F(x)$ as in (1), then $f(x)$ is said to be *integrable*.

1.3 GEOMETRIC INTERPRETATION

If $F'(x) = f(x)$ then $\int f(x) dx = F(x) + C$. Now if $F(x) + C$ is equated to y i.e. $y = F(x) + C$, then it represents the equation of curves for different values of C and these curves are parallel curves. So, *indefinite integrals represents a family or set of parallel curves in the xy - plane* (Fig. 1 next page).

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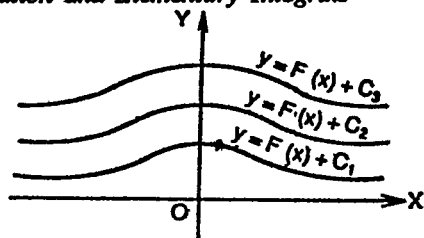


Fig 1

1.4 INTEGRABLE FUNCTIONS AND METHOD OF INTEGRATIONS

The conditions of integrability of functions will be discussed in details in chapter 8. Before that it can be said, from definition 1.1 and as mentioned in sec. 1.2, that for general values of x , the *continuous functions are integrable*. For example, algebraic polynomials, transcendental functions like trigonometric circular and hyperbolic, logarithmic, some rational and irrational functions of both algebraic and trigonometric are integrable. Of these functions, only trigonometric hyperbolic functions have little acquaintance. So, this kind of functions will be briefly discussed in sec. 1.5. All these functions are integrated using the following fundamental methods:

- (a) Direct integration
- (b) Expansion
- (c) Substitutions
- (d) Integration by parts
- (e) Partial fractions
- (f) Successive reductions
- (g) Differentiation of integrals

Among these methods *substitution* and *integration by parts* are the two general methods. Partial fraction is only an algebraic transformation. Others use those two methods and definition of integration.

1.5 DRAWBACKS IN THE STUDY OF INTEGRAL CALCULUS

The study of integral calculus is not so easy as differential calculus. Readers should be aware of the drawbacks or problems in the study of integral calculus, not encountered in differential calculus. The first problem is that all functions cannot be integrated using definitions because they have no elementary antiderivative for those functions. Examples are

$$\int (x^2 + 1)^{-1} dx, \int e^{-x^2} dx, \int e^{x^3} dx, \int \sqrt{a^2 \cos^2 x + b^2 \sin^2 x} dx \text{ etc.}$$

The second and most important problem is the non-existence of integrating rule for quotients $\int [U(x)/V(x)] dx$ as in differential calculus except the type $\int [f'(x)/f(x)] dx$, although both numerator and denominator

are separately integrable in elementary form. Examples are $\int \frac{e^x}{x+2} dx$, $\int \frac{\sin x}{x} dx$ etc. The third problem lies in the selection of first function (not to be integrated) and second function (to be integrated) in the product $\int U(x) V(x) dx$ for integration. Inaccurate selection sometimes creates massacre as in $\int (px+q) \sqrt{ax^2+bx+c} dx$, $\int \sqrt{x} \sin x dx$ etc.

These drawbacks are partly removed using method of substitutions. This method is again a problem to the readers and may be classified as the fourth drawbacks in the study of integral calculus. The readers need to identify a type of integral and its substitution and thus there arise hundreds of types with hundreds of substitutions available in tables of integral. To make the memorization of substitution easy, the author suggests to *memorize the procedure first and results in the second*. With this in mind the author prepares a short table illustrating procedures alongwith results.

The fifth drawback in the evaluation of some definite and improper integrals where the fundamental theorem of integral calculus i.e. $\int_a^b f(x) dx = F(b) - F(a)$ is not applicable directly. Examples are

$$\int_0^{\pi/2} \log \sin x dx, \int_0^{\pi} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \text{ etc.}$$

To overcome these problems, readers need to memorize a set of properties of definite integral and use them. This is also not easy to use. These properties are again not sufficient for some integrals which creates the sixth problem and constitutes a set of special functions such as gamma and beta functions, elliptic integrals and elliptic functions, $\operatorname{erf}(x) = \int_0^x e^{-t^2} dt$, $-\infty < x < \infty$,

$$\text{Exponential integral function, } E_i x = \int_{-\infty}^x \frac{e^t}{t} dt, \quad -\infty < x < 0,$$

$$\text{Sine integral function } S_i x = \int_0^x \frac{\sin t}{t} dt, \quad -\infty < x < \infty.$$

These functions also removes some deadlocks created by quotients and some elementary integrands.

Expansion of integrand and numerical techniques again removes lot of problems. Some special functions and only two numerical techniques are described in this book.

In the conclusion, readers should remember that lack of integrating rule of quotient and limitations of the use of product rule are responsible for generating a great varieties of techniques as well as special functions and numerical techniques too.

1.6 HYPERBOLIC FUNCTIONS

These functions are the linear combination [i.e. expression of the form $af(x) + bg(x)$ for two functions $f(x), g(x)$ and two constants a, b , general definition in sec. 1.8] of the well-known exponential functions, e^x and e^{-x} . These are essential in problems like hanging telegraph cable. These kinds of functions are called hyperbolic trigonometric functions in contrast to well known circular functions $\cos x, \sin x$ etc which are defined and characterised with respect to a circle. The definition and properties of hyperbolic functions are the subject matter of this section.

DEFINITION

$$\cosh x = (e^x + e^{-x})/2 \text{ and } \sinh x = (e^x - e^{-x})/2.$$

These are read as "cos and sin hyperbolic of x ". The remaining four functions are defined with the help of these two functions as follows

$$\operatorname{sech} x = \frac{2}{e^x + e^{-x}}, \operatorname{cosech} x = \frac{2}{e^x - e^{-x}}, \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \text{ and}$$

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}.$$

BASIC PROPERTY

From the above definitions,

$$\cosh^2 x - \sinh^2 x = \frac{(e^x + e^{-x})^2}{4} - \frac{(e^x - e^{-x})^2}{4} = 1.$$

$\therefore (a \cosh \theta, a \sinh \theta)$ is a point on the hyperbola $x^2 - y^2 = a^2$.

As $\cosh \theta$ and $\sinh \theta$ satisfies the equation of a hyperbola, so the name hyperbolic.

Relation between circular and hyperbolic functions

It is known (or sec. 2.16) that

$$\cos x = \frac{(e^{ix} + e^{-ix})}{2} \text{ and } \sin x = \frac{1}{2i} (e^{ix} - e^{-ix})$$

$$\therefore \cos ix = \frac{(e^{-x} + e^x)}{2} = \cosh x \text{ and}$$

$$\sin ix = \frac{i}{2} (e^x - e^{-x}) = i \sinh x.$$

Similarly, $\cosh ix = (e^{ix} + e^{-ix})/2 = \cos x$ and

$$\sinh ix = (e^{ix} - e^{-ix})/2 = i \sin x.$$

Observe that multiplication of the angle by the imaginary quantity 'i' converts circular to hyperbolic and vice-versa i.e., circular $\rightleftharpoons$ hyperbolic. Further note that 'i' multiplies the new function in case of sine only.

Formulae of hyperbolic functions

From the definition, it can be easily proved that

$$\cosh^2 x - \sinh^2 x = 1, \quad 1 - \tanh^2 x = \operatorname{sech}^2 x, \quad \coth^2 x - 1 = \operatorname{cosech}^2 x.$$

$$\text{Again, } \sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y,$$

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y.$$

The latter two formulae can be derived from circular function as follows :

$$\begin{aligned} \sinh(x+y) &= -i \sin(ix+iy) = -i \sin ix \cos iy - i \cos ix \sin iy \\ &= \sinh x \cosh y + \cosh x \sinh y. \end{aligned}$$

$$\text{Similarly } \sinh 2x = 2 \sinh x \cosh x \text{ and } \cosh 2x = \cosh^2 x + \sinh^2 x.$$

Relation between inverse hyperbolic and logarithmic functions

$$\text{Let } \sinh^{-1} x = \theta \quad \therefore x = \sinh \theta = (e^\theta - e^{-\theta})/2$$

$$\text{or } (e^\theta)^2 - 2xe^\theta - 1 = 0. \text{ It is a quadratic equation in } e^\theta.$$

Normally a quadratic equation has two roots but e^θ is never -ve.

$$\text{So } e^\theta = x + \sqrt{x^2 + 1} \text{ or } \theta = \log(x + \sqrt{x^2 + 1})$$

$$\text{i.e. } \sinh^{-1} x = \log(x + \sqrt{x^2 + 1}), \text{ for all } x.$$

$$\text{Similarly, } \cosh^{-1} x = \log(x + \sqrt{x^2 - 1}), \text{ for } x \geq 1$$

$$\tanh^{-1} x = \frac{1}{2} \log \frac{1+x}{1-x}, \quad x^2 < 1$$

$$\operatorname{sech}^{-1} x = \log \left(\frac{1}{x} + \sqrt{\frac{1}{x^2} - 1} \right), \quad 0 < x \leq 1$$

$$\operatorname{cosech}^{-1} x = \log \left(\frac{1}{x} + \sqrt{\frac{1}{x^2} + 1} \right), \quad x > 0$$

Comparison of circular and hyperbolic functions

The following table shows the differences between them.

Circular functions	Hyperbolic functions
1. $\sin(-x) = -\sin x$, $\cos(-x) = \cos x$	1. $\sinh(-x) = -\sinh x$, $\cosh(-x) = \cosh x$
2. These are periodic	2. Not periodic
3. $ \sin x $ or $ \cos x \leq 1$; $ \sec x \geq 1$; $-\infty < \tan x < \infty$	3. $ \sinh x \leq \infty$ and $\cosh x$ ranges from 1 to ∞ , $\operatorname{sech} x \geq 2$, $ \tanh x \leq 1$

4. No definite value when
 $x \rightarrow \pm \infty$.

4. For $x > 0$ and very big,
 $\cosh x \approx \sinh x \approx \frac{1}{2} e^x$,
 $\tanh x \approx \coth x \approx 1$,
 $\operatorname{sech} x \approx \operatorname{cosech} x \approx 2 e^{-x} \approx 0$
 and $x < 0$ with very big absolute
 values, $\cosh x \approx -\sinh x$,
 $\approx \frac{1}{2} e^{-x}$, $\tanh x \approx \coth x \approx -1$
 $\operatorname{sech} x \approx \operatorname{cosech} x \approx 2 e^x \approx 0$.

1.7 ELEMENTARY STANDARD INTEGRALS

Differential coefficients of elementary functions are determined by using the definition of differentiation. Integration of elementary functions are determined using the definition of integration which is inverse or reverse of differentiation. These integrals are known as *elementary standard integrals*. The integration of a given function is performed by using these integrals directly. So, the method is sometimes called *direct integration* method. A list of some standard integrals obtained directly from differentiation is given below :-

Derivatives	Integrals
1. $\frac{d}{dx}(x) = 1$	1. $\int 1 dx = x + C$
2. $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n, n \neq -1$	2. $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$
3. $\frac{d}{dx}(\log x) = \frac{1}{x}$	3. $\int \frac{1}{x} dx = \log x + C$
4. $\frac{d}{dx}(e^x) = e^x$	4. $\int e^x dx = e^x + C$
5. $\frac{d}{dx}\left(\frac{a^x}{\log a}\right) = a^x$	5. $\int a^x dx = \frac{a^x}{\log a} + C$
6. $\frac{d}{dx}(\sin x) = \cos x$	6. $\int \cos x dx = \sin x + C$
7. $\frac{d}{dx}(\cos x) = -\sin x$	7. $\int \sin x dx = -\cos x + C$
8. $\frac{d}{dx}(\tan x) = \sec^2 x$	8. $\int \sec^2 x dx = \tan x + C$

9. $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$	9. $\int \operatorname{cosec}^2 x \, dx = -\cot x + C$
10. $\frac{d}{dx}(\sec x) = \sec x \tan x$	10. $\int \sec x \tan x \, dx = \sec x + C$
11. $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$	11. $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + C$
12. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, x > 1$	12. $\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + C$
13. $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}, x > 1$	13. $\int \frac{dx}{\sqrt{1-x^2}} = -\cos^{-1} x + C$
14. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$	14. $\int \frac{dx}{1+x^2} = \tan^{-1} x + C$
15. $\frac{d}{dx}(\sinh x) = \cosh x$	15. $\int \cosh x \, dx = \sinh x + C$
16. $\frac{d}{dx}(\cosh x) = \sinh x$	16. $\int \sinh x \, dx = \cosh x + C$
17. $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$	17. $\int \operatorname{sech}^2 x \, dx = \tanh x + C$
18. $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}},$ $x \text{ is real}$	18. $\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1} x + C$
19. $\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}},$ $x > 1$	19. $\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1} x + C$
20. $\frac{d}{dx}(\tanh^{-1} x) = -\frac{1}{x^2-1}, x < 1$	20. $\int \frac{dx}{x^2-1} = -\tanh^{-1} x + C$

N.B. The variable x and the integral are both real (sec. 1.1). As $\log x$ is imaginary for $x < 0$, so $\log x$ in 3 of the above table is used to mean $\log |x|$ and subsequently argument of $\log$ will always be treated the same without using any sign of absolute value. This $\log x$ is the logarithm of x to the base "e" and angles in trigonometric functions are measured in radians unless otherwise stated

1.8 LINEARITY PROPERTY OF INTEGRAL

Theorem 1. Integral of the linear combination of the functions is the linear combinations of the integrals of those functions.

What is a linear combination of functions ? The linear combination of n -functions $f_1(x), f_2(x), \dots, f_n(x)$ is the sum of the product of the functions with equal numbers of constants $A_1, A_2, \dots, A_n$. The sum of the products $A_1 f_1(x) + A_2 f_2(x) + \dots + A_n f_n(x)$ is the linear combination of $f_1(x), f_2(x), \dots, f_n(x)$.

Proof. From differential calculus

$$[A_1 F_1(x)]' = A_1 F_1'(x) = A_1 f_1(x), \text{ (say) for const. } A_1.$$

$$\therefore \int (A_1 f_1(x)) dx = A_1 F_1(x) + C = A_1 \int f_1(x) dx + C$$

$$\therefore F_1'(x) = f_1(x)$$

$$\text{So, } \int f_1(x) dx = F_1(x) + C$$

coefficient of the integrand becomes the coefficient of the integral.

Similarly,

$$[A_1 f_1(x) + A_2 f_2(x)]' = A_1 F_1'(x) + A_2 F_2'(x)$$

$$= A_1 f_1(x) + A_2 f_2(x), \text{ (say)}$$

$$\therefore \int [A_1 f_1(x) + A_2 f_2(x)] dx = A_1 F_1(x) + A_2 F_2(x)$$

$$= A_1 \int f_1(x) dx + A_2 \int f_2(x) dx,$$

$$[\because F_1'(x) = f_1(x) \text{ and } F_2'(x) = f_2(x) \text{ etc.}]$$

Proceeding in this way, it can be written as,

$$\int (A_1 f_1(x) + A_2 f_2(x) + \dots + A_n f_n(x)) dx$$

$$= A_1 \int f_1(x) dx + A_2 \int f_2(x) dx + \dots + A_n \int f_n(x) dx$$

which proves the theorem.

Note 1 : This rule may not be applicable in case of infinite number of integrals.

Note 2 : Since sum of a finite number of constants is a constant, so addition of a single arbitrary constant will serve the purpose in case of finite number of integrals.

1.9 INTEGRATION BY DIRECT DEFINITION AND EXPANSION METHOD

In reality there are a limited number of functions that can be integrated directly from definition. Actually composite functions are simplified to elementary functions for direct integration using definition. This simplification can be carried out by expansion which may be either

a division in case of rational functions or a binomial, exponential and logarithmic expansion. So, integration by expansion means reduction of the given integral into a sum of simpler integrals. It may be mentioned here that there is *no general method* for integrating a fraction as in the case of differential calculus. But there are few fractions of specific type that can be integrated.

The two methods of integration are always used as parts of other methods to be discussed. The following examples illustrate the use of these methods.

Examples

1. Integrate $\frac{5x^3 - 3x^2 + 2x + 6}{x^2}$ w.r.t. x .

Here the integrand is an improper fraction.

Expanding by division, we get

$$\begin{aligned} \text{that } \int \frac{5x^3 - 3x^2 + 2x + 6}{x^2} dx &= 5 \int x dx - 3 \int dx + 2 \int \frac{dx}{x} + 6 \int x^{-2} dx + C \\ &= \frac{5}{2} x^2 - 3x + 2 \log x - \frac{6}{x} + C, \text{ using 1, 2, 3 of the table above.} \end{aligned}$$

2. Integrate $\sqrt{1 + \sin 2x}$ w.r.t. x .

There is no form like the given integrand. Let us remove the surd by using trigonometric identity,

$$\begin{aligned} \int \sqrt{1 + \sin 2x} dx &= \int \sqrt{\cos^2 x + \sin^2 x + 2 \sin x \cos x} dx + C \\ &= \int \sqrt{(\cos x + \sin x)^2} dx + C = \int (\cos x + \sin x) dx + C \\ &= \sin x - \cos x + C, \text{ using 6, 7 of the table above.} \end{aligned}$$

3. Compute $\int \left(\sqrt{\frac{1-x}{1+x}} + \frac{x}{\sqrt{1-x^2}} \right) dx$

Rationalising and adding,

$$\begin{aligned} \int \left(\sqrt{\frac{1-x}{1+x}} + \frac{x}{\sqrt{1-x^2}} \right) dx &= \int \frac{1-x+x}{\sqrt{1-x^2}} dx + C = \int \frac{dx}{\sqrt{1-x^2}} \\ &= \sin^{-1} x + C, \text{ (by 12 of table)} \end{aligned}$$

4. Perform the integration $\int \frac{dx}{\sqrt{x} + \sqrt{x-1}}$.

$$\text{Rationalising } \int \frac{dx}{\sqrt{x} + \sqrt{x-1}} = \int (\sqrt{x} - \sqrt{x-1}) dx + C_1$$

$$\begin{aligned}
 &= \int \sqrt{x} \, dx - \int \sqrt{x-1} \, dx + C_1 \\
 &= \frac{2}{3} x^{3/2} - \int \sqrt{x-1} \, dx + C_1
 \end{aligned}$$

(by 2 of table)

But $\int \sqrt{x-1} \, dx$ does not fit to any one of the table. Let us try to use definitions as follows:

Since differentiation reduces power, so the power of the function, whose first derivative is $\sqrt{x-1}$, is equal to

$$\frac{1}{2} + 1 = \frac{3}{2} \text{ i.e. } \frac{d}{dx}(x-1)^{3/2} = \frac{3}{2} \sqrt{x-1}.$$

$$\text{Therefore, by def. } \int \sqrt{x-1} \, dx = \frac{2}{3} (x-1)^{3/2} + C_2.$$

$$\text{Hence the required integral} = \frac{2}{3} x^{3/2} - \frac{2}{3} (x-1)^{3/2} + C, \quad C = C_1 + C_2.$$

5. Find the integral of $\int \frac{dx}{1 - \cos x}$.

To use the table, let us simplify the integrand.

$$\begin{aligned}
 \int \frac{dx}{1 - \cos x} &= \int \frac{1 + \cos x}{1 - \cos^2 x} dx + C = \int \frac{1 + \cos x}{\sin^2 x} dx + C \\
 &= \int \operatorname{cosec}^2 x \, dx + \int \operatorname{cosec} x \cot x \, dx + C \\
 &= -\cot x - \operatorname{cosec} x + C, \text{ by 9, 11 of the table.}
 \end{aligned}$$

6. Using definition, calculate, $\int (2x+3)^{-1/3} dx$.

The integrand does not belong to any one of the table. To use definition, let us find a function whose first derivative is the integrand. Since differentiation reduces the power by unity, so power of the original function $= -\frac{1}{3} + 1 = \frac{2}{3}$.

$$\text{Therefore, } \frac{d}{dx} (2x+3)^{2/3} = \frac{2}{3} (2x+3)^{-1/3}.$$

or integrating both sides,

$$\int \frac{2}{3} (2x+3)^{-1/3} dx = (2x+3)^{2/3} + C$$

$$\text{or } \int (2x+3)^{-1/3} dx = \frac{3}{4} (2x+3)^{2/3} + C$$

N.B. Using definition of integration is not always easy as seen from ex. 5 and 6 above. So, we need other procedures as substitution etc. next follows.

Exercises 1

Compute the following:

1. $\int x^3 dx,$
2. $\int \frac{dx}{\sqrt[3]{3x^2}},$
3. $\int \frac{x^2 + 2x - 1}{\sqrt{x}} dx,$
4. $\int \sqrt{x} (3x - 2) dx,$
5. $\int (x - 1)(x - 5) dx,$
6. $\int \frac{x^3 - x^2 + 2}{x + 1} dx,$
7. $\int \sqrt{x} (\sqrt{a} - \sqrt{x})^2 dx,$
8. $\int (5e^x + 6 \cos x - 7 \sec^2 x) dx,$
9. $\int \frac{2 \sec x + 3 \cot^2 x + 4 \cos^2 x}{\cos x} dx,$
10. $\int (\sqrt{x} + \frac{1}{\sqrt{x}}) dx,$
11. $\int (2 \sec^2 x - 3 - 3 \cot^2 x) dx,$
12. $\int \sin \frac{1}{2} x \cos \frac{1}{2} x dx,$
13. $\int \frac{1}{1 - \sin x} dx,$
14. $\int \left(\sqrt{\frac{x-1}{x+1}} - \frac{\lambda}{\sqrt{x^2-1}} \right) dx,$
15. $\int \frac{(1+\lambda)^2 - (1-\lambda)^2}{x[(1+\lambda)^2 + (1-\lambda)^2]} d\lambda,$
16. Use definition to calculate $\int \sqrt{x-2} dx,$
17. Use definition to integrate $\frac{1}{\sqrt{\lambda+1}}.$
18. Use definition to find $\int \frac{dx}{(2-3x)^{1/3}}.$
19. Use definition to compute $\int \frac{dx}{(3x-2)^2}.$

Carryout the following integrations:

20. $\int \tan^2 x dx,$
21. $\int \cot^2 x dx,$
22. $\int \frac{\cos 2x}{\sin^2 x \cos^2 x} d\lambda.$
23. $\int \cos x \cos 2x \cos 3x dx.$
24. $\int \frac{2^{x+1} - 5^{x-1}}{10^x} dx,$
25. $\int \cos x \sqrt{1 + \cos 2x} dx,$
26. $\int \operatorname{sech} x dx,$
27. $\int \tanh^2 x dx,$

28. $\int \frac{\cosh 2x}{\sinh^2 x \cosh^2 x} dx$

29. $\int \sqrt{\cosh 2x - 1} dx,$

30. $\int \sqrt{\cosh 2x + 1} dx,$

31. $\int x^2 dx,$

32. $\int \frac{dx}{(1 - 2x + 3x^2 + 4x^3 + \dots \infty)},$

33. $\int e^{-x^2} dx,$

34. $\int \frac{\cos 5x + \cos 4x}{1 - 2\cos 3x} dx,$

[Hints : multiply both numerator and denominator by $\sin 3x$]

35. $\int (f(x)g'(x) + f'(x)g(x)) dx,$ (') denotes order of derivative.

36. Integrate $\frac{1}{2}f'(x)$ with respect to x^4 where

(i) $f(x) = \tan^{-1} x + \log \sqrt{1+x} - \log \sqrt{1-x}.$

(ii) $f(x) = \tan^{-1} 2x + \frac{1}{2} \log (1+2x)/(1-2x).$

Answers

1. $\frac{1}{4}x^4 + c$

2. $[(9x)^{1/3} + c]$

3. $2\sqrt{x} \left(\frac{1}{5}x^2 + \frac{2}{3}x - 1 \right) + c$

4. $2x\sqrt{x} \left(\frac{3}{5}x - \frac{2}{3} \right) + c$

5. $\frac{1}{3}x^3 - 3x^2 + 5x + c$

6. $\left(\frac{1}{3}x^3 - x^2 + 2x + c \right)$

7. $\frac{2}{3}ax^{3/2} - \sqrt{ax^2} + \frac{2}{5}x^{5/2} + c$

8. $(5e^x + 6 \sin x - 7 \tan x + c)$

9. $2 \tan x - 3 \operatorname{cosec} x + 4 \sin x + c$

10. $\left(\frac{2}{3}x^{3/2} + 2\sqrt{x} + c \right)$

11. $(2 \tan x + 3 \cot x + c)$

12. $\left(-\frac{1}{2} \cos x + c \right)$

13. $(\tan x + \sec x + c)$

14. $(-\cosh^{-1} x + c)$

15. $(2 \tan^{-1} x + c)$

16. $\left(\frac{2}{3}(x-3)^{2/3} + c \right)$

17. $(2\sqrt{x+1} + c)$

18. $c - \frac{1}{2}(2-3x)^{2/3}$

19. $\frac{-1}{3(3x-2)} + c$

20. $(\tan x - x + c)$

21. $(c - x - \cot x)$

22. $(c - \cot x - \tan x)$

23. $\frac{1}{48}(2 \sin 6x + 3 \sin 4x + 6 \sin 2x + 12x) + c$

$$24. \left(-\frac{2}{\log 5} 5^{-x} + \frac{1}{5 \log 2} 2^{-x} + c \right)$$

$$25. \left(\frac{1}{2\sqrt{2}} \cos 2x + c \right)$$

$$27. (x - \tanh x + c)$$

$$29. (\sqrt{2} \cosh x + c)$$

$$31. \left(x + \frac{x^3 \log a}{3(1)} + \frac{x^5 (\log a)^2}{5(2)} + \dots \infty + c \right)$$

$$32. \left[\frac{1}{3}(1+x)^3 + c \right]$$

$$34. \left[-\left(\sin x + \frac{1}{2} \sin 2x \right) + c \right]$$

$$36. (i) \left[-\log(1-x^4) + c \right]$$

$$26. \left(\tan^{-1}(\sinh x) + c \right)$$

$$28. (\tanh x - \coth x + c)$$

$$30. (\sqrt{2} \sinh x + c)$$

$$33. \left(x - \frac{1}{3}x^3 + \frac{x^5}{8} - \dots \infty + c \right)$$

$$35. (f(x)g(x) + c)$$

$$36. (ii) \left[-32 \log(1-16x^4) + c \right]$$

$$24. \left(-\frac{2}{\log 5} 5^{-x} + \frac{1}{5 \log 2} 2^{-x} + c \right)$$

$$25. \left(\frac{1}{2\sqrt{2}} \cos 2x + c \right)$$

$$27. (x - \tanh x + c)$$

$$29. (\sqrt{2} \cosh x + c)$$

$$31. \left(x + \frac{x^3 \log a}{3(1)} + \frac{x^5 (\log a)^2}{5(2)} + \dots \infty + c \right)$$

$$32. \left[\frac{1}{3}(1+x)^3 + c \right]$$

$$34. \left[-\left(\sin x + \frac{1}{2} \sin 2x \right) + c \right]$$

$$36. (i) \left[-\log(1-x^4) + c \right]$$

$$26. \left(\tan^{-1}(\sinh x) + c \right)$$

$$28. (\tanh x - \coth x + c)$$

$$30. (\sqrt{2} \sinh x + c)$$

$$33. \left(x - \frac{1}{3}x^3 + \frac{x^5}{8} - \dots \infty + c \right)$$

$$35. (f(x)g(x) + c)$$

$$36. (ii) \left[-32 \log(1-16x^4) + c \right]$$

SUBSTITUTION (I)

2.1 INTRODUCTION

Substitution is basically the principal method of integration. Because proper substitution reduces a composite or even a complicated integrand to an elementary standard types of chapter 1. Certain types of rational and irrational integrand has specific substitution for their reduction to elementary standard integrals. These substitutions will be discussed in this chapter and in chapter 4. After the integration w.r.t. the *new variable*, the old one is to be *revived*.

2.2 CHANGE OF VARIABLES

Let $F(g(x))$ be a composite function i.e. a function of function. Therefore by chain rule of differentiation,

$$\begin{aligned}\frac{d}{dx} F(g(x)) &= \frac{d}{du} F(u) \cdot \frac{du}{dx}, \text{ when } u = g(x), \frac{du}{dx} = g'(x). \\ &= F'(u) \frac{du}{dx}\end{aligned}$$

$$\therefore \int F'(u) \frac{du}{dx} dx = \int F'(u) du = F(u) + C \quad (1)$$

$$\text{i.e.} \quad \int F'(g(x)) g'(x) dx = F(g(x)) + C \quad (2)$$

Therefore, the variable of integration x in left hand (L.H.) integral of (2) is changed to u in the middle integral of (1) to fit the definition. That is, by change of variables some complicated product and quotient functions can be transformed to elementary functions and can be easily integrated. This process of integration is known as *integration by substitution* or *change of variable*. This idea is used in the following sections. Any change in the variable must be recognised by the change of differential also.

2.3 STANDARD INTEGRAL OF THE TYPE $\int x^n dx$

Any integral of an expression with whole power falls into this standard type. In general, $\int (ax+b)^n dx$ is of this type. Because change of expression $(ax+b)$ to a single variable u and replacement of differential dx by the corresponding differential in u reduces $\int (ax+b)^n dx$ to this standard type; i.e. let us substitute $ax+b = u$.

Differentiating, $adx = du$ or $dx = \frac{1}{a} du$.

$$\begin{aligned}\therefore \int (ax+b)^n dx &= \frac{1}{a} \int u^n du + c \\ &= \frac{1}{a} \frac{1}{n+1} u^{n+1} + c, n \neq -1 \\ &= \frac{(ax+b)^{n+1}}{a(n+1)} + c, n \neq -1\end{aligned}$$

Therefore, it is clear from here that any deviation from the standard type needs a substitution of a new variable with a replacement of differential of the old variable by the new one.

2.4 STANDARD INTEGRAL OF THE TYPE $\int \frac{1}{x} dx$

The integral with fractional integrand having numerator as the differential coefficient of the denominator can be integrated easily by this type and the integral is the log of the denominator. More, generally,

$\int \frac{f'(x)}{f(x)} dx = \log f(x) + c$. Note that the integrand is a special kind of fraction. This result can be easily established. To change the variable of integration, let $f(x) = u$, $\therefore f'(x) dx = du$.

$$\begin{aligned}\therefore \int \frac{f'(x)}{f(x)} dx &= \int \frac{1}{u} du = \log u + c \\ &= \log f(x) + c\end{aligned}$$

2.5 STANDARD INTEGRALS $\int \tan x dx$ and $\int \cot x dx$

These two integrals can be put into the form 2.4 and are considered as standard integrals of elementary functions.

(A) The integral $\int \tan x dx = \int \frac{\sin x}{\cos x} dx = - \int \frac{du}{u}$ when $u = \cos x$.
Therefore, $\int \tan x dx = - \log u + c = - \log \cos x + c$
 $= \log \sec x + c$.

(B) Similarly, the integral $\int \cot x dx$
 $= \int \frac{\cos x}{\sin x} dx = \int \frac{du}{u}$ when $\sin x = u$
 $= \log u + c = \log \sin x + c$.

2.6 INTEGRAL OF THE TYPE $\int \frac{a \cos x + b \sin x}{c \cos x + d \sin x} dx$

The integrand is a fraction of trigonometric sines and cosines. In this case the numerator is expressed as a linear combination of the denominator and the differential coefficient of the denominator. Let,

$$a \cos x + b \sin x = l(c \cos x + d \sin x) + m \frac{d}{dx}(c \cos x + d \sin x)$$

This is an identity of sines and cosines. So, equating coefficients of $\cos x$ and $\sin x$, we get,

$$a = cl + dm \quad (i) \quad \text{and} \quad b = ld - mc \quad (ii)$$

Solving for l and m , we have,

$$l = \frac{ac + bd}{c^2 + d^2}, \quad m = \frac{ad - bc}{c^2 + d^2}.$$

Therefore, the given integral becomes,

$$\begin{aligned} \int \frac{a \cos x + b \sin x}{c \cos x + d \sin x} dx &= \frac{ac + bd}{c^2 + d^2} x \\ &+ \frac{ad - bc}{c^2 + d^2} \log(c \cos x + d \sin x) + k \quad \text{by 2.4.} \end{aligned}$$

Examples

1. Perform the integration $\int (3x+5)^{3/2} dx$.

This integral is *not* of elementary type. But the expression $(3x+5)$ has a power $3/2$. So, it seems to be of the form $\int x^n dx$ which needs a single variable of the type x and not an expression of x . To make the given expression $(3x+5)$ a single variable, let us substitute $3x+5 = u$ and the corresponding differential $3dx = du$ or $dx = (1/3) du$. Therefore,

$$\begin{aligned} \int (3x+5)^{3/2} dx &= (1/3) \int u^{3/2} du + c \quad \text{which is of form 2.3} \\ &= \frac{1}{3} \cdot \frac{2}{5} u^{5/2} + c = \frac{2}{15} (3x+5)^{5/2} + c. \end{aligned}$$

2. Compute $\int x^2 (2x^3+3)^{-2} dx$.

Here the integrand is a proper fraction. We have no way of integrating a fraction as differential calculus. But if observe, we shall see that differential of $(2x^3+3)$ involves $(x^2 dx)$. So, substitution is suggested by the change of variables (sec. 2.2). Hence fraction reduces to the type 2.3. Therefore, let us substitute $2x^3+3 = u$ or differentiating,

$$6x^2 dx = du \quad \text{or} \quad x^2 dx = \frac{1}{6} du$$

$$\begin{aligned}\therefore \int x^2 (2x^3 + 3)^{-2} dx &= \frac{1}{6} \int u^{-2} du + c = -\frac{1}{6} u^{-1} + c \text{ by 2.2} \\ &= -\frac{1}{6} (2x^3 + 3)^{-1} + c = -\frac{1}{6(2x^3 + 3)} + c\end{aligned}$$

3. Integrate $\cos^3 x \sin x$.

Here the integrand is a product. Of which one i.e. $\cos x$ possesses power and its differential coeff. is $(-\sin x)$. So, by change of variable (sec 2.2) product can be reduced to the type 2.3. Therefore, let us substitute $\cos x = u$ or differentiating, $(-\sin x) dx = du$.

$$\begin{aligned}\therefore \int \cos^3 x \sin x dx &= -\int u^3 du + c = -\frac{1}{4} u^4 + c \text{ by 2.3} \\ &= -\frac{1}{4} \cos^4 x + c\end{aligned}$$

4. Integrate $\cos^3 (2x+3) \sin(2x+3)$.

Here the integrand involves product and angles of trigonometric functions are expressions instead of a single variable. So, to make angles a function of a single variable, let us first substitute $2x+3 = u$ or differentiating $dx = \frac{1}{2} du$

$$\therefore \int \cos^3 (2x+3) \sin(2x+3) dx = \frac{1}{2} \int \cos^3 u \sin u du + c$$

which is identical to Ex. 3. Next substitution of $\cos u = p$ gives as in Ex. 3., $\int \cos^3 (2x+3) \sin(2x+3) dx = -\frac{1}{8} \cos^4 (2x+3) + c$.

5. Compute the integral $\int \frac{3x^2 + 4}{x^3 + 4x + 2} dx$.

This integrand is a fraction. But the differential coefficient of the denominator, $(x^3 + 4x + 2)$ is $(3x^2 + 4)$ which is the numerator. Therefore, by 2.4

$$\int \frac{3x^2 + 4}{x^3 + 4x + 2} dx = \log (x^3 + 4x + 2) + c.$$

6. Compute $\int \frac{dx}{\sin (2x-a) \cos (2x-b)}$

Note that the integrand is the reciprocal of a product and the angles of sine & cosine differ by a constant, $(a-b)$. So, the integral can be written as,

$$\begin{aligned}\int \frac{dx}{\sin (2x-a) \cos (2x-b)} \\ = \frac{1}{\cos (a-b)} \int \frac{\cos \{(2x-b) - (2x-a)\}}{\sin (2x-a) \cos (2x-b)} dx + c\end{aligned}$$

$$= \frac{1}{\cos(a-b)} \int \frac{\cos(2x-b) \cos(2x-a) + \sin(2x-b) \sin(2x-a)}{\sin(2x-a) \cos(2x-b)} dx + c$$

$$= \frac{1}{\cos(a-b)} \int \left\{ \frac{\cos(2x-a)}{\sin(2x-a)} + \frac{\sin(2x-b)}{\cos(2x-b)} \right\} dx + c$$

$$= \frac{1}{2 \cos(a-b)} \int \left\{ \frac{2 \cos(2x-a)}{\sin(2x-a)} - \frac{2 \sin(2x-b)}{\cos(2x-b)} \right\} dx + c, \text{balancing}$$

the differential coeffs., of denominator.

$$= \frac{1}{2 \cos(a-b)} \{ \log \sin(2x-a) - \log \cos(2x-b) \} + c \text{ by 2.4 or, 2.5}$$

$$= \frac{1}{2 \cos(a-b)} \log \left\{ \frac{\sin(2x-a)}{\cos(2x-b)} \right\} + c.$$

Exercises 2 (1) $\frac{1}{2u} (a+b \sin x)^2$

Perform the following :

$$1. \int x^2 (a + bx^3)^2 dx$$

$$2. \int \frac{4x^2}{3\sqrt{3-5x^3}} dx$$

$$3. \int \frac{(\sqrt{a}-\sqrt{x})^3}{\sqrt{x}} dx$$

$$4. \int \frac{2x+3}{(x^2+3x+1)^3} dx$$

$$5. \int \frac{1-2t}{\sqrt{3+t-t^2}} dt$$

$$6. \int \frac{(3+2 \log x)^2}{x} dx$$

$$7. \int (a + b \sin x) \cos x dx$$

$$8. \int \frac{e^x}{(ae^x + b)^2} dx$$

$$9. \int \frac{\cos ax}{\sqrt{b + c \sin ax}} dx$$

$$10. \int x^2 \sin x^3 \cos x^3 dx$$

$$11. \int \tan^3 x \sec^2 x dx$$

$$12. \int \frac{\sec^2 2x}{\sqrt{2+3 \tan 2x}} dx$$

$$13. \int \left(\frac{\sec x}{1 + \tan x} \right)^2 dx$$

$$14. \int \frac{\operatorname{cosec}^2(4x+5)}{\sqrt{1 - \cot(4x+5)}} dx$$

$$15. \int \frac{a \cos x + b \sin x}{(a \sin x - b \cos x)^2} dx$$

$$16. \int \left(e^x + \frac{1}{x} \right) (e^x + \log x) dx$$

$$17. \int \frac{\sec 2\theta \tan 2\theta}{\sqrt{3 \sec 2\theta - 4}} d\theta$$

$$18. \int \frac{\sin 2x}{(a^2 \sin^2 x + b^2 \cos^2 x)^3} dx$$

$$19. \int \frac{\sin x}{\sqrt{2+3 \tan^2 x}} dx$$

$$20. \int \frac{(2+3 \sin^{-1} x)^4}{\sqrt{1-x^2}} dx$$

$\hookrightarrow \text{put } \tan x = \frac{\sin x}{\cos x}$

$$\checkmark 21. \int \frac{\sin 2\theta + 3 \cos 3\theta}{\cos 2\theta - 2 \sin 3\theta} d\theta$$

$$\checkmark 22. \int \frac{e^{2x} - 1}{e^{2x} + 1} dx \quad \text{multiply by } e^{-2x}$$

$$\checkmark 23. \int \frac{\tan x}{1 + \log \sec x} dx$$

$$Q \checkmark 24. \int \frac{\sinh x + \cosh x}{\sin x + \cosh x} dx$$

$$\checkmark 25. \int \frac{\operatorname{cosec}^2 x}{1 + \cot x} dx$$

$$\checkmark 26. \int \frac{2x + 1 + (1/x)}{x^2 + x + \log x} dx$$

$$Q \checkmark 27. \int \frac{x+4}{2x+3} dx$$

$$\checkmark 28. \int \frac{dx}{(1+x^2) \tan^{-1} x}$$

$$Q \checkmark 29. \int \frac{3 \sin x}{\sin x + 2 \cos x} dx$$

$$Q \checkmark 30. \int \frac{\cos x + 4 \sin x}{3 \cos x + 2 \sin x} dx$$

$$Q \checkmark 31. \int \frac{\sin 2x}{(a + b \cos x)^2} dx$$

$$Q \checkmark 32. \int (a + b \cos x)^p \sin x dx$$

$$Q \checkmark 33. \int (a - b \tan x)^q \sec^2 x dx$$

$$Q \checkmark 34. \int \frac{(a + b \sin^{-1} x)^m}{\sqrt{1-x^2}} dx$$

$$Q \checkmark 35. \int \operatorname{cosec}^n x \cos x dx$$

$$Q \checkmark 36. \int x \cos^2 x^2 \sin x^2 dx$$

$$Q \checkmark 37. \int x^3 \tan^4 x^4 \sec^2 x^4 dx$$

$$Q \checkmark 38. \int \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x} \quad \text{sec}^2 \left(\frac{b}{a} \tan x \right)$$

$$Q \checkmark 39. \int \frac{\log x}{x \sqrt{1 + \log x}} dx$$

$$Q \checkmark 40. \int \frac{dx}{\cos(2x-a) \cos(2x+a)}$$

$$Q \checkmark 41. \int \frac{dx}{x + \sqrt{x}}$$

$$42. \int \frac{x}{\sqrt{x} + 1} dx \quad \text{u = } \sqrt{x} + 1$$

$$Q \checkmark 43. \int \frac{\sqrt{\tan x}}{\sin x \cos x} dx$$

$$Q \checkmark 44. \int \frac{dx}{\sin(x-a) \sin(x-b)}$$

$$Q \checkmark 45. \int \frac{1 + 2 \sin^2 x + 4 \sin x \cos x - \cos x + \sin x}{\sin x + \cos x} dx$$

$$Q \checkmark 46. \int \frac{x \sinh x + 1}{x \sin^{-1}(\cosh x + \log x) \sqrt{1 - (\cosh x + \log x)^2}} dx$$

$$Q \checkmark 47. \int \frac{dx}{\sin x + \tan x}$$

$$\checkmark 48. \int \frac{\sqrt{9x^2 - 1}}{x} dx \quad \text{sub } 2/3x$$

[Hints: Q. 38 : $a \tan x = bu$ and Q. 47 : $u = \tan \frac{x}{2}$]

$$\checkmark 49. \int \frac{1}{1 + e^x} dx$$

$$50. \int \sqrt{x + \sqrt{x^2 + 2}} dx$$

[Hints: Q. 50 :

$$x + \sqrt{x^2 + 2} = x + (x + i\sqrt{2})^{1/2} (x - i\sqrt{2})^{1/2} = \frac{1}{2} [2x + 2(x + i\sqrt{2})^{1/2}$$

$$(x - i\sqrt{2})^{1/2}] = \frac{1}{2} [(x + i\sqrt{2})^{1/2} + (x - i\sqrt{2})^{1/2}]^2$$

$$\text{Therefore } \int \sqrt{x + \sqrt{x^2 + 2}} \, dx = \frac{1}{\sqrt{2}} \int [(x + i\sqrt{2})^{1/2} + (x - i\sqrt{2})^{1/2}] \, dx$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{2}{3} [(x + i\sqrt{2})^{3/2} + (x - i\sqrt{2})^{3/2}] + c$$

$$= \frac{\sqrt{2}}{3} [(x + i\sqrt{2})^{1/2} + (x - i\sqrt{2})^{1/2}] [(x + i\sqrt{2}) + (x - i\sqrt{2}) - \sqrt{x^2 + 2}] + c$$

$$= \frac{\sqrt{2}}{3} \cdot \sqrt{2} \sqrt{(x + \sqrt{x^2 + 2})(2x - \sqrt{x^2 + 2})} + c$$

$$= \frac{2}{3} \cdot \frac{(x + \sqrt{x^2 + 2})(2x - \sqrt{x^2 + 2})}{\sqrt{(x + \sqrt{x^2 + 2})}} + c = \text{result.}]$$

Answers

1. $\left[\frac{1}{9b} (a + bx^3)^3 + c \right]$

2. $\left[c - \frac{2}{5} (3 - 5x^3)^{2/3} \right]$

3. $\left[c - \frac{1}{2} (\sqrt{a} - \sqrt{x})^4 \right]$

4. $\left[c - \frac{1}{2(x^2 + 3x + 1)^2} \right]$

5. $\left[2\sqrt{3 + t - t^2} + c \right]$

6. $\frac{1}{6} (3 + 2 \log x)^3 + c$

7. $\left[\frac{1}{2b} (a + b \sin x)^3 + c \right]$

8. $\left[c - \frac{1}{a(ae^x + b)} \right]$

9. $\left[\frac{2}{ac} \sqrt{b + c \sin ax} + k \right]$

10. $\left[\frac{1}{6} (\sin x^3)^2 + c \right]$

11. $\left[\frac{1}{4} \tan^4 x + c \right]$

12. $\left[\frac{1}{3} \sqrt{2 + 3 \tan 2x} + c \right]$

13. $\left[c - \frac{1}{1 + \tan x} \right]$

14. $\left[\frac{1}{2} \sqrt{1 - \cot(4x + 5)} + c \right]$

15. $\left[\frac{1}{b \cos x - a \sin x} + c \right]$

16. $\left[\frac{1}{2} (e^x + \log x)^2 + c \right]$

17. $\left[\frac{1}{3} \sqrt{3 \sec 2\theta - 4} + c \right]$

18. $\left[\frac{1}{2(b^2 - a^2)(a^2 \sin^2 x + b^2 \cos^2 x)^2} + c \right]$

19. $\left[\cos x \sqrt{2 + 3 \tan^2 x} + c \right]$

20. $\left[\frac{1}{15} (2 + 3 \sin^{-1} x)^5 + c \right]$

21. $\left[c - \frac{1}{2} \log(\cos 2\theta - 2 \sin 3\theta) \right]$

22. $\left[\log(e^x - e^{-x}) + c \right]$

Ans: $\int (e^x + e^{-x}) dx = e^x - e^{-x} + c$

23. $[\log(1 + \log \sec x) + c]$ 24. $[\log(\sin x + \cosh x) + c]$
 25. $[c - \log(1 + \cot x)]$ 26. $[\log(x^2 + x + \log x) + c]$
 27. $\left[\frac{x}{2} + \frac{5}{4} \log(2x + 3) + c\right]$ 28. $[\log(\tan^{-1} x) + c]$
 29. $\left[\frac{3}{5}x - \frac{6}{5} \log(\sin x + 2 \cos x) + c\right]$ 30. $\left[\frac{11}{13}x - \frac{10}{13} \log(3 \cos x + 2 \sin x) + c\right]$
 31. $\left[-\frac{2}{b^2} \left\{ \log(a + b \cos x) + \frac{a}{a + b \cos x} \right\} + c\right]$
 32. $\left[-\frac{1}{b(p+1)} (a + b \cos x)^{p+1} + c\right]$ 33. $\left[-\frac{1}{b(q+1)} (a - b \tan x)^{q+1} + c\right]$
 34. $\left[\frac{1}{b(m+1)} (a + b \sin^{-1} x)^{m+1} + c\right]$ 35. $\left[\frac{\operatorname{cosec}^{n-1} x}{1-n} + c\right]$
 36. $\left[-\frac{1}{6} \cos^3 x^2 + c\right]$ 37. $\left[\frac{1}{20} \tan^5 x^4 + c\right]$
 38. $\left[\frac{1}{ab} \tan^{-1} \left(\frac{a}{b} \tan x\right) + c\right]$ 39. $\frac{2}{3} [(\log x - 2) \sqrt{1 + \log x} + c]$
 40. $\left[\frac{1}{\lambda \sin 2a} \log \frac{\sec(2x+a)}{\sec(2x-a)} + c\right]$ 41. $[2 \log(\sqrt{x} + 1) + c]$
 42. $\left[\frac{2}{3} x^{3/2} - x + 2\sqrt{x} + \log(\sqrt{x} + 1) + c\right]$ 43. $[2\sqrt{\tan x} + c]$
 44. $\left[\frac{1}{\sin(b-a)} \log \frac{\sin(x-b)}{\sin(x-a)} + c\right]$
 45. $[\sin x - 3 \cos x - \log(\sin x + \cos x) + c]$ 46. $[\log \sin^{-1}(\cosh x + \log x) + c]$
 47. $\left[\frac{1}{2} \log \tan \frac{x}{2} - \frac{1}{4} \tan^2 \frac{x}{2} + c\right]$ 48. $[\sqrt{9x^2 - 1} - \sec^{-1} 3x + c]$
 49. $[x - \log(1 + e^x) + c]$ 50. $\left[\frac{2}{3} \frac{x^2 + \sqrt{x^2 + 2} - 2}{\sqrt{x + \sqrt{x^2 + 2}}} + c\right]$

2.7 INTEGRAL OF THE TYPES $\int e^x dx$ and $\int a^x dx$

Since $\int e^x dx = e^x + c$ and $\int a^x dx = \frac{a^x}{\log a} + c$

So integrals $\int e^{f(x)} f'(x) dx$ and $\int a^{f(x)} f'(x) dx$ can be integrated by the change of variables using substitution $f(x) = u$

2.8 INTEGRAL OF THE TYPES $\int \frac{x^m}{(a + bx)^n} dx$, $\int \frac{dx}{x^m(a + bx)^n}$

$$\int \frac{x^{2m+1}}{(\sqrt{a + bx^2})^n} dx \text{ and } \int \frac{dx}{x^n \sqrt{a + bx^2}}$$

(A) Integral $\int \frac{x^m}{(a+bx)^n} dx$, $m + ve$ integer.

If m is a positive integer, then substitute $(a+bx)=u$. Therefore,
 $x = \frac{u-a}{b}$. So,

$$\int \frac{x^m}{(a+bx)^n} dx = \frac{1}{b^{m+1}} \int \frac{(u-a)^m}{u^n} du.$$

As m is a positive integer, the numerator of the fractional integrand, namely $(u-a)^m$ can be finitely expanded by binomial theorem. Then by actual division by u^n , this can be integrated term by term.

(B) Integral $\int \frac{dx}{x^m(a+bx)^n}$, $(m+n) > 1$ and integer.

If $(m+n) > 1$ and an integer, then substitute $u = \frac{a+bx}{x}$.

$$\text{So, } x = \frac{a}{u-b} \quad \text{and} \quad dx = -\frac{a}{(u-b)^2} du$$

$$\begin{aligned} \therefore \int \frac{dx}{x^m(a+bx)^n} &= \int \frac{1}{x^{m+n}} \left(\frac{x}{a+bx} \right)^n dx \\ &= - \int \frac{(u-b)^{m+n}}{a^{m+n}} \cdot \frac{1}{u^n} \cdot \frac{a}{(u-b)^2} du \\ &= - \frac{1}{a^{m+n-1}} \int \frac{(u-b)^{m+n-2}}{u^n} du \end{aligned}$$

As $(m+n) > 1$ and an integer, so $m+n-2=0$ or a positive integer greater than 1. Hence by expanding the numerator and dividing by denominator, the integral can be easily integrated term by term.

(C) $\int \frac{x^{2m+1}}{(\sqrt{a+bx^2})^n} dx$, $m + ve$ integer.

$$\text{Substitute } u^2 = a+bx^2, \therefore x^2 = \frac{1}{b}(u^2-a) \text{ and } x dx = \frac{1}{b} u du$$

$$\begin{aligned} \therefore \int \frac{x^{2m+1}}{(\sqrt{a+bx^2})^n} dx &= \int x^{2m} \frac{x}{(\sqrt{a+bx^2})^n} dx \\ &= \frac{1}{b^m} \int \frac{(u^2-a)^m}{u^n} \cdot \frac{u}{b} du \\ &= \frac{1}{b^{m+1}} \int \frac{(u^2-a)^m}{u^{n-1}} du \end{aligned}$$

$\therefore m$ is a +ve integer, so $(u^2 - a)^m$ can be binomially expanded and after division by u^{n-1} , the integration can be performed term by term.

NOTE : More general integral $\int \frac{f(x^2)x}{(\sqrt{a+bx^2})^n} dx$ can also be integrated using the above substitution.

(D) Integral $\int \frac{dx}{x^n \sqrt{a+bx^2}}$, n + ve even integer. In this case, substitute

$$x = \frac{1}{u}. \text{ So } dx = -\frac{1}{u^2} du \text{ and}$$

$$\int \frac{dx}{x^n \sqrt{a+bx^2}} = -\int \frac{u^n \cdot u}{u^2 \sqrt{au^2+b}} du = -\int \frac{u^{n-1}}{\sqrt{au^2+b}} du$$

$\therefore n$ is a +ve even integer, so $(n-1)$ is a +ve integer and is of the type (C) above. Hence putting $au^2 + b = p^2$ this can be integrated as in (C).

Examples

1. Integrate $\frac{e^{\tan^{-1}x}}{1+x^2}$.

This is of the type 2.7. Let $\tan^{-1}x = u$ be the substitution. So, $\frac{1}{1+x^2} dx = du$. Therefore,

$$\int \frac{e^{\tan^{-1}x}}{1+x^2} dx = \int e^u du = e^u + c = e^{\tan^{-1}x} + c.$$

2. Compute $\int a^{\cos 2x} \sin 2x dx$.

This is also of the type 2.7. Let $\cos 2x = u$ be the substitution. So, $-2 \sin 2x dx = du$ or $\sin 2x dx = -\frac{1}{2} du$. Therefore,

$$\begin{aligned} \int a^{\cos 2x} \sin 2x dx &= -\frac{1}{2} \int a^u du = -\frac{1}{2} a^u / \log a + c \\ &= -\frac{1}{2} a^{\cos 2x} / \log a + c \end{aligned}$$

3. Evaluate the integral $\int \frac{x^3}{(a+bx)^4} dx$.

This is of the type 2.8 (A), because $m = 3$ is a +ve integer. So the substitution is $a + bx = u$ or $x = \frac{u-a}{b}$ and $dx = \frac{1}{b} du$. Therefore,

$$\begin{aligned}\int \frac{x^3}{(a+bx)^4} dx &= \int \frac{(u-a)^3}{b^3 u^4} \cdot \frac{1}{b} du = \frac{1}{b^4} \int \frac{(u-a)^3}{u^4} du \\&= \frac{1}{b^4} \int \frac{u^3 - 3u^2a + 3ua^2 - a^3}{u^4} du, \text{ Binomial expansion.} \\&= \frac{1}{b^4} \int \left(\frac{1}{u} - \frac{3a}{u^2} + \frac{3a^2}{u^3} - \frac{a^3}{u^4} \right) du, \text{ by division} \\&= \frac{1}{b^4} \left[\log u + \frac{3a}{u} - \frac{3a^2}{2u^2} + \frac{a^3}{3u^3} \right] + c \\&= \frac{1}{b^4} \left[\log(a+bx) + \frac{3a}{a+bx} - \frac{3a^2}{2(a+bx)^2} + \frac{a^3}{3(a+bx)^3} \right] + c\end{aligned}$$

4. Integrate $\frac{1}{x^{3/2}(a+bx)^{5/2}}$.

Here $m+n = \frac{3}{2} + \frac{5}{2} = 4 > 1$ and an integer. So, it is of type 2.8 (B).

Hence the substitution is $u = \frac{a+bx}{x}$ or $x = \frac{a}{u-b}$ and $dx = -\frac{a}{(u-b)^2} du$.

Therefore,

$$\begin{aligned}\int \frac{dx}{x^{3/2}(a+bx)^{5/2}} &= \int \frac{x^{5/2} dx}{x^4(a+bx)^{5/2}} = \int \frac{1}{x^4} \left(\frac{x}{a+bx} \right)^{5/2} dx \\&= \int \left(\frac{a-b}{a} \right)^4 \cdot \frac{1}{u^{5/2}} \cdot \frac{-a}{(u-b)^2} du \\&= -\frac{1}{a^3} \int \frac{(u-b)^2}{u^{5/2}} du = -\frac{1}{a^3} \int (u^{-1/2} - 2bu^{-3/2} + b^2u^{-5/2}) du \\&= -\frac{1}{a^3} \left[2\sqrt{u} + 4b \frac{1}{\sqrt{u}} - \frac{2b^2}{3} u^{-3/2} \right] + c \quad \text{where } u = \frac{a+bx}{x}.\end{aligned}$$

5. Compute the integral $\int \frac{x^3}{\sqrt{1-x^2}} dx$.

Here $m=1$ is a +ve integer, $2m+1=3$ and an odd integer and is of type 2.8 (C). So, the standard substitution is $1-x^2=u^2$ and $(-2x) dx = 2u du$. Therefore,

$$\int \frac{x^3}{\sqrt{1-x^2}} dx = -\int (1-u^2) du = \frac{1}{3} u^3 - u + c$$

$$= \frac{1}{3} (1-x^2)^{3/2} - \sqrt{1-x^2} + c$$

6. Evaluate $\int \frac{dx}{x^4 \sqrt{x^2-1}}$.

Here $n=4$, an even +ve integer and is of type 2.8 (D). So, the substitution is $x = \frac{1}{u}$ and $dx = -\frac{du}{u^2}$. Therefore,

$\int \frac{dx}{x^4 \sqrt{x^2-1}} = -\int \frac{u^3 du}{\sqrt{1-u^2}}$ which is the same as Ex. 5 and of type 2.8 (C). So, by Ex. 5,

$$\int \frac{dx}{x^4 \sqrt{x^2-1}} = -\frac{1}{3} (1-u^2)^{3/2} + \sqrt{1-u^2} + c$$

$$= \frac{\sqrt{x^2-1}}{3x^3} (2x^2+1) + c.$$

Exercises 2 (2)

Compute the following :

Q1. $\int e^{5x} dx$

Q2. $\int 5^x dx$

Q3. $\int \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}} dx$

Q4. $\int \frac{e^{\tan x}}{\cos^2 x} dx$

Q5. $\int \frac{e^{2x}}{e^x+1} dx$ *put $e^x = u$*

Q6. $\int \frac{e^{x/a} + e^{-x/a}}{e^{x/a} - e^{-x/a}} dx$

Q7. $\int \frac{dx}{e^x + e^{-x}}$ *put $e^x = u$*

Q8. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

Q9. $\int a^x e^x dx$ *put $a^x = u$*

Q10. $\int \frac{x^4}{e^x} dx$

Q11. $\int a^{3x+5} dx$

Q12. $\int \frac{x^4}{(a+bx)^2} dx$ *put $a+bx = u$*

Q13. $\int \frac{x^2}{a+bx} dx$ *put $a+bx = u$*

Q14. $\int \frac{x^2}{(a+bx)^3} dx$

Q15. $\frac{e^{2x}(e^2+1)}{(e^x+1)^2}$ *proceed*

$$Q15. \int \frac{x^2 dx}{\sqrt{2+3x}}$$

$$Q17. \int \frac{dx}{x^{3/2}\sqrt{2+3x}}$$

$$Q19. \int \frac{dx}{x^{5/3}(2-x)^{1/3}} \quad \text{put } \frac{1}{x} = u$$

$$Q21. \int \frac{dx}{2x^2(3+5x)^3}$$

$$Q23. \int \frac{x^5}{\sqrt{1+x^2}} dx$$

$$Q25. \int \frac{dx}{x^6 \sqrt{1+x^2}}$$

$$Q16. \int \frac{x dx}{(2+3x)^{1/3}} \quad \text{21. put } 3x = u$$

$$Q18. \int \frac{dx}{x^{5/2}\sqrt{1-x}} \quad \text{then } \frac{1}{x} = u$$

$$Q20. \int \frac{dx}{x(5-2x)^2}$$

$$Q22. \int \frac{dx}{x^4(2+3x)^3}$$

$$Q24. \int \frac{x^3 dx}{(2+3x^2)^{5/2}}$$

Answers

$$1. \left(\frac{1}{5} e^{5x} + c \right)$$

$$2. \left(\frac{1}{\log 5} 5^x + c \right)$$

$$3. \left(e^{\sin^{-1} x} + c \right)$$

$$4. \left(e^{\tan x} + c \right)$$

$$5. \left(e^x - \log(1+e^x) + c \right)$$

$$6. \left[a \log(e^{x/a} - e^{-x/a}) + c \right]$$

$$7. \left(\tan^{-1} e^x + c \right)$$

$$8. \left(2e^{\sqrt{x}} + c \right)$$

$$9. \left(\frac{1}{1+\log a} a^x e^x + c \right)$$

$$10. \left(c - \frac{1}{5} e^{-x^5} \right)$$

$$11. \left(\frac{1}{3 \log a} a^{3x+5} + c \right)$$

$$12. \left[\frac{1}{b^5} \left(\frac{u^3}{3} - 2au^2 + 6a^2u - 4a^3 \log u - \frac{a^4}{u} \right) + c, u = ax+b \right]$$

$$13. \left[\frac{1}{b^3} \left(\frac{u^2}{2} - 2au + a^2 \log u \right) + c, u = ax+b \right]$$

$$14. \left[\frac{1}{b^3} \left(\log u + \frac{2a}{u} - \frac{2a^3}{u^2} \right) + c, u = ax+b \right]$$

$$15. \left[\frac{1}{27} \left(\frac{2}{5} u^{5/2} - \frac{8}{3} u^{3/2} + 8u^{1/2} \right) + c, u = 2+3x \right]$$

$$16. \left[\frac{1}{9} \left(\frac{3}{5} u^{5/3} - 3u^{2/3} \right) + c, u = 3x+2 \right]$$

$$17. (c - \sqrt{3+2/x})$$

$$18. \left[c - \frac{2}{3} \left(\frac{1-x}{x} \right)^{3/2} - 2\sqrt{\frac{1-x}{x}} \right]$$

14. ans. $\frac{1}{b^3} \left(\log u + \frac{2a}{u} - \frac{a^2}{u^2} \right)$ answer. Deriv confirm.

$$\begin{aligned}
19. & \left[c - \frac{3}{4} \left(\frac{2-x}{x} \right)^{3/2} \right] & 20. & \left[c - \frac{1}{25} \left(\log \frac{5-2x}{x} - \frac{2x}{5-2x} \right) \right] \\
21. & \left[c - \frac{1}{162} \left(\frac{3+5x}{x} - 15 \log \frac{3+5x}{x} - \frac{75x}{3+5x} + \frac{125x^2}{2(3+5x)^2} \right) \right] \\
22. & \left[c - \frac{1}{64} \left(\frac{u^3}{3} - \frac{15}{2} u^2 + 90u - 270 \log u - 405 \frac{1}{u} + \frac{243}{2} \frac{1}{u^2} \right), \quad u = \frac{2+3x}{x} \right] \\
23. & \left[\frac{1}{5} (1+x^2)^{5/2} - \frac{2}{3} (1+x^2)^{3/2} + \sqrt{1+x^2} + c \right] \\
24. & \left[c - \frac{4+9x^2}{27(2+3x)^{3/2}} \right] & 25. & \left[c - \frac{\sqrt{x^2+1}}{15x} \left(8 - \frac{4}{x^2} + \frac{3}{x^4} \right) \right]
\end{aligned}$$

2.9 STANDARD INTEGRALS OF TRIGONOMETRIC FUNCTIONS

From definition the following standard integrals of trigonometric circular functions are known and are tabulated in sec. 1.7:

(i) $\int \sin x \, dx = -\cos x + c$

(ii) $\int \cos x \, dx = \sin x + c$

(iii) $\int \sec^2 x \, dx = \tan x + c$

(iv) $\int \operatorname{cosec}^2 x \, dx = -\cot x + c$

Also from sec. 2.5, the following two integrals are standard,

(v) $\int \tan x \, dx = \log \sec x + c$

(vi) $\int \cot x \, dx = \log \sin x + c$

Still there are two more standard integrals of trigonometric circular functions. They are

(vii) $\int \sec x \, dx = \log (\sec x + \tan x) + c$

$$= \log \tan \left(\frac{\pi}{4} + \frac{x}{2} \right) + c = \log \frac{1 + \tan(x/2)}{1 - \tan(x/2)}$$

(viii) $\int \operatorname{cosec} x \, dx = \log \tan \frac{x}{2} + c.$

The last two are being established below :

Proof (vii). Let $I = \int \sec x \, dx = \int \frac{\sec x (\sec x + \tan x)}{\sec x + \tan x} dx$
 $= \log (\sec x + \tan x) + c,$

$\therefore$ numerator is the diff. coeff. of the denominator. This result can be further simplified as

$$\begin{aligned}
 I &= \log \left(\frac{1 + \sin x}{\cos x} \right) + c \\
 &= \log \frac{(\cos(x/2) + \sin(x/2))^2}{\cos^2(x/2) - \sin^2(x/2)} + c = \log \frac{\cos(x/2) + \sin(x/2)}{\cos(x/2) - \sin(x/2)} + c \\
 &= \log \frac{1 + \tan(x/2)}{1 - \tan(x/2)} + c \\
 &= \log \tan \left(\frac{\pi}{4} + \frac{x}{2} \right) + c
 \end{aligned}$$

$$\begin{aligned}
 \text{(viii)} \quad \int \operatorname{cosec} x \, dx &= \int \frac{1}{\sin x} \, dx = \int \frac{1}{2 \sin(x/2) \cos(x/2)} \, dx \\
 &= \frac{1}{2} \int \frac{\sec^2(x/2)}{\tan(x/2)} \, dx \\
 &= \log \tan \frac{x}{2} + c,
 \end{aligned}$$

$\therefore$ numerator is the diff. coeff. of denominator.

Examples

1. Integrate $\tan(ax+b)$.

To fit the integral as standard, the angle of the integrand $(ax+b)$ is to be expressed as a single variable and hence substitute $ax+b=u$ and $dx=\frac{1}{a}du$. So,

$$\begin{aligned}
 \int \tan(ax+b) \, dx &= \frac{1}{a} \int \tan u \, du + c = \frac{1}{a} \sec^2 u + c \\
 &= \frac{1}{a} \sec^2(ax+b) + c
 \end{aligned}$$

2. Perform the integral of $\int x^2 \cos^2(2x^3+3) \, dx$.

Substitute $2x^3+3=u$ or $x^2 \, dx = \frac{1}{6} du$.

$$\begin{aligned}
 \text{So, } \int x^2 \cos^2(2x^3+3) \, dx &= \frac{1}{6} \int \cos^2 u \, du + c = \frac{1}{12} \int (1 + \cos 2u) \, du + c \\
 &= \frac{1}{12} \left\{ \int du + \int \cos 2u \, du \right\} + c = \frac{1}{12} \left\{ u + \frac{1}{2} \sin 2u \right\} + c \\
 &= \frac{1}{12} \left\{ (2x^3+3) + \frac{1}{2} \sin 2(2x^3+3) \right\} + c.
 \end{aligned}$$

3. Compute $\int \frac{\sqrt{3+4 \tan \theta}}{\cos^2 \theta} d\theta$.

Let $3 + 4 \tan \theta = u$. So, $\sec^2 \theta d\theta = \frac{1}{4} du$.

$$\begin{aligned} \therefore \int \frac{\sqrt{3+4 \tan \theta}}{\cos^2 \theta} d\theta &= \frac{1}{4} \int \sqrt{u} du + c = \frac{1}{6} u^{3/2} + c \\ &= \frac{1}{6} (3 + 4 \tan \theta)^{3/2} + c. \end{aligned}$$

Exercises 2 (3)

Find the following integrals :

Q1. $\int \sin p\theta d\theta$ 2. $\int \cos (2x+3) dx$

3. $\int x \tan (ax^2 + b) dx$ 4. $\int \sec^2 ax dx$

5. $\int \frac{dx}{\cos^2 x - \sin^2 x}$ 6. $\int \frac{d\theta}{3 \sin \theta - 4 \sin^3 \theta}$

7. $\int (2x+3) \operatorname{cosec} (x^2+3x+2) dx$ 8. $\int \frac{\cos 2x}{\sin x} dx$

9. $\int \frac{\cos 2x}{\cos x} dx$ 10. $\int \frac{\sin x}{\sin 2x} dx$

11. $\int t \sec 3t^2 \tan 3t^2 dt$

Q12. $\int (ax+b) \operatorname{cosec} (ax^2+2bx+c) \cot (ax^2+2bx+c) dx$

Q13. $\int \frac{dx}{\sin x \cos^2 x}$ Q14. $\int \frac{dx}{\sin^2 x \cos^2 x}$

Q15. $\int \frac{dx}{\sin x \cos^3 x}$ Q16. $\int \frac{3dx}{\cos^2 (5x+6)}$

Q17. $\int \frac{2x^2 dx}{\sin^2 (x^3+9)}$ Q18. $\int \tan (3x+5) dx$

Q19. $\int \left(e^x + \frac{1}{x} \right) \cot (e^x + \log x) dx$ Q20. $\int (\tan \theta + \cot \theta)^2 d\theta$

Q21. $\int (\sec \phi - \tan \phi)^2 d\phi$ Q22. $\int \frac{dx}{1 - \cos x}$

Q23. $\int \frac{\sin x}{\sin 3x \sin 2x} dx$

Q24. $\int \frac{\cos x}{\cos 3x \sin 2x} dx$ put $\cos x = \cos(3x-2x)$
proceed

Q25. $\int \frac{dx}{\cos 3x - \cos x}$

26. $\int \frac{dx}{1 - \sin x}$

write $\sin x = \sin(3x-2x)$

$$27. \int \frac{\sin \sqrt{x}}{\sqrt{x}} dx$$

$$28. \int \sin^2 2\theta \, d\theta$$

$$29. \int (\sec x + 2)^2 dx$$

$$30. \int \frac{e^x dx}{\tan(e^x + a)}$$

$$31. \int [\operatorname{cosec}(3\theta + 2) - 1]^2 d\theta$$

$$32. \int \frac{\sin^4 x}{\cos^8 x} dx = \int \tan^4 x (1 + \tan^2 x) \sec^2 x$$

$$33. \int \frac{\cos^2 x}{\sin^6 x} dx = \int \cot^2 x (1 + \cot^2 x) \operatorname{cosec}^2 x dx = \int z^2 (1 + z^2) \cdot dz$$

Answers

$$1. \left(c - \frac{1}{p} \cos p\theta \right)$$

$$2. \left[\frac{1}{2} \sin(2x + 3) + c \right]$$

$$3. \left[\frac{1}{2a} \log \sec(ax^2 + b) + c \right]$$

$$4. \left[\frac{1}{a} \tan ax + c \right]$$

$$5. \left[\frac{1}{2} \log(\sec 2x + \tan 2x) + c \right]$$

$$6. \left(\frac{1}{3} \log \tan \frac{3\theta}{2} + c \right)$$

$$7. \left(\log \tan \frac{x^2 + 3x + 2}{2} \right) + c$$

$$8. \left(\log \tan \frac{x}{2} + 2 \cos x + c \right)$$

$$9. [2 \sin x - \log(\sec x + \tan x) + c]$$

$$11. \left(\frac{1}{6} \sec 3t^2 + c \right)$$

$$10. \left[\frac{1}{2} \log(\sec x + \tan x) + c \right]$$

$$13. \left(\log \tan \frac{x}{2} + \sec x + c \right)$$

$$12. \left[c_1 - \frac{1}{2} \operatorname{cosec}(ax^2 + 2bx + c) \right]$$

$$15. \left(\log \tan x + \frac{1}{2} \tan^2 x + c \right)$$

$$14. (\tan x - \cot x + c)$$

$$17. \left[c - \frac{2}{3} \cot(x^3 + 9) \right]$$

$$16. \left[\frac{3}{5} \tan(5x + 6) + c \right]$$

$$19. [\log \sin(e^x + \log x) + c]$$

$$18. \left[\frac{1}{3} \log \sec(3x + 5) + c \right]$$

$$21. [2(\tan \phi - \sec \phi) - \phi + c]$$

$$20. (\tan \theta - \cot \theta + c)$$

$$23. \left(\frac{1}{2} \log \sin 2x - \frac{1}{3} \log \sin 3x + c \right)$$

$$22. [c - (\cot x + \operatorname{cosec} x)]$$

$$25. \left[\frac{1}{4} \left\{ \operatorname{cosec} x - \log(\sec x + \tan x) \right\} + c \right]$$

$$24. \left(\frac{1}{2} \log \sin 2x + \frac{1}{3} \log \sec 3x + c \right)$$

$$27. (c - 2 \cos \sqrt{x})$$

$$26. (\sec x + \tan x + c)$$

$$29. [\tan x + 4 \log(\sec x + \tan x) + 4x + c]$$

$$28. \left(\frac{\theta}{2} - \frac{1}{8} \sin 4\theta + c \right)$$

$$31. \left[\theta - \frac{1}{3} \cot(3\theta + 2) - \frac{2}{3} \log \tan \frac{3\theta + 2}{2} + c \right]$$

$$30. [\log \sin(e^x + a) + c]$$

$$32. \left[\frac{1}{5} \tan^5 x + \frac{1}{7} \tan^7 x + c \right]$$

$$33. \left(c - \frac{1}{3} \cot^3 x - \frac{1}{5} \cot^5 x \right)$$

2.10 INTEGRAL OF THE TYPES

$$\int \sin px \cos qx \, dx, \int \sin px \cos qx \, dx, \int \cos px \cos qx \, dx$$

These integrals are very useful. They arise in many physical and mathematical applications in science and engineering educations such as Fourier series, alternating current theory, heat transfer problems, bending of beams, cable stress analysis in suspension bridges etc. They are easily integrated by converting the products to sums using addition formulae. Thus :

$$\begin{aligned} \text{(A)} \quad \int \sin px \cos qx \, dx &= \frac{1}{2} \int \{ \sin (p+q)x + \sin (p-q)x \} \, dx + c \\ &= -\frac{\cos (p+q)x}{2(p+q)} - \frac{\cos (p-q)x}{2(p-q)} + c \end{aligned}$$

$$\begin{aligned} \text{(B)} \quad \int \sin px \sin qx \, dx &= \frac{1}{2} \int \{ \cos (p-q)x - \cos (p+q)x \} \, dx + c \\ &= \frac{\sin (p-q)x}{2(p-q)} - \frac{\sin (p+q)x}{2(p+q)} + c \end{aligned}$$

$$\begin{aligned} \text{(C)} \quad \int \cos px \cos qx \, dx &= \frac{1}{2} \int \{ \cos (p+q)x + \cos (p-q)x \} \, dx + c \\ &= \frac{\sin (p+q)x}{2(p+q)} + \frac{\sin (p-q)x}{2(p-q)} + c. \end{aligned}$$

2.11 INTEGRAL OF THE TYPE $\int \sin^m x \cos^n x \, dx$

General discussion of this type of integral will be given in chapter 6 on reduction formulae. Here only the following *particular* cases will be discussed sequentially :

(i) Either m or n is +ve odd integer or both m and n are +ve odd integers;

(ii) Both m and n are +ve even integer;

(iii) $(m+n)$ is a -ve even integer;

(iv) Either m or n is zero i.e. type $\int \sin^m x \, dx$ or $\int \cos^n x \, dx$ only.

Case (i) In this case, $u = \sin x$ or $\cos x$ is substituted according as n or m is +ve odd integer. Let m be a +ve odd integer. Let $m = 2r + 1$, r is a +ve integer. Then $\int \sin^m x \cos^n x \, dx$

$$= \int \sin^{2r+1} x \cos^n x \, dx = \int (1 - \cos^2 x)^r \cos^n x \sin x \, dx$$

$$= - \int (1 - u^2)^r u^n \, du, \text{ when } u = \cos x$$

$\therefore r$ is a +ve integer; so $(1-u^2)^r$ can be expanded binomially and can be integrated term by term. Similarly, if n is +ve odd integer, separate one $\cos x$, express the remaining cosines into sines and substitute $u = \sin x$. Then proceed as before. Again if both m and n are +ve odd integer then any one of the above two procedures is enough for integration.

Case (ii) If m and n both +ve even integer, then express the product as a linear expression of multiple angles of sines and cosines. This can be done either by using trigonometric identities $\cos 2x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x$ or by the use of De Moivre's theorem. To use the latter procedure, take

$\cos x + i \sin x = y$. So by De Moivre's theorem

$$\cos x - i \sin x = \frac{1}{y}, \quad \cos nx + i \sin nx = y^n \quad \text{and} \quad \cos nx - i \sin nx = \frac{1}{y^n}$$

Hence adding and subtracting the above relations,

$$\cos x = \frac{1}{2} \left(y + \frac{1}{y} \right), \quad \sin x = \frac{1}{2i} \left(y - \frac{1}{y} \right), \quad 2 \cos nx = y^n + \frac{1}{y^n}$$

$$\text{and} \quad 2i \sin nx = y^n - \frac{1}{y^n}$$

Now replace sine and cosine in the product by their new expressions in terms of y . After simplification into the pairs like $\left(y^n + \frac{1}{y^n} \right)$ or $\left(y^n - \frac{1}{y^n} \right)$, $n = 1, 2, 3, \dots$, replace them by their counterparts of cosines and sines of multiple angles and integrate termwise.

Case (iii) $(m+n)$ is -ve even integer.

From the given product one $\sec^2 x$ is to be separated and the remaining sines and cosines can be expressed in terms of tangents. Then substitute for $\tan x = u$. Let $m+n = -2r$, r is a +ve integer. So,

$$\begin{aligned} & \int \sin^m x \cos^n x \, dx \\ &= \int \tan^m x \cos^{m+n} x \, dx = \int \tan^m x \cos^{-2r} x \, dx \\ &= \int \tan^m \sec^{2r} x \, dx = \int \tan^m x \sec^{2r-2} x \cdot \sec^2 x \, dx \\ &= \int \tan^m x (1 + \tan^2 x)^{r-1} \sec^2 x \, dx \\ &= \int u^m (1 + u^2)^{r-1} du, \quad \text{when } u = \tan x. \end{aligned}$$

$\therefore r$ is a +ve integer, so $r-1=0$ or a +ve integer greater than 1. Expanding binomially, termwise integration is possible.

Case (iv) Either m or n is zero i.e. Type $\int \sin^m x \, dx$ or $\int \cos^n x \, dx$ only.

If $m=0$ and n is even or vice-versa then follow procedure (ii) i.e. even powers should be converted to multiple angles. If $m=0$ and n is odd or vice-versa then follow procedure (i): See examples.

2.12 INTEGRAL OF THE TYPE $\int \frac{dx}{a \cos x + b \sin x}$, a, b const.

Here is an exceptional substitution for constants $a=r \cos \theta$ and $b=r \sin \theta$ or the reverse, so that, the expression $(a \cos x + b \sin x)$ simplifies either to $r \cos (x + \theta)$ or $r \sin (x + \theta)$ where $r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1} \frac{b}{a}$.

With these substitution the integral reduces either to $\frac{1}{r} \int \sec (x + \theta) dx$ or

or $\frac{1}{r} \int \operatorname{cosec} (x + \theta) dx$ and hence can be easily integrated.

2.13 INTEGRAL OF THE TYPES $\int \tan^n x dx$, $\int \cot^n x dx$, $n \neq 1$.

If $n \neq 1$ is a +ve integer then these integrals are integrated by reducing the exponent. To do that one $\tan^2 x$ or $\cot^2 x$ is separated and is expressed in terms of $\sec^2 x$ or $\operatorname{cosec}^2 x$ and then use substitution for $\tan x$ or $\cot x$ as the case may be. Continue the process until the exponent, n is exhausted. Let

$$\begin{aligned} I_n &= \int \tan^n x dx = \int \tan^{n-2} x \tan^2 x dx = \int \tan^{n-2} x (\sec^2 x - 1) dx \\ &= \int \tan^{n-2} x \sec^2 x dx - \int \tan^{n-2} x dx = \frac{\tan^{n-1} x}{n-1} - I_{n-2}, n \neq 1. \end{aligned}$$

It is a reduction formula. Repeated use of this formula gives the integral. Similarly,

$$I_n = \int \cot^n x dx = -\frac{\cot^{n-1} x}{n-1} - I_{n-2}, n \neq 1.$$

2.14 INTEGRAL OF THE TYPES $\int \sec^n x dx$, $\int \operatorname{cosec}^n x dx$.

General case will be discussed in chapter 6. Here the case of +ve even integer will be discussed. Let $n=2r$, r is any +ve integer. One $\sec^2 x$ or $\operatorname{cosec}^2 x$ is separated and remaining even powered $\sec x$ or $\operatorname{cosec} x$ is expressed in terms of $\tan^2 x$ or $\cot^2 x$ and then use substitution for $\tan x$ or $\cot x$ as the case may be. So,

$$\begin{aligned} \int \sec^n x dx &= \int \sec^{2r} x dx = \int \sec^{2r-2} x \sec^2 x dx \\ &= \int (1 + \tan^2 x)^{r-1} \sec^2 x dx = \int (1 + u^2)^{r-1} du, \end{aligned}$$

when $u = \tan x$.

This can be integrated after binomial expansion, as $(r-1)$ is some integer. Similarly,

$$\int \operatorname{cosec}^r x \, dx = \int \operatorname{cosec}^{2r} x \, dx = - \int (1+u^2)^{r-1} du \text{ when } u = \cot x.$$

2.15 INTEGRAL OF THE TYPES

$$\int \tan^m x \sec^n x \, dx, \int \cot^m x \operatorname{cosec}^n x \, dx$$

General case will be discussed in chapter 6. *Special cases* like

- (i) n +ve even integer,
- (ii) Both m and n are +ve odd integers,
- (iii) m +ve even and n +ve odd integers.
- (iv) m +ve odd integer,

Case (i) Since n is a +ve even integer, separation of one $\sec^2 x$ or $\operatorname{cosec}^2 x$ and transformation of the remaining function of $\sec^2 x$ or $\operatorname{cosec}^2 x$ to that of $\tan^2 x$ or $\cot^2 x$ reduces the integral to simple form on the substitution $u = \tan x$ or $\cot x$.

Case (ii) Since m and n are both +ve odd integers, separate one $(\tan x \sec x)$ or $(\cot x \operatorname{cosec} x)$ and write $\tan^{m-1} x = (\sec^2 x - 1)^{\frac{m-1}{2}}$ or $\cot^{m-1} x = (\operatorname{cosec}^2 x - 1)^{\frac{m-1}{2}}$. Then use substitution for $\sec x$ or $\operatorname{cosec} x$ and integrate easily.

Case (iii) Since m is a +ve even integer, so $\tan^m x = (\sec^2 x - 1)^{m/2}$ and $\cot^m x = (\operatorname{cosec}^2 x - 1)^{m/2}$. The integral then reduces to the powers of $\sec x$ or $\operatorname{cosec} x$ i.e. to the type 2.14.

Case (iv) Since m is odd +ve integer, this is identical to the case (ii). Separate one $\tan x$ or $\cot x$ and transform the remaining even powered $\tan x$ or $\cot x$ to $\sec x$ or $\operatorname{cosec} x$ as the case may be. Then substitute for $\sec x$ or $\operatorname{cosec} x$.

2.16 TRIGONOMETRIC HYPERBOLIC FUNCTIONS AS INTEGRANDS.

The types from 2.10 to 2.15 used trigonometric circular functions as integrands. In each of them if the integrand is replaced by corresponding trigonometric hyperbolic functions i.e. $\int \sinh px \cosh qx \, dx$ etc.,

$\int \sinh^m x \cosh^n x \, dx, \int \frac{dx}{a \cosh x + b \sinh x}, \int \tanh^m x \operatorname{sech}^n x \, dx$ etc. The procedures remain same. De Moivre's theorem used in case (ii) sec. 2.11 for the conversion of power to multiple angles cannot be applicable here. This is the only exception.

Examples

1. Integrate $\sin^{11}\theta$.

This is of the type 2.11(iv) because $n=0, m=11$ which is an odd +ve integer. So, accordingly one $\sin \theta$ is to be separated and the remaining even powered sine is to be expressed in terms of cosine. i.e.

$$\begin{aligned}\int \sin^{11}\theta d\theta &= \int \sin^{10}\theta \sin \theta d\theta + c = \int (\sin^2\theta)^5 \sin \theta d\theta + c \\ &= \int (1 - \cos^2\theta)^5 \sin \theta d\theta + c \\ &= -\int (1 - u^2)^5 du + c \text{ when } \cos \theta = u. \\ &= -\int (1 - 5u^2 + 10u^4 - 10u^6 + 5u^8 - u^{10}) du + c \\ &= -\left(u - \frac{5}{3}u^3 + 2u^5 - \frac{10}{7}u^7 + \frac{5}{9}u^9 - \frac{1}{11}u^{11}\right) + c \\ &\quad \text{where } u = \cos \theta.\end{aligned}$$

2. Compute $\int \cos^5 \theta \sin^2 \theta d\theta$.

This is also of type 2.11 (i) & $m=2, n=5$. So one cosine is to be separated and the remaining even powered cosines is to be converted into sines i.e.

$$\begin{aligned}\int \cos^5 \theta \sin^2 \theta d\theta &= \int \cos^4 \theta \sin^2 \theta \cos \theta d\theta + c \\ &= \int (1 - \sin^2 \theta)^2 \sin^2 \theta \cos \theta d\theta + c = \int (1 - u^2)^2 u^2 du \\ &\quad \text{when } \sin \theta = u \\ &= \int (u^2 - 2u^4 + u^6) du + c = \frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7 + c \\ &= \frac{1}{3}\sin^3 \theta - \frac{2}{5}\sin^5 \theta + \frac{1}{7}\sin^7 \theta + c.\end{aligned}$$

3. Integrate $\sin^4 x \cos^2 x$.

Here both the exponents are even. So, this is of the type 2.11 (ii) and the powers or exponents are to be converted to multiple angles. Three procedures are described below :

Method 1

Powers are being transformed into angles using trigonometric formulae as follows :

$$\begin{aligned}\int \sin^4 x \cos^2 x dx &= \int (\sin x \cos x)^2 \sin^2 x dx + c \\ &= \int \left(\frac{1}{2} \sin 2x\right)^2 \frac{1 - \cos 2x}{2} dx + c \\ &= \frac{1}{8} \int \left(\frac{1 - \cos 4x}{2}\right) (1 - \cos 2x) dx + c\end{aligned} \quad (i)$$

$$\begin{aligned}
&= \frac{1}{16} \int (1 - \cos 2x - \cos 4x + \cos 2x \cos 4x) dx + c \\
&= \frac{1}{16} \int \left\{ 1 - \cos 2x - \cos 4x + \frac{1}{2} (\cos 6x + \cos 2x) \right\} dx + c \\
&= \frac{1}{16} \left(x - \frac{1}{2} \sin 2x - \frac{1}{4} \sin 4x + \frac{1}{12} \sin 6x + \frac{1}{4} \sin 2x \right) + c \\
&= \frac{1}{32} \left(2x - \frac{1}{2} \sin 2x - \frac{1}{2} \sin 4x + \frac{1}{6} \sin 6x \right) + c
\end{aligned}$$

Method 2

Same trigonometric procedures as in method 1 but a different approach.

$$\begin{aligned}
\int \sin^4 x \cos^2 x dx &= \int (\sin^2 x)^2 \frac{1}{2} (1 + \cos 2x) dx + c \\
&= \int \frac{1}{4} (1 - \cos 2x)^2 \frac{1}{2} (1 + \cos 2x) dx + c \quad \text{(ii)} \\
&= \frac{1}{8} \int (1 - \cos^2 2x) (1 - \cos 2x) + c = \frac{1}{8} \int \sin^2 2x (1 - \cos 2x) + c \\
&= \frac{1}{16} \int (1 - \cos 4x) (1 - \cos 2x) + c \\
&= \text{(identical with (i) in method 1)}
\end{aligned}$$

(ii) can also be proceeded in another way of decomposing the perfect squared term. Verification remains to readers.

Method 3

Use of De Moivre's theorem. Let $\cos x + i \sin x = y$.

$$\therefore \cos x - i \sin x = \frac{1}{y} \quad \text{Also } \cos nx + i \sin nx = y^n$$

$$\text{and } \cos nx - i \sin nx = \frac{1}{y^n} \quad \text{Adding and subtracting,}$$

$$\cos x = \frac{1}{2} \left(y + \frac{1}{y} \right), \quad \sin x = \frac{1}{2i} \left(y - \frac{1}{y} \right), \quad 2 \cos nx = y^n + \frac{1}{y^n}$$

$$\text{and } 2i \sin nx = y^n - \frac{1}{y^n}$$

$$\begin{aligned}
\therefore \sin^4 x \cos^2 x &= \left(\frac{1}{2i} \right)^4 \left(y - \frac{1}{y} \right)^4 \left(\frac{1}{2} \right)^2 \left(y + \frac{1}{y} \right)^2 \\
&= \frac{1}{2^6} \left(y - \frac{1}{y} \right)^2 \left(y^2 - \frac{1}{y^2} \right)^2 \\
&= \frac{1}{2^6} \left(y^3 - \frac{1}{y} - y + \frac{1}{y^3} \right)^2 \\
&= \frac{1}{2^6} \left[\left(y^6 + \frac{1}{y^6} \right) - 2 \left(y^4 + \frac{1}{y^4} \right) - \left(y^2 + \frac{1}{y^2} \right) + 4 \right]
\end{aligned}$$

$$= \frac{1}{2^6} [2 \cos 6x - 4 \cos 4x - 2 \cos 2x + 4]$$

$$\begin{aligned} \therefore \int \sin^4 x \cos^2 x \, dx &= \frac{1}{32} \int (\cos 6x - 2 \cos 4x - \cos 2x + 2) \, dx + c \\ &= \frac{1}{32} \left(\frac{1}{6} \sin 6x - \frac{1}{2} \sin 4x - \frac{1}{2} \sin 2x + 2x \right) + c \end{aligned}$$

4. Find the integral $\int \sin^{-5/3} x \cos^{-1/3} x \, dx$.

Here $m+n = -\frac{5}{3} - \frac{1}{3} = -2$, a -ve even integer. So, it is of type

2.11 (iii). Therefore,

$$\begin{aligned} \int \sin^{-5/3} x \cos^{-1/3} x \, dx &= \int \frac{1}{\sin^{5/3} x \cos^{1/3} x} \, dx + c \\ &= \int \frac{1}{\tan^{5/3} x \cos^{1/3+5/3} x} \, dx + c = \int \frac{\sec^2 x}{\tan^{5/3} x} \, dx + c. \\ &= \int \frac{1}{u^{5/3}} \, du + c \text{ when } \tan x = u. \\ &= -\frac{2}{3} u^{-3/2} + c = -\frac{2}{3} \tan^{-3/2} x + c \end{aligned}$$

5. Compute $\int \frac{dx}{3 \sin x + 4 \cos x}$.

This is of type 2.12. So, Let $3 = r \cos \theta$ and $4 = r \sin \theta$,

$\therefore r = 5, \tan \theta = \frac{4}{3}$. Therefore,

$$\begin{aligned} \int \frac{dx}{3 \sin x + 4 \cos x} &= \frac{1}{r} \int \frac{dx}{\sin(x+\theta)} + c \\ &= \frac{1}{r} \int \operatorname{cosec}(x+\theta) \, d\theta + c = \frac{1}{r} \log \tan \frac{1}{2}(x+\theta) + c \\ &= \frac{1}{5} \log \tan \left(\frac{x}{2} + \frac{1}{2} \tan^{-1} \frac{4}{3} \right) + c \end{aligned}$$

Note : If the substitutions are reversed i.e. $3 = r \sin \theta$ and $4 = r \cos \theta$. Then the result will be $\frac{1}{5} \log \tan \left(\frac{\pi}{4} + \frac{x}{2} + \frac{1}{2} \tan^{-1} \frac{4}{3} \right) + c$.

Exercises 2 (4)

Compute the following :

- | | |
|--|------------------------------------|
| ✓1. $\int \sin 2x \cos 3x \, dx$ | ✓2. $\int \cos 2nt \cos 5nt \, dt$ |
| 3. $\int \sin x \sin 2x \sin 3x \, dx$ | ✓4. $\int \sin^2 x \cos^2 x \, dx$ |

$$\text{Ex 5. } \int (\sin \theta - \cos 2\theta)^2 d\theta$$

$$\text{Ex 6. } \int \sin^5 x dx$$

$$17. \sec^{2/3} \theta \cdot \csc^{7/3} \theta$$

$$\text{Ex 7. } \int \cos^5 \left(\frac{3}{2}x + 4 \right) dx$$

$$\text{Ex 8. } \int \sin^3 x \cos^2 x dx$$

$$= \cos^{2/3} \theta \cdot \sin^{-4/3} \theta$$

$$\text{Ex 9. } \int \sin^3 \theta \cos^3 \theta d\theta$$

$$\text{Ex 10. } \int \sin^3 3\theta \cos^4 3\theta d\theta$$

$$= \tan^{-4/3} \theta \cdot \sec^{2/3} \theta$$

$$\text{Ex 11. } \int \cos^5 x \sin^4 x dx$$

$$\text{Ex 12. } \int \sin^{5/2} x \cos^3 x dx$$

$$13. \int \sin^2 x \cos^6 x dx$$

$$14. \int \sin^8 x dx$$

$$\text{Ex 15. } \int \frac{\cos^3 \theta}{\sin^5 \theta} d\theta$$

$$\text{Ex 16. } \int \frac{1}{\sin^3 \theta \cos^5 \theta} d\theta$$

$$\text{Ex 17. } \int \sec^{2/3} \theta \csc^{4/3} \theta d\theta$$

$$\text{Ex 18. } \int \frac{d\theta}{\sqrt{\sin^5 \theta \cos^7 \theta}} = \int \sin^{-5/2} \theta \cdot \cos^{-7/2} \theta$$

$$\text{Ex 19. (i) } \int \frac{dx}{\sin x + \cos x}$$

$$(ii) \int \frac{dx}{\sqrt{x} + \sqrt{1-x}}$$

[Hint : $x = \sin^2 \theta$].

$$\text{Ex 20. } \int \frac{dt}{(5 \sin t - 12 \cos t)^2}$$

$$\text{Ex 21. } \int \cot^3 \theta d\theta$$

$$\text{Ex 22. } \int \csc^4 \frac{x}{4} dx$$

$$\text{Ex 23. } \int \sec^6 x dx$$

$$24. \int \tan^2 x \sec^4 x dx$$

$$\text{Ex 25. } \int \tan^5 3\theta d\theta$$

$$26. \int \frac{\sin^{3/2} x}{\cos^{11/2} x} dx$$

$$\text{Ex 27. } \int (\tan bt - \cot bt)^3 dt$$

first report
then proceed

$$28. \int \left(\frac{\sec ax}{\tan ax} \right)^4 dx$$

$$\text{Ex 29. } \int \tan^n x \sec^4 x dx$$

$$30. \int (\cos x + 2 \cos 2x)^2 dx$$

$$\text{Ex 31. } \int \frac{\tan^5 2\theta}{\sec^3 2\theta} d\theta = \frac{\sin^5 2\theta}{\cos^5 2\theta} \text{ prove}$$

$$\text{Ex 32. } \int \frac{\sin 2x}{\sin^4 x + \cos^4 x} dx$$

$$\text{Ex 33. } \int \frac{dx}{\sqrt[3]{\sin^{11} x \cos x}}$$

$$34. \int \frac{x^5}{\sqrt{a^2 + x^2}} dx$$

$$\text{Ex 35. } \int \tan^4 x \sec^4 x dx$$

$$\text{Ex 36. } \int \cot^4 x \csc^4 x dx$$

$$\text{Ex 37. } \int \cot^3 x \csc^3 x dx$$

$$38. \int \tan^3 x \sec^3 x \, dx$$

$$39. \int \frac{\sin x}{\sqrt{1 + \sin x}} \, dx$$

$$40. \int (x-3)^3 \sqrt{6x-x^2} \, dx$$

$$41. \int (x+7)^5 \sqrt{-x^2-14x-45} \, dx$$

$$42. (i) \int \sinh^4 x \, dx$$

$$(ii) \int \sinh^3 x \cosh^4 x \, dx$$

$$(iii) \int \tanh^5 x \, dx$$

Answers

$$1. \left(\frac{1}{2} \cos x - \frac{1}{10} \cos 5x + c \right)$$

$$2. \left[\frac{1}{2} \left(\frac{1}{7n} \sin 7nt + \frac{1}{3n} \sin 3nt \right) + c \right]$$

$$3. \left[c - \frac{1}{8} \cos 2x - \frac{1}{16} \cos 4x + \frac{1}{24} \cos 6x \right]$$

$$4. \left(\frac{1}{8} x - \frac{1}{32} \sin 4x + c \right)$$

$$5. \left[\theta - \frac{1}{4} \sin 2\theta + \frac{1}{8} \sin 4\theta + \frac{1}{3} \cos 3\theta - \cos \theta + c \right]$$

$$6. \left(\frac{2}{3} \cos^3 x - \cos x - \frac{1}{5} \cos^5 x + c \right)$$

$$7. \left[\frac{2}{3} \sin \left(\frac{3}{2} x + 4 \right) - \frac{4}{9} \sin^3 \left(\frac{3}{2} x + 4 \right) + \frac{2}{15} \sin^5 \left(\frac{3}{2} x + 4 \right) + c \right]$$

$$8. \left(\frac{1}{5} \cos^5 x - \frac{1}{3} \cos^3 x + c \right)$$

$$9. \left(\frac{1}{4} \sin^4 \theta - \frac{1}{6} \sin^6 \theta + c \right)$$

$$10. \left[\frac{1}{3} \left(\frac{1}{7} \cos^7 3\theta - \frac{1}{5} \cos^5 3\theta \right) + c \right]$$

$$11. \left(\frac{1}{5} \sin^5 x - \frac{2}{7} \sin^7 x + \frac{1}{9} \sin^9 x + c \right)$$

$$12. \left(\frac{2}{7} \sin^{7/2} x - \frac{2}{11} \sin^{11/2} x + c \right)$$

$$13. \left[c + \frac{1}{128} \left(5x + 2 \sin 2x - \sin 4x - \frac{2}{3} \sin 6x - \frac{1}{8} \sin 8x \right) \right]$$

$$14. \left[\frac{1}{64} \left(\frac{1}{8} \sin 8x - \frac{4}{3} \sin 6x + 7 \sin 4x - 28 \sin 2x + 35x \right) + c \right]$$

$$15. \left(\frac{1}{2} \operatorname{cosec}^2 \theta - \frac{1}{4} \operatorname{cosec}^4 \theta + c \right) = C - \frac{1}{4} \cot^4 \theta$$

$$16. \left[c - \frac{1}{2} \cot^2 \theta + 3 \log \tan \theta + \frac{3}{2} \tan^2 \theta + \frac{1}{4} \tan^4 \theta \right]$$

$$17. (c - 3 \tan^{-1} 3)$$

$$18. \left[2 \cot^{3/2} \theta \left(\frac{1}{5} \tan^4 \theta - 2 \tan^2 \theta - \frac{1}{3} \right) + c \right]$$

$$19. (i) \left[\frac{1}{\sqrt{2}} \log \tan \left(\frac{1}{2} x + \frac{1}{8} \pi \right) + c \right]$$

$$(ii) \left[\sqrt{x} - \sqrt{1-x} - \frac{1}{\sqrt{2}} \log \tan \left(\frac{1}{2} \sin^{-1} \sqrt{x} + \frac{\pi}{8} \right) + c \right]$$

20. $\left[\frac{1}{169} \tan(t + 22^\circ 37') + c \right]$
22. $\left[c - \frac{4}{3} \cot^3 \frac{1}{4}x - 4 \cot \frac{1}{4}x \right]$
24. $\left(\frac{1}{3} \tan^3 x + \frac{1}{5} \tan^5 x + c \right)$
26. $\left(\frac{2}{5} \tan^{5/2} x + \frac{2}{9} \tan^{9/2} x + c \right)$
28. $\left[c - \frac{1}{a} \left(\cot ax + \frac{1}{3} \cot^3 ax \right) \right]$
30. $\left[\frac{5}{2}x + 2 \sin x + \frac{1}{4} \sin 2x + \frac{2}{3} \sin 3x + \frac{1}{2} \sin 4x + c \right]$
31. $\left(\frac{1}{2} \sec 2\theta + \cos 2\theta - \frac{1}{6} \cos^3 2\theta + c \right)$
33. $\left[c - \frac{3(1 + 4 \tan^2 x)}{8 \tan^2 x \sqrt[3]{\tan^2 x}} \right]$
35. $\left(c + \frac{1}{7} \tan^7 x + \frac{1}{5} \tan^5 x \right)$
37. $\left(c - \frac{1}{5} \operatorname{cosec}^5 x + \frac{1}{3} \operatorname{cosec}^3 x \right)$
39. $\left[2\sqrt{1 - \sin x} + \sqrt{2} \log \frac{\sqrt{2} + 1 - \tan(x/4)}{\sqrt{2} - 1 + \tan(x/4)} + c \right]$
40. $\left[\frac{1}{15} (6x - x^2)^{5/2} - (6x - x^2)^{3/2} + c \right]$
41. $\left[-(16/3)(-x^2 - 14x - 45)^{3/2} \right.$
 $\left. + (8/5)(-x^2 - 14x - 45)^{5/2} - (1/7)(-x^2 - 14x - 45)^{7/2} + c \right]$
42. (i) $\left[\frac{3}{8}x - \frac{1}{4} \sinh 2x + \frac{1}{32} \sinh 4x + c \right]$
 (ii) $\left[\frac{1}{7} \cosh^7 x - \frac{1}{5} \cosh^5 x + c \right]$ (ii) $\left[\log \cosh x - \frac{1}{2} \tanh^2 x - \frac{1}{4} \tanh^4 x + c \right]$
21. $\left(c - \frac{1}{2} \cot^2 \theta - \log \sin \theta \right)$
23. $\left(\tan x + \frac{2}{3} \tan^2 x + \frac{1}{5} \tan^5 x + c \right)$
25. $\left(\frac{1}{12} \tan^4 3\theta - \frac{1}{6} \tan^2 3\theta + \log \sec 3\theta + c \right)$
27. $\left[\frac{1}{2b} (\tan^2 bt + \cot^2 bt) + \frac{4}{b} \log \sin 2bt + c \right]$
29. $\left(\frac{1}{n+1} \tan^{n+1} x + \frac{1}{n+3} \tan^{n+3} x + c \right)$
32. $\left[\tan^{-1}(\tan^2 x) + c \right]$
34. $\frac{\sqrt{a^2 + x^2}}{15} (8a^4 - 4a^2 x^2 + 3x^4) + c$
36. $\left(-\frac{1}{7} \cot^7 x - \frac{1}{5} \cot^5 x + c \right)$
38. $\left(\frac{1}{5} \sec^5 x - \frac{1}{3} \sec^3 x + c \right)$

2.17 STANDARD INTEGRAL OF THE TYPES

$$\int \frac{dx}{a^2 \pm x^2}, \int \frac{dx}{x^2 - a^2}$$

$$(A) \int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$$

The integrand is the reciprocal of the sum of two squares of a constant and a variable. This can be integrated either by using definition or by substitution.

By definition :

$$\frac{d}{dx} \left(\tan^{-1} \frac{x}{a} \right) = \frac{1}{a} \cdot \frac{1}{1 + \frac{x^2}{a^2}} = \frac{a}{a^2 + x^2}$$

$$\therefore \int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$$

By substitution : Let $x = a \tan \theta$. So $dx = a \sec^2 \theta d\theta$

$$\begin{aligned} \therefore \int \frac{1}{a^2 + x^2} dx &= \int \frac{a \sec^2 \theta}{a^2 (1 + \tan^2 \theta)} d\theta + c = \frac{1}{a} \int d\theta + c \\ &= \frac{1}{a} \theta + c = \frac{1}{a} \tan^{-1} \frac{x}{a} + c. \end{aligned}$$

$$(B) \int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \frac{a+x}{a-x} + c$$

The integrand here is the reciprocal of the subtraction of the square of the variable from that of a constant. The expression $(a^2 - x^2)$ can be factorized into $(a-x)(a+x)$ and the integrand can be decomposed into partial fractions i.e.

$$\begin{aligned} \int \frac{dx}{a^2 - x^2} &= \int \frac{1}{(a-x)(a+x)} dx + c = \frac{1}{2a} \int \frac{(a-x) + (a+x)}{(a-x)(a+x)} dx + c \\ &= \frac{1}{2a} \int \left(\frac{1}{a+x} + \frac{1}{a-x} \right) dx + c \\ &= \frac{1}{2a} (\log(a+x) - \log(a-x)) + c \\ &= \frac{1}{2a} \log \frac{a+x}{a-x} + c \end{aligned}$$

(C) $\int \frac{dx}{x^2 - a^2}$. This is same as (B) except the position of the variable. So, the procedure remains the same.

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \int \left(\frac{1}{x-a} - \frac{1}{x+a} \right) dx + c = \frac{1}{2a} \log \frac{x-a}{x+a} + c.$$

2-18 INTEGRAL OF TYPE $\int \frac{dx}{ax^2 + bx + c}$.

The integrand is the reciprocal of a general quadratic. Suppose, the quadratic *cannot* be factorized into linear factors with real coefficients. In such a case, $(ax^2 + bx + c)$ can be expressed as the sum or difference of two squares and the integral reduces to any one of 2-17, i.e.

$$\int \frac{dx}{ax^2 + bx + c} = \frac{1}{a} \int \frac{dx}{\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}}$$

This is of the type 2-17 (A) or (C) according as $4ac > b^2$ or $4ac < b^2$ and is of type 2-17 (B) if $a < 0$.

2-19 INTEGRAL OF THE TYPE $\int \frac{px + q}{ax^2 + bx + c} dx$.

The integrand is a proper fraction of which numerator is a linear expression while denominator is a quadratic and *cannot* be factorized into linear factor with real coefficients. In such case, the numerator is expressed as the sum of the differential coefficients of the denominator and a constant i.e.

$$\begin{aligned} \int \frac{px + q}{ax^2 + bx + c} dx &= \int \frac{\frac{p}{2a} \frac{d}{dx} (ax^2 + bx + c) + \left(q - \frac{pb}{2a}\right)}{ax^2 + bx + c} dx \\ &= \frac{p}{2a} \int \frac{\frac{d}{dx} (ax^2 + bx + c)}{ax^2 + bx + c} + \left(q - \frac{pb}{2a}\right) \int \frac{1}{ax^2 + bx + c} dx \end{aligned}$$

Of these two integrals first one is of type $\int \frac{f'(x)}{f(x)} dx = \log f(x)$, (sec. 2-4) and second is of type 2-18 i.e. any of one the three in 2-17. Therefore,

$$\begin{aligned} \int \frac{px + q}{ax^2 + bx + c} dx &= \frac{p}{2a} \log(ax^2 + bx + c) \\ &\quad + \left(q - \frac{pb}{2a}\right) \int \frac{dx}{ax^2 + bx + c}. \end{aligned}$$

N.B. If the expression $ax^2 + bx + c$ in 2-17 and 2-18 can be factorized with real coeffs. then these can be better handled with partial fractions (ch. 3)

2-20 INTEGRAL OF THE TYPES $\int \frac{dx}{a \sin x + b}$, $\int \frac{dx}{a \cos x + b}$

$$\begin{aligned} \int \frac{dx}{a \cos x + b \sin x + c} \text{ and } \int \frac{dx}{a + b \cosh x}, \int \frac{dx}{a + b \sinh x}, \\ \int \frac{dx}{a \cosh x + b \sinh x + c} \end{aligned}$$

These six integrals can be simplified to 2-17 and so to say 2-18 by using:

$$(i) \sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}, \cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}$$

$$1 = \cos^2 \frac{x}{2} + \sin^2 \frac{x}{2}$$

$$\text{for the first three and } \sinh x = 2 \sinh \frac{x}{2} \cosh \frac{x}{2},$$

$$\cosh x = \cosh^2 \frac{x}{2} + \sinh^2 \frac{x}{2}, 1 = \cosh^2 \frac{x}{2} - \sinh^2 \frac{x}{2}$$

for the last three integrals i.e. existing angles are expressed into half-angles, changing to tangents and substituting for tangents circular or hyperbolic as the case may be; i.e.

$$(A) \int \frac{dx}{a \cos x + b} = \int \frac{dx}{a \left(\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} \right) + b \left(\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} \right)}$$

$$= \int \frac{dx}{(a+b) \cos^2 \frac{x}{2} + (b-a) \sin^2 \frac{x}{2}} = \int \frac{\sec^2 \frac{x}{2} dx}{(a+b) + (b-a) \tan^2 \frac{x}{2}}$$

$$= \int \frac{2 du}{(a+b) + (b-a) u^2} \text{ where } u = \tan \frac{x}{2}$$

Similarly, other two of the first three.

$$(B) \int \frac{dx}{a + b \sinh x} = \int \frac{dx}{a \left(\cosh^2 \frac{x}{2} - \sinh^2 \frac{x}{2} \right) + 2 \sinh \frac{x}{2} \cosh \frac{x}{2}}$$

$$= \int \frac{\operatorname{sech}^2 \frac{x}{2} dx}{a \left(1 - \tanh^2 \frac{x}{2} \right) + 2 \tanh \frac{x}{2}} = \int \frac{2 du}{a (1 - u^2) + 2u} \text{ where } u = \tanh \frac{x}{2}.$$

Similarly, other two of the last three.

2.21 INTEGRAL OF TYPES $\int \frac{dx}{a + b \cos^2 x}, \int \frac{dx}{a + b \sin^2 x},$

$$\int \frac{dx}{a \cos^2 x + b \sin^2 x}, \int \frac{dx}{a + b \cosh^2 x}, \int \frac{dx}{a + b \sinh^2 x},$$

$$\int \frac{dx}{a \cosh^2 x + b \sinh^2 x}.$$

All the six integrals can be integrated in the same way. Dividing both numerator and denominator by $\cos^2 x$ or $\cosh^2 x$ (as the case may be), express the denominators of each integral into $\tan^2 x$ or $\tanh^2 x$ and substitute for $\tan x$ or $\tanh x$. Then use 2.17.

2.22 INTEGRAL OF THE TYPES

$$\int \frac{\sin^n x}{a + b \cos x} dx \quad \text{and} \quad \int \frac{\cos^n x}{a + b \sin x} dx.$$

Here n is assumed to be +ve integer. So, there arises two cases

(i) n is even, (ii) n is odd. Let $I = \int \frac{\sin^n x}{a + b \cos x} dx$

Case. (i) Let $n = 2k, k > 0$ is an integer. Therefore,

$$I = \int \frac{\sin^{2k} x}{a + b \cos x} dx = \int \frac{(1 - \cos^2 x)^k}{a + b \cos x} dx.$$

Now the integrand is a proper fraction. It can be expanded binomially and by division, then term by term integration is possible after transforming powers to angles.

Case. (ii) Let $n = 2k + 1, k \geq 0$ and integer. Therefore,

$$I = \int \frac{\sin^{2k+1} x}{a + b \cos x} dx.$$

Substitution of $a + b \cos x = u$ gives $I = - \int \frac{1}{bu} \left\{ 1 - \left(\frac{u-a}{b} \right)^2 \right\}^k du.$

Binomial expansion enables one to integrate term by term.

The procedures are valid for the second integral also.

2.23 INTEGRAL OF THE TYPES $\int \frac{a \cos x + b \sin x + c}{d \cos x + e \sin x + f} dx$

$$\text{and} \quad \int \frac{a \cosh x + b \sinh x + c}{d \cosh x + e \sinh x + f} dx.$$

Both the integrals can be integrated in the same way by expressing numerator as a linear combination of the denominator, differential coefficient of the denominator and a constant as follows: (for 2nd integral) Let $a \cosh x + b \sinh x + c$

$$= [l(d \cosh x + e \sinh x + f) + m \frac{d}{dx} (d \cosh x + e \sinh x + f) + n]$$

This is an identity of $\sinh$ and $\cosh$. Therefore, equating coefficients of $\cosh x, \sinh x$ and constants, we have,

$$a = ld + me, \quad b = le + md, \quad c = lf + n.$$

Solving for l, m and n , we get,

$$l = \frac{ad - be}{d^2 - e^2}, \quad m = \frac{bd - ae}{d^2 - e^2}, \quad n = \frac{(cd - af)d + (bf - ce)e}{d^2 - e^2}$$

$$\therefore \int \frac{a \cosh x + b \sinh x + c}{d \cosh x + e \sinh x + f} dx = lx + m \log (d \cosh x + e \sinh x + f) \\ + n \int \frac{dx}{d \cosh x + e \sinh x + f} + k$$

The integral on the right is of type six in 2-20. Similarly,

$$\int \frac{a \cos x + b \sin x + c}{d \cos x + e \sin x + f} dx = lx + m \log (d \cos x + e \sin x + f) \\ + n \int \frac{dx}{d \cos x + e \sin x + f} + k$$

and the integral on the right is of type three in 2-20.

2-24 INTEGRAL OF $\int \sec x dx$ IN TERMS OF $\sin x$.

$$\text{The integral, } \int \sec x dx = \int \frac{1}{\cos x} dx + c = \int \frac{\cos x}{\cos^2 x} dx + c$$

$$= \int \frac{\cos x}{1 - \sin^2 x} dx + c = \int \frac{du}{1 - u^2} + c \text{ when } \sin x = u$$

$$= \frac{1}{2} \log \frac{1+u}{1-u} + c = \frac{1}{2} \log \frac{1+\sin x}{1-\sin x} + c$$

2-25 EXPONENTIAL FORM OF Sine AND Cosine.

Using $a=1$ in 2-17 (B).

$$\int \frac{dx}{1-x^2} = \frac{1}{2} \log \frac{1+x}{1-x} + c$$

Again using $x=iu$, $i=\sqrt{-1}$, we have,

$$i \int \frac{du}{1+u^2} = \frac{1}{2} \log \frac{1+iu}{1-iu} + c$$

$$\text{or, } \int \frac{du}{1+u^2} = \frac{1}{2i} \log \frac{1+iu}{1-iu} + c$$

$$\text{or, } \tan^{-1} u = \frac{1}{2i} \log \frac{1+iu}{1-iu} + c, \text{ from 2-17 (A)}$$

Let $u = \tan \theta$ or $\theta = \tan^{-1} u$. Therefore,

$$\theta = \frac{1}{2i} \log \frac{1+i \tan \theta}{1-i \tan \theta} + c$$

Here $c=0$ when $\theta=0$.

$$\text{So, } \theta = \frac{1}{2i} \log \frac{\cos \theta + i \sin \theta}{\cos \theta - i \sin \theta}$$

or,

$$e^{2i\theta} = (\cos \theta + i \sin \theta)^2$$

$$\therefore e^{i\theta} = \cos \theta + i \sin \theta, \text{ (taking square-root)}$$

(1)

Replacing i by $(-i)$, we get

$$e^{-i\theta} = \cos \theta - i \sin \theta \quad (2)$$

The equations (1) and (2) are known as *Euler's equations*. Adding and subtracting (1) & (2) $\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$ and $\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

These are *exponential forms* of cosine and sine respectively.

Examples

1. Integrate $\frac{1}{x^2 + 64}$

This integral is of type 2-17 (A). So, Let $x = 8 \tan \theta$.

$\therefore dx = 8 \sec^2 \theta d\theta$. Therefore,

$$\begin{aligned} \int \frac{dx}{x^2 + 64} &= \int \frac{dx}{x^2 + 8^2} + c = \int \frac{8 \sec^2 \theta d\theta}{8^2(1 + \tan^2 \theta)} + c \\ &= \frac{1}{8} \int d\theta + c = \frac{1}{8} \theta + c = \frac{1}{8} \tan^{-1} \frac{x}{8} + c \end{aligned}$$

2. Compute $\int \frac{dx}{64 - x^2}$

This integral is of type 2-17 (B). Therefore,

$$\begin{aligned} \int \frac{dx}{64 - x^2} &= \int \frac{dx}{(8-x)(8+x)} + c = \frac{1}{2 \cdot 8} \int \frac{(8-x) + (8+x)}{(8-x)(8+x)} dx + c \\ &= \frac{1}{16} \int \left(\frac{1}{8-x} + \frac{1}{8+x} \right) dx + c \\ &= \frac{1}{16} [\log(8+x) - \log(8-x)] + c \\ &= \frac{1}{16} \log \frac{8+x}{8-x} + c \end{aligned}$$

3. Find $\int \frac{dx}{x^2 - 64}$.

This integral is of type 2-17 (C). So,

$$\begin{aligned} \int \frac{dx}{x^2 - 64} &= \int \frac{dx}{(x-8)(x+8)} + c = \frac{1}{2 \cdot 8} \int \frac{-(x-8) + (x+8)}{(x-8)(x+8)} dx + c \\ &= \int \frac{1}{16} \left(\frac{1}{x-8} - \frac{1}{x+8} \right) dx + c \\ &= \frac{1}{16} (\log(x-8) - \log(x+8)) + c \\ &= \frac{1}{16} \log \frac{x-8}{x+8} + c. \end{aligned}$$

4. Find the integral $\int \frac{\sin \theta + \cos \theta}{5 - \sin 2\theta} d\theta$.

At this stage, type cannot be identified. So, let us simplify the integrand as

$$\begin{aligned} \int \frac{\sin \theta + \cos \theta}{5 - \sin 2\theta} d\theta &= \int \frac{(\sin \theta + \cos \theta) d\theta}{4 + \cos^2 \theta + \sin^2 \theta - 2 \sin \theta \cos \theta} + c \\ &= \int \frac{\sin \theta + \cos \theta}{4 + (\sin \theta - \cos \theta)^2} d\theta + c \\ &= \int \frac{du}{4 + u^2} + c \text{ when } \sin \theta - \cos \theta = u \\ &= \frac{1}{2} \tan^{-1}(u/2) + c \text{ by 2-17 (A)} \\ &= \frac{1}{2} \tan^{-1} \left(\frac{\sin \theta - \cos \theta}{2} \right) + c. \end{aligned}$$

5. Compute $\int \frac{dx}{3 + 4 \cosh 2x}$.

This is of the type 2-20 (4). Therefore, expressing

$1 = \cosh^2 x - \sinh^2 x$ and $\cosh 2x = \cosh^2 x + \sinh^2 x$ in terms of half the given angle, we get

$$\begin{aligned} \int \frac{dx}{3 + 4 \cosh 2x} &= \int \frac{dx}{3 (\cosh^2 x - \sinh^2 x) + 4 (\cosh^2 x + \sinh^2 x)} + c \\ &= \int \frac{dx}{7 \cosh^2 x + \sinh^2 x} + c = \int \frac{\operatorname{sech}^2 x dx}{7 + \tanh^2 x} + c \\ &= \int \frac{du}{(\sqrt{7})^2 + u^2} + c \text{ when } \tanh x = u \\ &= \frac{1}{\sqrt{7}} \tan^{-1} \left(\frac{u}{\sqrt{7}} \right) + c = \frac{1}{\sqrt{7}} \tan^{-1} \left(\frac{\tanh x}{\sqrt{7}} \right) + c \end{aligned}$$

6. Integrate $\frac{4x-3}{9x^2+4}$.

This is of the type 2-19. Therefore,

$$\begin{aligned} \int \frac{4x-3}{9x^2+4} dx &= \int \frac{\frac{2}{9} \frac{d}{dx} (9x^2+4) - 3}{9x^2+4} dx + c \\ &= \frac{\frac{2}{9} \int \frac{d}{dx} (9x^2+4)}{9x^2+4} dx - 3 \int \frac{1}{9x^2+4} dx + c \end{aligned}$$

$$\begin{aligned}
 &= \frac{2}{9} \log(9x^2 + 4) - \frac{1}{3} \int \frac{1}{x^2 + \left(\frac{2}{3}\right)^2} dx + c. \\
 &= \frac{2}{9} \log(9x^2 + 4) - \frac{1}{2} \tan^{-1} \left(\frac{3x}{2} \right) + c
 \end{aligned}$$

Exercises 2 (5) (P)

Find the following integrals :

Ex 1. $\int \frac{dx}{x^2 + 16}$

Ex 2. $\int \frac{dx}{4x^2 - 25}$

Ex 3. $\int \frac{dt}{4 - 9t^2}$

Ex 4. $\int \frac{e^x}{1 + e^{2x}} dx$

Ex 5. $\int \frac{a^2 x}{x^4 + b^2} dx$

Ex 6. $\int \frac{4x^2}{3 + 5x^6} dx$

Ex 7. $\int \frac{d\theta}{e^{\theta} + e^{-\theta}}$

Ex 8. $\int \frac{\cos \theta}{a^2 + b^2 \sin^2 \theta} d\theta$

Ex 9. $\int \frac{dx}{x^2 + 4x + 5}$

Ex 10. $\int \frac{dx}{2x - x^2 - 10}$

Ex 11. $\int \frac{dx}{2x^2 - 2x + 1}$

Ex 12. $\int \frac{x dx}{x^4 - 4x^2 + 13}$

Ex 13. $\int \frac{e^x dx}{e^{2x} - 6e^x + 16}$

Ex 14. $\int \frac{\sin \theta}{1 + \cos^2 \theta} d\theta$

Ex 15. $\int \frac{\cos \theta - \sin \theta}{5 + \sin 2\theta} d\theta$

Ex 16. $\int \frac{d\theta}{\cos^2 \theta (1 + \sec^2 \theta + 2 \tan \theta)}$

Ex 17. $\int \frac{dx}{1 + \cos^2 x}$

Ex 18. $\int \frac{x^2 + \sin^2 x}{1 + x^2} \sec^2 x dx$

Ex 19. $\int \frac{u^2 du}{4 - (3u^3 - 1)^2}$

Ex 20. $\int \frac{dx}{1 + x - x^2}$

Ex 21. $\int \frac{\sin x dx}{\cos^2 x + 6 \cos x + 5}$

Ex 22. $\int \frac{x dx}{x^4 - 4x^2 + 3}$

23. $\int \frac{x}{(x-3)\sqrt{x+1}} dx$

24. $\int \frac{x+1}{x\sqrt{x+2}} dx$

25. $\int \frac{dx}{x[(\log x)^2 + 8 \log x - 9]}$

Ex 26. $\int \frac{2+3x}{1+x^2} dx$

$$Q\cancel{P} 27. \int \frac{x+1}{x^2+2x+5} dx$$

$$28. \int \frac{3x-2}{1-6x-9x^2} dx$$

$$29. \int \frac{4-3x}{1+x+x^2} dx$$

$$Q\cancel{P} 30. \int \frac{dx}{2+3 \cos x}$$

$$Q\cancel{P} 31. \int \frac{dx}{3+2 \cos 2x}$$

$$Q\cancel{P} 32. \int \frac{dx}{3 \sin x + 4}$$

$$\cancel{Q\cancel{P}} 33. \int \frac{dx}{(5+4 \cos x)^2}$$

$$34. \int \frac{dx}{1+3 \sin^2 x}$$

$$Q\cancel{P} 35. \int \frac{dx}{1+\sin x+\cos x}$$

$$Q\cancel{P} 36. \int \frac{dx}{1-2 \cos x+3 \sin x}$$

$$Q\cancel{P} 37. \int \frac{dx}{4+\cos^2 x}$$

$$38. \int \frac{\sin^2 x}{1+2 \cos x} dx$$

$$39. \int \frac{dx}{3+4 \sinh x}$$

$$40. \int \frac{d\theta}{\sin^4 \theta - \cos^4 \theta}$$

$$41. \int \frac{\cos x + 2 \sin x + 3}{4 \cos x + 5 \sin x + 6} dx$$

$$42. \int \frac{\cosh x + 2 \sinh x + 3}{4 \cosh x + 5 \sinh x + 6} dx$$

$$43. \text{ Show that } \int \frac{dx}{x\sqrt{(x^2-x+2)}} = \frac{1}{\sqrt{2}} \log \frac{\sqrt{(x^2-x+2)} + x - \sqrt{2}}{\sqrt{(x^2-x+2)} + x + \sqrt{2}} + c$$

using the substitution $u = x + \sqrt{(x^2-x+2)}$.

$$44. \text{ Show that } \int \frac{dx}{x\sqrt{(x^2+2x-1)}} = 2 \tan^{-1} \{x + \sqrt{(x^2+2x-1)}\} + c \text{ using}$$

the substitution $u = x + \sqrt{(x^2+2x-1)}$.

[Hints: For 43 and 44 $\frac{du}{u^2-2} = \frac{dx}{x\sqrt{(x^2-x+2)}}$ and

$$\frac{2 du}{u^2+1} = \frac{dx}{x\sqrt{(x^2+2x-1)}} \text{ respectively].}$$

Answers

$$1. \left(\frac{1}{4} \tan^{-1} \frac{x}{4} + c \right)$$

$$2. \left(\frac{1}{20} \log \frac{2x-5}{2x+5} + c \right)$$

$$3. \left(\frac{1}{12} \log \frac{2+3t}{2-3t} + c \right)$$

$$4. (\tan^{-1} e^x + c)$$

$$5. \left(\frac{a^2}{2b} \tan^{-1} \frac{x^2}{b} + c \right)$$

$$6. \left(\frac{4}{3\sqrt{15}} \tan^{-1} \frac{\sqrt{5}x^3}{\sqrt{3}} + c \right)$$

7. $\left[\tan^{-1} e^{\theta} + c \right]$
9. $\left[\tan^{-1} (x+2) + c \right]$
11. $\left[\tan^{-1} (2x-1) + c \right]$
13. $\left[\frac{1}{\sqrt{7}} \tan^{-1} \frac{e^x - 3}{\sqrt{7}} + c \right]$
15. $\left[\frac{1}{2} \tan^{-1} \left(\frac{\sin \theta + \cos \theta}{2} \right) + c \right]$
17. $\left[\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\tan x}{\sqrt{2}} \right) + c \right]$
19. $\left[\frac{1}{36} \log \frac{1+3e^3}{3-3e^3} + c \right]$
21. $\left[\frac{1}{4} \log \frac{5+\cos x}{1+\cos x} + c \right]$
23. $\left[2\sqrt{x+1} + \frac{3}{2} \log \frac{\sqrt{x+1}-2}{\sqrt{x+1}+2} + c \right]$
25. $\left[\frac{1}{10} \log \frac{\log x-1}{\log x+9} + c \right]$
27. $\left[\frac{1}{2} \log (x^2+2x+5) + c \right]$
28. $\left[c - \frac{1}{6} \log (1-6x-9x^2) - \frac{3}{2\sqrt{2}} \log \left(\frac{3x+1+\sqrt{2}}{\sqrt{2}-1-3x} \right) \right]$
29. $\left[c - \frac{3}{2} \log (1+x+x^2) + \frac{11}{\sqrt{3}} \tan^{-1} \frac{2x+1}{\sqrt{3}} \right]$
30. $\left[\frac{1}{\sqrt{5}} \log \left(\frac{\sqrt{5} + \tan (x/2)}{\sqrt{5} - \tan (x/2)} \right) + c \right]$
31. $\left[\frac{1}{\sqrt{5}} \tan^{-1} \left(\frac{1}{\sqrt{5}} \tan x \right) + c \right]$
33. $\left[\frac{10}{27} \tan^{-1} \left(\frac{\tan (x/2)}{3} \right) - \frac{4}{9} \frac{\sin x}{5+4 \cos x} + c \right]$
35. $\left[\log \{1 + \tan (x/2)\} + c \right]$
37. $\left[\frac{1}{2\sqrt{5}} \tan^{-1} \left(\frac{2 \tan x}{\sqrt{5}} \right) + c \right]$
39. $\left[\frac{1}{5} \log \frac{1+3 \tanh (x/2)}{9-3 \tanh (x/2)} + c \right]$
8. $\left[\frac{1}{ab} \tan^{-1} \left(\frac{b \sin \theta}{a} \right) + c \right]$
10. $\left[\frac{1}{3} \tan^{-1} \left(\frac{1-x}{3} \right) + c \right]$
12. $\left[\frac{1}{6} \tan^{-1} \frac{x^2+2}{3} + c \right]$
14. $[c - \tan^{-1} (\cos \theta)]$
16. $[\tan^{-1} (1 + \tan \theta) + c]$
18. $(\tan x - \tan^{-1} x + c)$
20. $\left[\frac{1}{\sqrt{5}} \log \frac{\sqrt{5}-1+2x}{\sqrt{5}+1-2x} + c \right]$
22. $\left[\frac{1}{4} \log \frac{x^2-3}{x^2-1} + c \right]$
24. $\left[2\sqrt{x+2} + \sqrt{2} \log \left(\frac{\sqrt{x+2}-\sqrt{2}}{x} \right) + c \right]$
26. $\left[2 \tan^{-1} x + \frac{3}{2} \log (1+x^2) + c \right]$
32. $\left[\frac{2}{\sqrt{7}} \tan^{-1} \left(\frac{4 \tan (x/2)+3}{\sqrt{7}} \right) + c \right]$
34. $\left[\frac{1}{2} \tan^{-1} (2 + \tan x) + c \right]$
36. $\left[\frac{\sqrt{3}}{6} \log \frac{\sqrt{3} \{ \tan (x/2)+1 \} - 2}{\sqrt{3} \{ \tan (x/2)+1 \} + 2} + c \right]$
38. $\left[\frac{1}{4} x - \frac{1}{2} \sin x - \frac{\sqrt{3}}{4} \log \frac{\sqrt{3} + \tan (x/2)}{\sqrt{3} - \tan (x/2)} + c \right]$
40. $\left[\frac{1}{2} \log \frac{1+\tan \theta}{\tan \theta - 1} + c \right]$

$$41. \left[\frac{14}{41}x - \frac{3}{41} \log (4 \cos x + 5 \sin x + 6) + \frac{39}{41} \cdot \frac{1}{\sqrt{5}} \log \left(\frac{2 \sin \frac{x}{2} + (5 - \sqrt{5}) \cos \frac{x}{2}}{2 \sin \frac{x}{2} + (5 + \sqrt{5}) \cos \frac{x}{2}} \right) + c \right]$$

$$42. \left[\frac{2}{3}x - \frac{1}{3} \log (4 \cosh x + 5 \sinh x + 6) + \frac{1}{3} \cdot \frac{1}{\sqrt{5}} \log \left(\frac{(3\sqrt{5} - 5) \cosh (x/2) + 2 \sinh (x/2)}{(3\sqrt{5} + 5) \cosh (x/2) - 2 \sinh (x/2)} \right) + c \right]$$

2.26 INTEGRAL OF THE TYPE $\int \frac{ax^2 \pm c}{a^2x^4 + bx^2 + c^2} dx$.

Observe that numerator is two degree less than that of denominator. Dividing both numerator and denominator by x^2 , reduce the degree of each by two, i.e.

$$\int \frac{ax^2 \pm c}{a^2x^4 + bx^2 + c^2} dx = \int \frac{a \pm \frac{c}{x^2}}{a^2x^2 + b + \frac{c^2}{x^2}} dx + k_1,$$

[Dividing both numerator and denominator by x^2]

$$= \int \frac{a \pm \frac{c}{x^2}}{(ax \mp \frac{c}{x})^2 + (b \mp 2ac)} dx = k_1,$$

$$= \int \frac{du}{u^2 \pm k^2} + k_1, \text{ where } ax \mp \frac{c}{x} = u \text{ and } k = \sqrt{b \mp 2ac}.$$

Now this is of type 2.17(A), (B) or (C)

N.B. ($\pm$) sign of numerator corresponds to ($\mp$) sign of denominator.

2.27 INTEGRAL OF THE TYPES

$$\int \frac{dx}{a^2x^4 + bx^2 + c^2} \text{ and } \int \frac{x^2 dx}{a^2x^4 + bx^2 + c^2}.$$

These are special cases of 2.26. In the first one, the numerator simplifies with $a = 0$ and $c = 1$ and in the second $a = 1$, $c = 0$. The numerators of these integrals can be expressed as the sum and the difference of the numerator of 2.26. Thus

$$\begin{aligned} \int \frac{1}{a^2x^4 + bx^2 + c^2} dx &= \frac{1}{2c} \int \frac{(ax^2 + c) - (ax^2 - c)}{a^2x^4 + bx^2 + c^2} dx \\ &= \frac{1}{2c} \int \frac{ax^2 + c}{a^2x^4 + bx^2 + c^2} dx - \frac{1}{2c} \int \frac{ax^2 - c}{a^2x^4 + bx^2 + c^2} dx \end{aligned}$$

$$\begin{aligned}\text{and } \int \frac{x^2}{a^2 x^2 + bx^2 + c^2} dx &= \frac{1}{2a} \int \frac{(ax^2 + c) + (ax^2 - c)}{a^2 x^2 + bx^2 + c^2} dx \\ &= \frac{1}{2a} \int \frac{ax^2 + c}{a^2 x^2 + bx^2 + c^2} dx + \frac{1}{2a} \int \frac{ax^2 - c}{a^2 x^2 + bx^2 + c^2} dx\end{aligned}$$

Now use the procedure of 2.26 repeatedly.

Examples

1. Integrate $\frac{x^2 + 1}{x^4 + 1}$.

This is of type 2.26. Here $a=1, b=0, c=1$. So, dividing the numerator and denominator by x^2 ,

$$\begin{aligned}\int \frac{x^2 + 1}{x^4 + 1} dx &= \int \frac{1 + \frac{1}{x^2}}{x^2 + \frac{1}{x^2}} dx + c = \int \frac{1 + \frac{1}{x^2}}{\left(x - \frac{1}{x}\right)^2 + (\sqrt{2})^2} dx + c \\ &= \int \frac{du}{u^2 + (\sqrt{2})^2} + c \quad \text{where } x - \frac{1}{x} = u \\ &= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{u}{\sqrt{2}} \right) + c = \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x^2 - 1}{\sqrt{2}x} \right) + c\end{aligned}$$

2. Compute $\int \sqrt{\tan x} dx$.

This integral *cannot* be identified at this stage. Let us try to eliminate the surd. So, let $\tan x = u^2$. Therefore,

$$\sec^2 x dx = 2u du \quad \text{or } (1 + \tan^2 x) dx = 2u du \quad \text{or } dx = \frac{2u}{1 + u^4} du$$

$$\therefore \int \sqrt{\tan x} dx = \int \sqrt{u^2} \cdot \frac{2u}{1 + u^4} du + c = \int \frac{2u^2}{u^4 + 1} du + c$$

$$\begin{aligned}I &= \int \frac{2u^2}{u^4 + 1} du + c = \int \frac{(u^2 + 1) + (u^2 - 1)}{u^4 + 1} du + c \\ &= \int \frac{u^2 + 1}{u^4 + 1} du + \int \frac{u^2 - 1}{u^4 + 1} du + c = I_1 + I_2 + c\end{aligned}$$

Here I_1 & I_2 are of type 2.26.

From Ex. 1 above, $I_1 = \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{u^2 - 1}{\sqrt{2}u} \right) + c_1$. Also,

$$\begin{aligned}
 I_2' &= \int \frac{1 - \frac{1}{u^2}}{u^2 + \frac{1}{u^2}} du + c_2 = \int \frac{1 - \frac{1}{u^2}}{(u + \frac{1}{u})^2 - (\sqrt{2})^2} du + c_2 \\
 &= \frac{1}{2\sqrt{2}} \log \frac{u + \frac{1}{u} - \sqrt{2}}{u + \frac{1}{u} + \sqrt{2}} + c_2 = \frac{1}{2\sqrt{2}} \log \frac{u^2 - \sqrt{2}u + 1}{u^2 + \sqrt{2}u + 1} + c_2 \\
 \therefore I &= I_1 + I_2 = \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\tan x - 1}{\sqrt{2} \tan x} \right) + \frac{1}{2\sqrt{2}} \log \frac{\tan x - \sqrt{2} \tan x + 1}{\tan x + \sqrt{2} \tan x + 1} + c
 \end{aligned}$$

✎ Exercises 2 (6)

Compute the following integrals :

1. $\int \frac{x^3 + 1}{x^4 + 4} dx$

2. $\int \frac{x^2 - 1}{x^4 + 1} dx$

3. $\int \frac{dx}{\cos^4 x + \sin^4 x + \cos^2 x \sin^2 x}$

4. $\int \frac{x^2 + 1}{x^4 + x^2 + 1} dx$

5. $\int \frac{1}{x^4 + x^2 + 1} dx$

6. $\int \frac{x^2 + 3x + 1}{x^4 - x^2 + 1} dx$ *partial*

7. $\int \frac{x^2 - 1}{x^4 + x^2 + 1} dx$

8. (i) $\int \frac{x^2}{x^4 + a^4} dx = \frac{1}{2} (I_1 + I_2)$ and

(ii) $\int \frac{dx}{x^4 + a^4} = \frac{1}{2a^2} (I_1 - I_2)$ where $I_1 = \int \frac{x^2 + a^2}{x^4 + a^4} dx$ and

$I_2 = \int \frac{x^2 - a^2}{x^4 + a^4} dx.$

9. $\int \frac{dx}{\cos^4 x + \sin^4 x - \cos^2 x \sin^2 x}$

10. $\int \sqrt{\cot x} dx$

11. $\int \frac{\sin^2 x}{\cos^4 x + \sin^4 x} dx$

12. $\int \frac{1}{\sqrt{2 - \tan \theta}} d\theta$

13. $\int \frac{dx}{\sqrt{1 + \cot x}}$

14. $\int \frac{dx}{(x^4 + 1)^2}$

15. $\int \frac{dx}{(x^4 + 1)^3}$

Answers

$$1. \left[\frac{1}{4} \log(x^4 + 4) + \frac{1}{16} \left[2 \tan^{-1} \left(\frac{x^2 - 2}{2x} \right) - \log \frac{x^2 - 2x + 2}{x^2 + 2x + 2} \right] + c \right]$$

$$2. \left[\frac{1}{2\sqrt{2}} \log \frac{x^2 - \sqrt{2}x + 1}{x^2 + \sqrt{2}x + 1} + c \right] \quad 3. \left[\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\tan x - \cot x}{\sqrt{3}} \right) + c \right]$$

$$4. \left[\frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{x^2 - 1}{\sqrt{3}x} \right) + c \right]$$

$$5. \left[\frac{1}{2\sqrt{3}} \tan^{-1} \left(\frac{x^3 - 1}{\sqrt{3}x} \right) - \frac{1}{4} \log \frac{x^2 - x + 1}{x^2 + x + 1} + c \right]$$

$$6. \left[\tan^{-1} \left(\frac{x^2 - 1}{x} \right) + \sqrt{3} \tan^{-1} \left(\frac{2x^2 - 1}{\sqrt{3}} \right) + c \right] \quad 7. \left[\frac{1}{2} \log \frac{x^2 - x + 1}{x^2 + x + 1} + c \right]$$

$$8. (i) \left[\frac{\sqrt{2}}{4a} \left\{ \tan^{-1} \left(\frac{x^2 - a^2}{a\sqrt{2}x} \right) + \frac{1}{2} \log \left(\frac{x^2 - \sqrt{2}ax + a^2}{x^2 + \sqrt{2}ax + a^2} \right) \right\} + c \right]$$

$$(ii) \left[\frac{\sqrt{2}}{4a} \left\{ \tan^{-1} \left(\frac{x^2 - a^2}{a\sqrt{2}x} \right) - \frac{1}{2} \log \left(\frac{x^2 - \sqrt{2}ax + a^2}{x^2 + \sqrt{2}ax + a^2} \right) \right\} + c \right]$$

$$9. \left[\tan^{-1}(\tan x - \cot x) + c \right]$$

$$10. \left[\frac{1}{2\sqrt{2}} \log \frac{t^2 + \sqrt{2}t + 1}{t^2 - \sqrt{2}t + 1} - \frac{1}{\sqrt{2}} \tan^{-1} \frac{t^2 - 1}{\sqrt{2}t} + c, \cot x = t^2 \right]$$

$$11. \left[\frac{1}{4\sqrt{2}} \left\{ \tan^{-1} \frac{t^2 - 1}{\sqrt{2}t} + \log \frac{t^2 - \sqrt{2}t + 1}{t^2 + \sqrt{2}t + 1} \right\} + c, \tan x = t \right]$$

$$12. \left[\text{Put } (2 - \tan \theta) = u^2, \frac{1}{10\sqrt{6}} \tan^{-1} \frac{\tan \theta + 3}{\sqrt{6}(2 - \tan \theta)} + \frac{1}{10\sqrt{14}} \left[\log \frac{7 - \tan \theta - \sqrt{14}(2 - \tan \theta)}{7 - \tan \theta + \sqrt{14}(2 - \tan \theta)} \right] \right]$$

$$13. \left[\frac{1}{4\sqrt{2}} \tan^{-1} \frac{1 - \cot x}{\sqrt{2}(1 + \cot x)} + \frac{1}{4\sqrt{6}} \log \frac{\cot x + 3 - \sqrt{6}(1 + \cot x)}{\cot x + 3 + \sqrt{6}(1 + \cot x)} + c \right]$$

$$14. \left[-\frac{x}{4(x^4 + 1)} + \frac{3}{8\sqrt{2}} \tan^{-1} \left(\frac{x^2 - 1}{x\sqrt{2}} \right) - \frac{3}{16\sqrt{2}} \log \frac{x^2 + 1 - x\sqrt{2}}{x^2 + 1 + x\sqrt{2}} + c \right]$$

$$[\text{Hints: } I = \int \frac{x^4 + 1 - x^4}{(x^4 + 1)^2} dx = \int \frac{dx}{x^4 + 1} - \int x \cdot \frac{x^3}{(x^4 + 1)^2} dx.]$$

Integrate by parts the second integral

$$\therefore I = \frac{3}{4} \int \frac{dx}{x^4 + 1} - \frac{x}{4(x^4 + 1)}$$

$$15. \left[\frac{x}{8(x^2 + 1)^2} + \frac{7}{8} \left\{ -\frac{x}{4(x^4 + 1)} + \frac{3}{8\sqrt{2}} \tan^{-1} \frac{x^2 - 1}{x\sqrt{2}} - \frac{3}{16\sqrt{2}} \log \frac{x^2 + 1 - x\sqrt{2}}{x^2 + 1 + x\sqrt{2}} \right\} + c \right]$$

PARTIAL FRACTIONS

3.1. INTRODUCTION

There is no rule for integrating a fraction as is for differentiation. Partial fractions (PF) are used to integrate rational functions, $\frac{P(x)}{Q(x)}$ where both $P(x)$ and $Q(x)$ are polynomials in x without having no common factors between them but $Q(x)$ can be factorized into linear and quadratic factors, repeated and non-repeated. This fraction is called proper if the degree of $P(x)$ is less than that of $Q(x)$, otherwise improper.

3.2. THE RULES FOR PARTIAL FRACTIONS

Before performing PF, the given fraction should be made proper by actual division if it is improper. A proper fraction can be decomposed into PF according to the following table. The proofs of how this table is created are obtained in text books on Advanced Higher Algebra.

Nature of factors in $Q(x)$	Form of PF corresponding to each factor of $Q(x)$
(i) Non-repeated linear factor, $(ax + b)$	(i) $\frac{A}{ax + b}$
(ii) Repeated linear factor, $(ax + b)^n$	(ii) $\sum_{r=1}^n \frac{A_r}{(ax + b)^r}$
(iii) Non-repeated quadratic factor, $(ax^2 + bx + c)$	(iii) $\frac{Ax + B}{ax^2 + bx + c}$
(iv) Repeated quadratic factor $(ax^2 + bx + c)^n$	(iv) $\sum_{r=1}^n \frac{A_r x + B_r}{(ax^2 + bx + c)^r}$

DETERMINATION OF THE CO-EFFICIENTS OF PF

The coefficients of PF can be determined in two ways (i) algebraic method and (ii) derivative method. In algebraic method, there is no general formula for the determination of the coefficients of PF. Yet, it has a wide user. On the other hand derivative method has formula for that (sec 3.3). Algebraic method is as follows :

- (i) Remove the denominators by multiplying throughout by $Q(x)$.

- (ii) Giving values to x or equating coefficients of x , form a set of simultaneous equations of the coeffs. and then solve for the coefficients.

3.3. DERIVATIVES TO DETERMINE THE COEFFICIENTS OF PF

This method will be described in details for the four different cases one after another to derive formula for each.

(i) Non-repeated linear factors

Let $P(x)/Q(x)$ be a proper fraction and let α_1 be a non-repeated real root of $Q(x)$. So, $Q(x) \equiv (x - \alpha_1)f(x)$. Therefore,

$\frac{P(x)}{Q(x)} = \frac{A_1}{x - \alpha_1} + \frac{F(x)}{f(x)}$ where $\frac{F(x)}{f(x)}$ is the sum of the remaining fractions.

$$\therefore P(x) = A_1 f(x) + (x - \alpha_1) F(x)$$

$$\text{or } P(\alpha_1) = A_1 f(\alpha_1) \quad (1)$$

Again, $Q(x) = (x - \alpha_1)f(x)$

$$\therefore Q'(x) = (x - \alpha_1)f'(x) + f(x), \text{ prime denotes differentiation w.r.t. } x.$$

$$\text{or } Q'(\alpha_1) = f(\alpha_1) \quad (2)$$

$$\text{From (1) and (2), } A_1 = \frac{P(\alpha_1)}{Q'(\alpha_1)}$$

Similarly, if $\alpha_2, \dots, \alpha_n$ are non-repeated real roots of $Q(x)$, then

$$A_2 = \frac{P(\alpha_2)}{Q'(\alpha_2)}, \dots, A_n = \frac{P(\alpha_n)}{Q'(\alpha_n)}.$$

$$\therefore A_r = \frac{P(\alpha_r)}{Q'(\alpha_r)}, \quad r = 1, 2, \dots, n. \quad (3)$$

(ii) Repeated linear factors

Let α be a n -times repeated real root of $Q(x)$.

Then $(x - \alpha)^n$ is a n -times repeated linear factor of $Q(x)$.

Let $\frac{P(x)}{Q(x)} = \sum_{r=1}^n \frac{A_r}{(x - \alpha)^r} + \frac{f_1(x)}{f(x)}$, where $\frac{f_1(x)}{f(x)}$ is the sum of the

remaining fractions, Let $\frac{P(x)}{Q(x)}(x - \alpha)^n = F(x)$.

Therefore multiplying throughout by $(x - \alpha)^n$, we have,

$$F(x) = \sum_{r=1}^n A_r (x - \alpha)^{n-r} + \frac{f_1(x)}{f(x)} (x - \alpha)^n \quad (1)$$

$$\therefore F(\alpha) = A_n.$$

To determine the remaining coefficients, $A_r, r = 1, 2, \dots (n-1)$, (1) is differentiated successively and in each derivative $x = \alpha$ is used to get the following :

$$F'(\alpha) = A_{n-1}, F''(\alpha) = (2!) A_{n-2}, F'''(\alpha) = (3!) A_{n-3},$$

$$\dots\dots\dots F^{n-1}(\alpha) = ((n-1)!) A_1.$$

$$\therefore A_r = \frac{F^{(n-r)}(\alpha)}{(n-r)!}, r = 1, 2, \dots, n.$$

(iii) Non-repeated quadratic factor

Let $(x^2 + px + q)$ be a non-repeated quadratic factor of $Q(x)$ and let $\alpha = \frac{1}{2}(-p + i\sqrt{4q - p^2})$ and $\beta = \frac{1}{2}(-p - i\sqrt{4q - p^2})$ be two imaginary zeros of this factor.

Let $\frac{P(x)}{Q(x)} = \frac{Ax+B}{x^2+px+q} + \frac{F(x)}{f(x)}$ where $\frac{F(x)}{f(x)}$ is the sum of the remaining fractions.

$$\therefore P(x) = (Ax+B)f(x) + (x^2+px+q)F(x) \quad (1)$$

or $P(\alpha) = (A\alpha+B)f(\alpha),$

and $P(\beta) = (A\beta+B)f(\beta), [\because \alpha^2 + p\alpha + q = 0 \text{ and } \beta^2 + p\beta + q = 0]$

$$\therefore \left. \begin{aligned} A &= \frac{1}{\alpha - \beta} \left[\frac{P(\alpha)}{f(\alpha)} - \frac{P(\beta)}{f(\beta)} \right] \\ \text{and } B &= \frac{1}{\beta - \alpha} \left[\frac{\beta P(\alpha)}{f(\alpha)} - \frac{\alpha P(\beta)}{f(\beta)} \right] \end{aligned} \right\} \quad (2)$$

Again $Q(x) = (x^2 + px + q)f(x)$

$$Q'(x) = (2x+p)f(x) + (x^2+px+q)f'(x)$$

$$\therefore \left. \begin{aligned} Q'(\alpha) &= (2\alpha+p)f(\alpha) = i\sqrt{4q-p^2}f(\alpha) = (\alpha-\beta)f(\alpha) \\ \text{and } Q'(\beta) &= (2\beta+p)f(\beta) = -i\sqrt{4q-p^2}f(\beta) = (\beta-\alpha)f(\beta) \end{aligned} \right\} \quad (3)$$

Using (3) in (2),

$$A = \frac{P(\alpha)}{Q'(\alpha)} + \frac{P(\beta)}{Q'(\beta)} \text{ and } B = - \left[\frac{\alpha P(\beta)}{Q'(\beta)} + \frac{\beta P(\alpha)}{Q'(\alpha)} \right] \quad (4)$$

Using (2) or (4) the coefficients A and B can be determined.

(iv) Repeated quadratic factors.

Let $(x^2 + px + q)^n$ be a n -times repeated quadratic factors of $Q(x)$ and it has two imaginary roots $\alpha = \frac{1}{2}(-p + i\sqrt{4q - p^2})$

and $\beta = \frac{1}{2}(-p - i\sqrt{4q - p^2}).$

Let $\frac{P(x)}{Q(x)} = \sum_{r=1}^n \frac{A_r x + B_r}{(x^2 + px + q)^r} + \frac{F(x)}{f(x)}$ where $\frac{F(x)}{f(x)}$ is the sum of the remaining fractions.

$$\therefore P(x) = \sum_{r=1}^n (A_r x + B_r) (x^2 + px + q)^{n-r} f(x) + (x^2 + px + q)^n F(x) \quad (1)$$

and let $Q_n = (x^2 + px + q)^n f(x)$

The n th derivative of (2) at $x = \alpha$ and $x = \beta$ are given by

$$\left. \begin{aligned} Q_n^{(n)}(\alpha) &= n! (2\alpha + p)^n f(\alpha) \\ Q_n^{(n)}(\beta) &= n! (2\beta + p)^n f(\beta) \end{aligned} \right\} \quad (3)$$

Also (1) and its successive derivatives at $x = \alpha$ and $x = \beta$ are

$$\left. \begin{aligned} P(\alpha) &= (A_n \alpha + B_n) f(\alpha) \\ P(\beta) &= (A_n \beta + B_n) f(\beta) \end{aligned} \right\} \quad (4)$$

$$\left. \begin{aligned} P'(\alpha) &= [(A_n \alpha + B_n) f(\alpha)]' + (A_{n-1} \alpha + B_{n-1}) f(\alpha) \\ &\quad \cdot [(\alpha^2 + p\alpha + q)^{n-r}]'_{r=n-1} \\ P'(\beta) &= [(A_n \beta + B_n) f(\beta)]' + (A_{n-1} \beta + B_{n-1}) f(\beta) \\ &\quad \cdot [(\beta^2 + p\beta + q)^{n-r}]'_{r=n-1} \end{aligned} \right\} \quad (5)$$

$$\left. \begin{aligned} P''(\alpha) &= [(A_n \alpha + B_n) f(\alpha)]'' + 2C_1 [(A_{n-1} \alpha + B_{n-1}) f(\alpha)]' \\ &\quad \cdot [(\alpha^2 + p\alpha + q)^{n-r}]'_{r=n-1} + (A_{n-2} \alpha + B_{n-2}) f(\alpha) \\ &\quad \cdot [(\alpha^2 + p\alpha + q)^{n-r}]''_{r=n-2} \\ P''(\beta) &= [(A_n \beta + B_n) f(\beta)]'' + 2C_1 [(A_{n-1} \beta + B_{n-1}) f(\beta)]' \\ &\quad \cdot [(\beta^2 + p\beta + q)^{n-r}]'_{r=n-1} + (A_{n-2} \beta + B_{n-2}) f(\beta) \\ &\quad \cdot [(\beta^2 + p\beta + q)^{n-r}]''_{r=n-2} \end{aligned} \right\} \quad (6)$$

$$P^{(n-m)}(\alpha) = \left. \begin{aligned} &[u_r^{(n-m)}(\alpha)]_{r=n} + {}^{n-m}C_1 [u_r^{(n-m-1)}(\alpha) v_r'(\alpha)]_{r=n-1} \\ &+ {}^{n-m}C_2 [u_r^{(n-m-2)}(\alpha) v_r''(\alpha)]_{r=n-2} + \dots \\ &+ {}^{n-m}C_{n-m-1} [u_r'(\alpha) v_r^{(n-m-1)}(\alpha)]_{r=m+1} \\ &+ [u_r(\alpha) v_r^{(n-m)}(\alpha)]_{r=m} \end{aligned} \right\} \quad (7a)$$

and

$$P^{(n-m)}(\beta) = \left. \begin{aligned} &[u_r^{(n-m)}(\beta)]_{r=n} + {}^{n-m}C_1 [u_r^{(n-m-1)}(\beta) v_r'(\beta)]_{r=n-1} \\ &+ {}^{n-m}C_2 [u_r^{(n-m-2)}(\beta) v_r''(\beta)]_{r=n-2} + \dots \\ &+ {}^{n-m}C_{n-m-1} [u_r'(\beta) v_r^{(n-m-1)}(\beta)]_{r=m+1} + [u_r(\beta) v_r^{(n-m)}(\beta)]_{r=m} \end{aligned} \right\} \quad (7b)$$

where $u_r(x) = (A_r x + B_r)f(x)$ and $v_r(x) = (x^2 + px + q)^{n-r}$. The, equations (7a), (7b) are identical to Liebnitz formula for n th derivative of a product except the first term in which $[v_r]_{r=n} = 1$.

Again, from (3),

$$Q_1^{(1)} = 1! (2\alpha + p) f(\alpha) = [v_r'(\alpha)]_{r=n-1} f(\alpha)$$

and in the same way,

$$\left. \begin{aligned} Q_{n-m}^{(n-m)}(\alpha) &= (n-m)! (2\alpha + p)^{n-m} f(\alpha) \\ &= \left[v_r^{(n-m)}(\alpha) \right]_{r=m} f(\alpha) \end{aligned} \right\} \quad (8)$$

Similar equations like (8) can be written corresponding to $x = \beta$.

From (4),

$$\left. \begin{aligned} A_n &= \frac{1}{\alpha - \beta} \left[\frac{P(\alpha)}{f(\alpha)} - \frac{P(\beta)}{f(\beta)} \right] = \frac{P(\alpha)}{Q_1^{(1)}(\alpha)} + \frac{P(\beta)}{Q_1^{(1)}(\beta)} \\ B_n &= \frac{1}{\beta - \alpha} \left[\frac{\beta P(\alpha)}{f(\alpha)} - \frac{\alpha P(\beta)}{f(\beta)} \right] = - \left[\frac{\beta P(\alpha)}{Q_1^{(1)}(\alpha)} + \frac{\alpha P(\beta)}{Q_1^{(1)}(\beta)} \right] \end{aligned} \right\} \quad (9)$$

by using (8) and (3) of (iii). From (5),

$$\begin{aligned} A_{n-1} &= \frac{1}{\alpha - \beta} \left[\frac{P'(\alpha) - [(A_n \alpha + B_n) f(\alpha)]'}{f(\alpha) [(2\alpha + p\alpha + q)^{n-1}]'_{r=n-1}} - \frac{P'(\beta) - [(A_n \beta + B_n) f(\beta)]'}{f(\beta) [(2\beta + p\beta + q)^{n-1}]'_{r=n-1}} \right] \\ &= \frac{1}{\alpha - \beta} \left[\frac{P'(\alpha) - [(A_n \alpha + B_n) f(\alpha)]'}{f(\alpha) (2\alpha + p)^2} - \frac{P'(\beta) - [(A_n \beta + B_n) f(\beta)]'}{f(\beta) (2\beta + p)^2} \right] \\ &= \frac{P'(\alpha) - [(A_n \alpha + B_n) f(\alpha)]'}{f(\alpha) (2\alpha + p)^2} + \frac{P'(\beta) - [(A_n \beta + B_n) f(\beta)]'}{f(\beta) (2\beta + p)^2} \\ &= 2 \left[\frac{P'(\alpha) - [(A_n \alpha + B_n) f(\alpha)]'}{Q_2^{(2)}(\alpha)} + \frac{P'(\beta) - [(A_n \beta + B_n) f(\beta)]'}{Q_2^{(2)}(\beta)} \right] \quad (10a) \end{aligned}$$

$$\beta_{n-1} = -2 \left[\beta \frac{P'(\alpha) - [(A_n \alpha + B_n) f(\alpha)]'}{Q_2^{(2)}(\alpha)} + \alpha \frac{P'(\beta) - [(A_n \beta + B_n) f(\beta)]'}{Q_2^{(2)}(\beta)} \right] \quad (10b)$$

Proceeding in this way, from (7)

$$\begin{aligned} A_m \alpha + B_m &= [P^{(n-m)}(\alpha) - [u_r^{(n-m)}(\alpha)]_{r=n} \\ &\quad - (n-m) C_1 [u_r^{(n-m-1)}(\alpha) v_r'(\alpha)]_{r=n-1} \dots \\ &\quad - (n-m) C_{n-m-1} [u_r'(\alpha) v_r^{(n-m-1)}(\alpha)]_{r=m+1}] / [f(\alpha) [v_r^{(n-m)}(\alpha)]_{r=m}] \\ A_m \beta + B_m &= [P^{(n-m)}(\beta) - [u_r^{(n-m)}(\beta)]_{r=n} \\ &\quad - (n-m) C_1 [u_r^{(n-m-1)}(\beta) v_r'(\beta)]_{r=n-1} \dots \\ &\quad - (n-m) C_{n-m-1} [u_r'(\beta) v_r^{(n-m-1)}(\beta)]_{r=m+1}] / [f(\beta) [v_r^{(n-m)}(\beta)]_{r=m}] \end{aligned}$$

where $m = n, n-1, n-2, \dots, 2, 1$

$$\left. \begin{aligned} A_m \alpha + B_m &= \frac{M(\alpha)}{f(\alpha) [v_r^{(n-m)}(\alpha)]_{(r=m)}} \\ A_m \beta + B_m &= \frac{M(\beta)}{f(\beta) [v_r^{(n-m)}(\beta)]_{(r=m)}} \end{aligned} \right\} \quad (11)$$

where $m = n, n-1, n-2, \dots, 2, 1$.

$$M(x) = [P^{(n-m)}(x) - [u_r^{(n-m)}(x)]_{r=n} - (n-m) C_1 [u_r^{(n-m-1)}(x) v_r'(x)]_{r=n-1} \dots - (n-m) C_{n-m-1} [u_r'(x) v_r^{(n-m-1)}(x)]_{r=m+1}]$$

$\therefore$ From (11), we have,

$$\begin{aligned} A_m &= \frac{1}{\alpha - \beta} \left[\frac{M(\alpha)}{f(\alpha) [v_r^{(n-m)}(\alpha)]_{(r=m)}} - \frac{M(\beta)}{f(\beta) [v_r^{(n-m)}(\beta)]_{(r=m)}} \right] \\ &= \frac{1}{\alpha - \beta} \left[\frac{M(\alpha)}{f(\alpha) ((n-m)!) (2\alpha + p)^{n-m}} - \frac{M(\beta)}{f(\beta) ((n-m)!) (2\beta + p)^{n-m}} \right] \\ &= \frac{M(\alpha)}{f(\alpha) ((n-m)!) (2\alpha + p)^{n-m+1}} + \frac{M(\beta)}{f(\beta) ((n-m)!) (2\beta + p)^{n-m+1}} \\ &= (n-m+1) \left[\frac{M(\alpha)}{Q_{n-m+1}^{(n-m+1)}(\alpha)} + \frac{M(\beta)}{Q_{n-m+1}^{(n-m+1)}(\beta)} \right] \end{aligned} \quad (12)$$

$$\text{and } B_m = -(n-m+1) \left[\frac{\beta M(\alpha)}{Q_{n-m+1}^{(n-m+1)}(\alpha)} + \frac{\alpha M(\beta)}{Q_{n-m+1}^{(n-m+1)}(\beta)} \right],$$

$m = n-1, n-2, \dots, 2, 1$

Equations (9) and (12) give all the required coefficients.

2. Find the integral $\int \frac{dx}{(x^2+a^2)(x^2+b^2)}$.

Let $\frac{1}{(x^2+a^2)(x^2+b^2)} = \frac{A_1}{y+a^2} + \frac{A_2}{y+b^2}$ where $x^2=y$

$$\therefore A_1 = \frac{1}{(y+b^2)+(y+a^2)} \Big|_{y=-a^2} = \frac{1}{b^2-a^2} = -\frac{1}{a^2-b^2}$$

and $A_2 = \frac{1}{(y+b^2)+(y+a^2)} \Big|_{y=-b^2} = \frac{1}{a^2-b^2}$.

$$\begin{aligned} \therefore \int \frac{dx}{(x^2+a^2)(x^2+b^2)} &= \frac{1}{a^2-b^2} \int \left[\frac{1}{x^2+b^2} - \frac{1}{x^2+a^2} \right] dx + c \\ &= \frac{1}{a^2-b^2} \left[\frac{1}{b} \tan^{-1} \frac{x}{b} - \frac{1}{a} \tan^{-1} \frac{x}{a} \right] + c. \end{aligned}$$

3. Perform the integration $\int \frac{x dx}{(x^2+a^2)(x^2+b^2)}$.

Let $x^2=y$ or $2x dx = dy$

$$\therefore \int \frac{x dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{2} \int \frac{dy}{(y+a^2)(y+b^2)} + c.$$

Let $\frac{P(y)}{Q(y)} = \frac{1}{(y+a^2)(y+b^2)} = \frac{A_1}{y+a^2} + \frac{A_2}{y+b^2}$

$$\therefore A_1 = \frac{1}{(y+b^2)+(y+a^2)} \Big|_{y=-a^2} = \frac{1}{b^2-a^2} = -\frac{1}{a^2-b^2}$$

and $A_2 = \frac{1}{(y+b^2)+(y+a^2)} \Big|_{y=-b^2} = \frac{1}{a^2-b^2}$

$$\begin{aligned} \therefore \int \frac{dy}{(y+a^2)(y+b^2)} &= \frac{1}{a^2-b^2} \int \left[\frac{1}{y+b^2} - \frac{1}{y+a^2} \right] dy + c. \\ &= \frac{1}{a^2-b^2} \log \frac{y+b^2}{y+a^2} + c. \end{aligned}$$

$$\therefore \int \frac{x dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{2(a^2-b^2)} \log \frac{x^2+b^2}{x^2+a^2} + c.$$

N.B. Probs. like Ex. 2, 3. may be

$$\int \frac{x^2}{x^4+x^2-2} dx; \int \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)(x^2+c^2)} \text{ with}$$

Ans. $\frac{1}{3} \left[\sqrt{2} \tan^{-1} \frac{x}{\sqrt{2}} + \frac{1}{2} \log \frac{x-1}{x+1} \right]$ and

$\sum \frac{a}{(a^2 - b^2)(c^2 - a^2)} \tan^{-1} \frac{x}{a} + k$ and also 5, 6, 7, 8, 12, 13, 17 in Exercise 3 are of this type.

4. Find the integral of $\int \frac{8x^3 + 7}{(x+1)(2x+1)^3} dx$ corresponding to repeated factors.

Here the coeff. of x in the repeated factor $(2x+1)^3$ is not unity.

So, let us make it unity. Therefore, integrand $\frac{8x^3 + 7}{8(x+1)(x+\frac{1}{2})^3}$

Hence $F(x) = \frac{8x^3 + 7}{8(x+1)} = x^2 - x + 1 - \frac{1}{8(x+1)}$ and $\alpha = -\frac{1}{2}, n = 3$.

$\therefore F'(x) = 2x - 1 + \frac{1}{8(x+1)^2}$ and $F''(x) = 2 - \frac{2}{8(x+1)^3}$.

$\therefore$ The coeffs. of the PFS corresponding to the linear multiple factor is $A_r = \frac{F^{3-r}(-1/2)}{(3-r)!}, r = 1, 2, 3$

$\therefore A_1 = 0, A_2 = -\frac{3}{2}$ and $A_3 = \frac{3}{2}$. So the sum of the required

$$PFS = \frac{A_1}{x + \frac{1}{2}} + \frac{A_2}{\left(x + \frac{1}{2}\right)^2} + \frac{A_3}{\left(x + \frac{1}{2}\right)^3} = \frac{-3/2}{\left(x + \frac{1}{2}\right)^2} + \frac{3/2}{\left(x + \frac{1}{2}\right)^3}$$

$\therefore$ The required integral $= \int \left[-\frac{3/2}{\left(x + \frac{1}{2}\right)^2} + \frac{3/2}{\left(x + \frac{1}{2}\right)^3} \right] dx$

$$= \frac{3}{2x+1} - \frac{3}{(2x+1)^2} + C.$$

5. Find the integral of $\int \frac{x dx}{(1+x)(x^2+1)}$ corresponding to quadratic factor.

Here $Q(x) \equiv (1+x)(x^2+1), P(x) \equiv x$ and roots $\alpha = i, \beta = -i$.

Therefore, $A = \frac{P(\alpha)}{Q'(\alpha)} + \frac{P(\beta)}{Q'(\beta)} = \frac{i}{2(i-1)} + \frac{i}{2(i+1)} = \frac{1}{2}$

and $B = -\left[\frac{\alpha P(\beta)}{Q'(\beta)} + \frac{\beta P(\alpha)}{Q'(\alpha)} \right] = -\left[\frac{i-i}{-2(i+1)} + \frac{-i-i}{2(i-1)} \right] = \frac{1}{2}$

∴ *PF.* corresponding to the quadratic factor of the integrand

$$= \frac{Ax+B}{x^2+1} = \frac{1}{2} \frac{x+1}{x^2+1}$$

∴ The reqd. integral

$$\begin{aligned} &= \frac{1}{2} \int \frac{x+1}{x^2+1} dx = \frac{1}{2} \int \frac{x}{x^2+1} dx + \frac{1}{2} \int \frac{1}{x^2+1} dx \\ &= \frac{1}{4} \log(1+x^2) + \frac{1}{2} \tan^{-1} x + C. \end{aligned}$$

6. Compute $\int \frac{dx}{x(x^2+1)^3}$.

$$\text{Let } \frac{1}{x(x^2+1)^3} = \frac{A}{x} + \frac{A_1x+B_1}{x^2+1} + \frac{A_2x+B_2}{(x^2+1)^2} + \frac{A_3x+B_3}{(x^2+1)^3}.$$

Here $P(x) = 1$ and $Q_3(x) = x(x^2+1)^3$. Hence $A = P(0)/Q'(0) = 1$

Therefore $Q_1 = x(x^2+1)$ and $Q_1^{(1)}(x) = (x^2+1) + 2x^2$,

$$Q_2 = x(x^2+1)^2 \text{ and } Q_2^{(1)} = (x^2+1)^2 + 4x^2(x^2+1),$$

$$Q_2^{(2)} = 4x(x^2+1) + 8x(x^2+1) + 8x^3 \text{ etc.}$$

Also for quadratic parts, $f(x) = x, \alpha = i, \beta = -i, n = 3$.

$$\therefore A_3 = \left[-\frac{1}{2} + \frac{1}{-2} \right] = -1, B_3 = -\left[\frac{-i1}{-2} + \frac{i1}{-2} \right] = 0$$

[Using eqn. (9) sec. 3:3 (iv)]

$$\begin{aligned} \text{Also, } A_2 = (3-2+1) &\left[\frac{0 - (-1i+0)1 - (-1)i}{-8i} \right. \\ &\left. + \frac{0 - (-1-i+0)1 - (-1)(-i)}{8i} \right] \end{aligned}$$

$$= -1 \text{ where } M(x) = P'(x) - [(A_3x+B_3)f(x)]'$$

and $B_2 = 0$. Similarly, $A_1 = -1, B_1 = 0$ where

$$M(x) = P''(x) - [(A_3x+B_3)f(x)]''$$

$$- {}^2C_1 [(A_2x+B_2)f(x)]' [(x^2+1)]'_{r=2}.$$

$$\begin{aligned} \therefore \int \frac{dx}{x(x^2+1)^3} &= \int \left[\frac{1}{x} - \frac{x}{x^2+1} - \frac{x}{(x^2+1)^2} - \frac{x}{(x^2+1)^3} \right] dx \\ &= \log x - \frac{1}{2} \log(x^2+1) + \frac{1}{2(x^2+1)} + \frac{1}{4(x^2+1)^2} + C \\ &= \log \frac{x}{\sqrt{x^2+1}} + \frac{1}{2(x^2+1)} + \frac{1}{4(x^2+1)^2} + C \end{aligned}$$

Exercises 3

Perform integration of the following :

1. $\int \frac{1}{1+e^x} dx$

2. $\int \frac{x dx}{x^2+2x-3}$

3. $\int \frac{x^3}{(x+1)^3} dx$

4. $\int \frac{1}{(1-x^2)^2} dx$

5. $\int \frac{dx}{x^4-1}$

6. $\int \frac{dx}{x^4+5x^2+4}$

7. $\int \frac{x dx}{x^4-1}$

8. $\int \frac{3x^2-2}{x^4-3x^2-4} dx$

9. $\int \frac{x^2+x-1}{x^3+x^2-6x} dx$

10. $\int \frac{(2x^3+2x^2+4x+1)}{x^2+x+1} dx$

11. $\int \frac{3x^2-1}{x^2-3x+2} dx$

12. $\int \frac{x^2 dx}{x^4-x^2-12}$

13. $\int \frac{dx}{1-x^6}$

14. $\int \frac{x^7 dx}{x^{12}-1}$

15. $\int \frac{3x+1}{(x-1)^2(x+3)} dx$

16. $\int \frac{x^5 dx}{(a^2-x^2)^2}$

17. $\int \frac{x^3 dx}{(a+cx^2)^4}$

18. $\int \frac{x^5 dx}{(1+x^2)^3}$

19. $\int \frac{dx}{(1+x^2)(1-x)}$

20. $\int \frac{1}{x^4+1} dx$

21. $\int \frac{dx}{(\cos^2 x + 4 \sin x - 5) \cos x}$

22. $\int \frac{dx}{\sin x (2 + \cos x - 2 \sin x)}$

23. $\int \frac{dx}{\sin x (2 \cos^2 x - 1)}$

24. $\int \frac{dx}{\sin x + \sin 2x}$

25. $\int \frac{1 + \sin x}{\sin x (1 + \cos x)} dx$

26. $\int \frac{\sec x}{1 + \operatorname{cosec} x} dx$

27. $\int \frac{\sin^2 x \cos x}{\sin x + \cos x} dx$

28. Find the integral of Problem 13 corresponding to the quadratic parts only.

29. Verify problems 13, 17 after calculating separately the integrals corresponding to non-repeated and repeated parts of the PFs.
30. Verify problems 3, 6 after calculating separately the integrals corresponding to linear and quadratic parts of the PFs.
31. Find the integral corresponding to the quadratic part of the PF of integrand in prob. 10.
32. Work out probs. 8 (i), 10, 11 and examples 1, 2 of chapter 2 using partial fractions.

Answers

[Hints : Use $e^x = u$,] [Prob. 49 Exercise 2 (i)]

2. $\frac{3}{4} \log(x+3) + \frac{1}{4} \log(x-1) + C$
3. $\left[\text{Hint : Expand } \left(1 - \frac{1}{x+1}\right)^3, x - 3 \log(x+1) - \frac{3}{(x+1)} + \frac{1}{2(x+1)^2} + C \right]$
4. $\left[\text{Hints : Expand } \left(\frac{1}{2(x+1)} - \frac{1}{2(x-1)}\right)^2, -\frac{1}{4(x+1)} - \frac{1}{4(x-1)} - \frac{1}{4} \log \frac{x+1}{x-1} + C \right]$
5. $\frac{1}{4} \log \frac{x-1}{x+1} - \frac{1}{2} \tan^{-1} x + C$
6. $\frac{1}{3} \tan^{-1} x - \frac{1}{6} \tan^{-1} \frac{x}{2} + C$
7. $\frac{1}{4} \log \frac{x^2-1}{x^2+1} + C$
8. $\frac{1}{2} \log \frac{x-2}{x+2} + \tan^{-1} x + C$
9. $\frac{1}{6} \log x + \frac{1}{2} \log(x-2) + \frac{1}{3} \log(x+3) + C$
10. $x^2 + \log(x^2 + x + 1) + C$
11. $3x + 11 \log(x-2) - 2 \log(x-1) + C$
12. $\frac{\sqrt{3}}{7} \tan^{-1} \frac{x}{\sqrt{3}} + \frac{1}{7} \log \frac{x-2}{x+2} + C$
13. $\frac{1}{6} \log(1+x) - \frac{1}{6} \log(1-x) + \frac{1}{2\sqrt{3}} \tan^{-1} \left(\frac{2x-1}{\sqrt{3}}\right) + \frac{1}{12} \log \frac{1+x+x^2}{1-x+x^2} + \frac{1}{2\sqrt{3}} \tan^{-1} \left(\frac{2x+1}{\sqrt{3}}\right) + C$
14. Hints : Put $x^4 = u$.

$$\frac{1}{24} \log \frac{(x^4-1)^2}{x^8+x^4+1} + \frac{1}{4\sqrt{3}} \tan^{-1} \frac{2x^4+1}{\sqrt{3}} + C$$
15. $\frac{1}{2} \log \frac{x-1}{x+3} - \frac{1}{x-1} + C$
16. $\frac{a^4}{2(a^2-x^2)} + \frac{x^2}{2} + a^2 \log(a^2-x^2) + C$

$$17. -\frac{1}{4c^2(a+cx^2)^2} + \frac{a}{6c^2(a+cx^2)^3} + K$$

$$18. \frac{1}{x^2+1} - \frac{1}{4(x^2+1)^2} + \frac{1}{2} \log(x^2+1) + C$$

$$19. \frac{1}{8} \log \frac{1+x}{1-x} - \frac{1}{4(1+x)} - \frac{1}{4(1+x)^2} + C$$

$$20. x^4 + 1 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1), \text{ Ex. 2 (6).}$$

Prob. 8 (ii).

$$21. \text{Hints : Put } \sin x = u \text{ after multiplying both numerator and denominator by } \cos x.$$

$$\log \left[(1 - \sin x)^{1/2} (1 + \sin x)^{-1/18} (2 - \sin x)^{-4/9} \right] + \frac{1}{3(2 - \sin x)} + C$$

$$22. \text{Hints : Put } \tan \frac{x}{2} = u.$$

$$\frac{1}{3} \log \tan \frac{x}{2} + \frac{5}{3} \log \left(\tan \frac{x}{2} - 3 \right) - \log \left(\tan \frac{x}{2} - 1 \right) + C$$

$$23. \text{Hints : Put } \cos x = u.$$

$$\frac{1}{\sqrt{2}} \log \frac{1 + \sqrt{2} \cos x}{1 - \sqrt{2} \cos x} + \log \tan \frac{x}{2} + C$$

$$24. \text{Hints : Put } \cos x = u.$$

$$-\frac{1}{3} \log \frac{(1 + 2 \cos x)^2}{\sin x (1 + \cos x)} + C$$

$$25. \frac{1}{2} \log \tan \frac{x}{2} + \tan \frac{x}{2} + \frac{1}{4} \tan^2 \frac{x}{2} + C$$

$$26. \text{Hints : } \sin x = u.$$

$$\text{Ans. } \frac{1}{4} \left[\frac{2}{1 + \sin x} + \log \frac{1 + \sin x}{1 - \sin x} \right] + C$$

$$27. \text{Hints : Put } \tan x = u.$$

$$\frac{1}{4} \log (\sin x + \cos x) - \frac{1}{4} \cos x (\sin x + \cos x) + C$$

$$28. \frac{1}{2\sqrt{3}} \left(\tan^{-1} \frac{2x-1}{\sqrt{3}} + \tan^{-1} \frac{2x+1}{\sqrt{3}} \right) + \frac{1}{12} \log \frac{1+x+x^2}{1-x+x^2} + C$$

$$31. \frac{\sqrt{3}}{7} \tan^{-1} \frac{x}{\sqrt{3}} + C.$$

SUBSTITUTION (II)

4.1 INTRODUCTION

Integration of some rational functions have been discussed in chapter 2 using substitutions. Similarly, in this chapter integration of some standard irrational functions will be discussed using substitutions. These integrals are again considered standard integrals, because many functions can be put into these forms and can be easily integrated. These will be described in the following sections one after another.

4.2 INTEGRAL OF THE TYPES $\int \frac{1}{\sqrt{x^2 \pm a^2}} dx$ and

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx.$$

(i) Consider the integral $\int \frac{1}{\sqrt{x^2 + a^2}} dx$ first. Note that the integrand

is the reciprocal of the square root of the sum of two squares of which one is a constant and other is a variable. In this case substitution of $x = a \sinh \theta$ or $x = a \tan \theta$ removes the surd. Let us illustrate the first substitution. Therefore, $dx = a \cosh \theta d\theta$. So,

$$\begin{aligned} \int \frac{1}{\sqrt{x^2 + a^2}} dx &= \int \frac{a \cosh \theta}{a \sqrt{1 + \sinh^2 \theta}} d\theta \\ &= \int d\theta = \theta + c = \sinh^{-1} \frac{x}{a} + c. \end{aligned}$$

If $x = a \tan \theta$ is used then $\int \frac{1}{\sqrt{x^2 + a^2}} dx = \log (x + \sqrt{x^2 + a^2}) + c.$

But from sec. 1.6 $\sinh^{-1}(x/a) = \log (x + \sqrt{x^2 + a^2}) - \log a$ Merging $(-\log a)$ with the arbitrary constant, it is seen that all the substitutions give the unique integral.

A third and most tricky substitution is $\sqrt{x^2 + a^2} = u - x$. so,

$$\frac{du}{u} = \frac{dx}{\sqrt{x^2 + a^2}}.$$

Therefore,
$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \int \frac{du}{u} = \log u + c$$

$$= \log (x + \sqrt{x^2 + a^2}) + c.$$

(ii) Consider the integral $\int \frac{1}{\sqrt{x^2 - a^2}} dx$. The integrand here is the reciprocal of the square root of the difference of the square of a constant from that of a variable. Here the fruitful substitution is $x = a \cosh \theta$ or $x = a \sec \theta$ which removes the surd. Therefore, using $x = a \cosh \theta$, $dx = a \sinh \theta d\theta$,

$$\begin{aligned} \int \frac{1}{\sqrt{x^2 - a^2}} dx &= \int \frac{a \sinh \theta}{a \sqrt{\cosh^2 \theta - 1}} d\theta \\ &= \int d\theta = \theta + c = \cosh^{-1} \frac{x}{a} + c. \end{aligned}$$

Similarly, $x = a \sec \theta$ gives $\int \frac{1}{\sqrt{x^2 - a^2}} dx = \log (x + \sqrt{x^2 - a^2}) + c$

and the substitution $\sqrt{x^2 - a^2} = u - x$, gives the same result.

As in (i), $\int \frac{1}{\sqrt{x^2 - a^2}} dx = \cosh^{-1} \frac{x}{a} + c = \log (x + \sqrt{x^2 - a^2}) + c$ is unique.

(iii) Consider the integral $\int \frac{dx}{\sqrt{a^2 - x^2}}$. The integrand is the reciprocal of the difference of the square of a variable from that of a constant. Just opposite to the form of (ii) above. Here the only substitution is $x = a \sin \theta$ which removes the surd. Therefore, $dx = a \cos \theta d\theta$ and

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{a \cos \theta d\theta}{\sqrt{a^2 (1 - \sin^2 \theta)}} = \int d\theta = \theta + c = \sin^{-1} \frac{x}{a} + c.$$

4.3 INTEGRAL OF THE TYPE $\int \frac{dx}{\sqrt{ax^2 + bx + c}}$, $a, b, c = \text{const.}$

Assume that the general quadratic $ax^2 + bx + c$ cannot be factored into real factors. But this expression can be written as a sum or difference of two squares as in 4.2 and the three possibilities can be handled as in 4.2. It is advisable to make the coefficient of x^2 unity by taking 'a' outside the radical.

4.4 INTEGRAL OF THE TYPES $\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$

$$\text{and } \int \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}$$

These two integrals can be integrated using substitution $x-\alpha=u^2$ which reduces the integral to type 4.2. Therefore, $x=u^2+\alpha$ and $dx=2udu$.

$$\begin{aligned}\therefore \int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}} &= \int \frac{2udu}{u\sqrt{u^2+(\alpha-\beta)}} \\ &= 2 \log(u + \sqrt{u^2 + \alpha - \beta}) + c \\ &= 2 \log(\sqrt{x-\alpha} + \sqrt{x-\beta}) + c\end{aligned}$$

$$\begin{aligned}\text{and } \int \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}} &= \int \frac{2du}{\sqrt{(\beta-\alpha)-u^2}} \\ &= 2 \sin^{-1} \frac{u}{\sqrt{\beta-\alpha}} + c = 2 \sin^{-1} \sqrt{\frac{x-\alpha}{\beta-\alpha}} + c.\end{aligned}$$

4.5 INTEGRAL OF THE TYPE $\int \frac{dx}{(ax+b)\sqrt{cx+d}}$

To remove the surd, substitute $cx+d=u^2$ and $dx = \frac{2u}{c} du$.

$$\text{Therefore, } \int \frac{dx}{(ax+b)\sqrt{cx+d}} = \frac{2}{a} \int \frac{du}{u^2 + \left(\frac{bc-ad}{a}\right)} \quad \text{which is of the}$$

types in sec. 2.17.

4.6 INTEGRAL OF THE TYPE $\int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$

The numerator $(px+q)$ of the integrand is to be expressed as the sum of the differential coeffs. of the denominator and a constant, when the quadratic has no real zeros, i.e.

$$\begin{aligned}\int \frac{px+q}{\sqrt{ax^2+bx+c}} dx &= \int \frac{\frac{p}{2a} \frac{d}{dx}(ax^2+bx+c) + \left(q - \frac{pb}{2a}\right)}{\sqrt{ax^2+bx+c}} dx \\ &= \frac{p}{a} \sqrt{ax^2+bx+c} + \frac{2qa-pb}{2a} \int \frac{dx}{\sqrt{ax^2+bx+c}}\end{aligned} \quad (i)$$

The integral on the right can be integrated using 4.3.

Alternatively.

$$\text{Let } \int \frac{px+q}{\sqrt{ax^2+bx+c}} dx = l\sqrt{ax^2+bx+c} + m \int \frac{dx}{\sqrt{ax^2+bx+c}} \quad (\text{ii})$$

Differentiating both sides w.r.t. x , we get,

$$\frac{px+q}{\sqrt{ax^2+bx+c}} = l \cdot \frac{2ax+b}{2\sqrt{ax^2+bx+c}} + m \cdot \frac{1}{\sqrt{ax^2+bx+c}}$$

Now multiplying throughout by $\sqrt{ax^2+bx+c}$,

$$\text{we get } px+q = \frac{l}{2}(2ax+b) + m.$$

Equating coefficients of x 's and constants and solving the simultaneous equations, $l = \frac{p}{a}$ and $m = q - \frac{pb}{2a}$. With these values of l and m , (ii) becomes (i) and hence use 4.3.

4.7 INTEGRAL OF THE TYPE $\int \frac{\sum_{r=0}^n a_r x^r}{\sqrt{ax^2+bx+c}} dx$.

• Alternative procedures of 4.6 is applicable here.

$$\begin{aligned} \text{Let } & \int \frac{\sum_{r=0}^n a_r x^r}{\sqrt{ax^2+bx+c}} dx \\ &= \left(\sum_{r=0}^{n-1} A_{n-r-1} x^r \right) \sqrt{ax^2+bx+c} + A_n \int \frac{dx}{\sqrt{ax^2+bx+c}} \end{aligned}$$

Differentiating and equating coeffs. of x 's, it is possible to know the constants, A 's as before and use of 4.3 gives the integral.

4.8 INTEGRAL OF THE TYPE $\int \frac{dx}{(px+q)^r \sqrt{ax^2+bx+c}}$, $r = 1, 2, 3, \dots, \infty$.

Here the standard substitution is $px+q = \frac{1}{u}$.

So, $x = \frac{1}{p} \left(\frac{1}{u} - q \right)$ and $dx = -\frac{1}{u^2 p} du$. Hence the given integral is

$$-\int \frac{u^{r-1} du}{\sqrt{(aq^2 - pbq + cp^2) u^2 - (2aq - pb) u + a}}$$

The integral can be obtained using 4.3.

4.9 INTEGRAL OF THE TYPE $\int \frac{dx}{(ax^2 + b)\sqrt{cx^2 + e}}$

The substitution $x = \frac{1}{u}$ transforms the integral to $\int \frac{-udu}{(a + bu^2)\sqrt{c + eu^2}}$.

To get rid of the surd, we next substitute $c + eu^2 = p^2$ and the integral again transforms to $-\int \frac{dp}{bp^2 + (ae - bc)}$. Now use of 2.17 integrates the

integrand. Alternative substitution is $\frac{cx^2 + e}{ax^2 + b} = u^2$, See Ex. 9.

4.10 INTEGRAL OF THE TYPES $\int \frac{dx}{x(a + bx^n)}$

$$\text{and } \int \frac{dx}{x\sqrt{a + bx^n}}.$$

Both the integrals can be integrated by putting $x^n = \frac{1}{u^2}$, so that

$$\frac{dx}{x} = -2 \frac{du}{nu}. \text{ Therefore,}$$

$$\begin{aligned} \int \frac{dx}{x(a + bx^n)} &= \frac{-2}{n} \int \frac{u du}{au^2 + b} = -\frac{1}{na} \log (au^2 + b) + C \\ &= \frac{1}{na} \log \frac{x^n}{a + bx^n} + C. \end{aligned}$$

$$\text{and } \int \frac{dx}{x\sqrt{a + bx^n}} = \frac{-2}{n} \int \frac{du}{\sqrt{au^2 + b}} \text{ which can be easily integrated}$$

using 4.2.

4.11 INTEGRAL OF THE TYPE $\int \frac{dx}{(a + cx^2)^{3/2}}$

This can be integrated by using two substitutions. First substitute $x = \frac{1}{u}$ to transform it to $\int \frac{dx}{(a + cx^2)^{3/2}} = -\int \frac{udu}{(au^2 + c)^{3/2}}$ and then use $au^2 + c = p$.

$$\text{So, } \int \frac{dx}{(a + cx^2)^{3/2}} = \frac{1}{a(au^2 + c)^{1/2}} + C = \frac{x}{a(a + cx^2)^{1/2}} + C.$$

4.12 INTEGRAL OF THE TYPE $\int \frac{dx}{(ax^2 + bx + c)\sqrt{(ex + f)}}$

Substitution of $ex + f = u^2$ changes the integral to

$$\int \frac{1}{a\left(\frac{u^2-f}{e}\right)^2 + b\left(\frac{u^2-f}{e}\right) + c} \cdot \frac{1}{u} \cdot \frac{2u}{e} du$$

$$= 2e \int \frac{du}{au^4 - (2af - be)u^2 + (af^2 - bef + ce^2)}$$

It can be integrated after factorization and decomposition into partial fractions.

4.13 INTEGRAL OF THE TYPE $\int \frac{dx}{(Ax^2 + Bx + C)\sqrt{ax^2 + bx + c}}$

Here both the expressions inside and outside the radical are quadratic. Assume that they have no real zeros. In that case the problem reduces to simpler types 4.4, 4.5. Also assume that the expressions have uncommon complex zeros.

In such case substitute $x = \frac{pt + q}{t + 1}$ or $dx = \frac{p - q}{(t + 1)^2} dt$

where p and q are two distinct real roots of

$$(Ab - Ba)m^2 - 2(Ca - Ac)m + (Bc - Cb) = 0$$

[For proof of this statement, see p 39 Integral Calculus, Ferrar, 1958, Oxford]. Then the integral takes the form

$$\int \frac{dx}{(Ax^2 + Bx + C)\sqrt{ax^2 + bx + c}} = \int \frac{(t + 1)(p - q)}{(Pt^2 + Qt + R)\sqrt{Ut^2 + Vt + W}} dt$$

where

$$P = Ap^2 + Bp + C, Q = 2Apq + B(p + q) + 2C, R = Aq^2 + Bq + C,$$

$$U = ap^2 + bp + c, V = 2apq + b(p + q) + 2c, W = aq^2 + bq + c.$$

Next substitute, $Ut^2 + Vt + W = u^2$ to remove radical. Now the integral reduces to 2.17 and can be integrated easily. This procedure is also

applicable to the type $\int \frac{(px + q) dx}{(Ax^2 + Bx + C)\sqrt{ax^2 + bx + c}}$ if the quadratics have no real zeros.

N.B. (i). If $ax^2 + bx + c \neq 0$ has real distinct roots say m_1, m_2 i.e. $ax^2 + bx + c = a(x - m_1)(x - m_2)$ then substitution of $x - m_1$ or $x - m_2 = u$

reduces the integral to 4.9 with the PF of the resulting expression for $\frac{1}{Ax^2+Bx+C}$ in terms of u . Example is 61 in exercise.

(ii) Substitution $u = \sqrt{ax} + \sqrt{ax^2+bx+c}$ may be sometime useful if the resulting fourth degree expression in the denominator can be easily factorized. Example is 57 in exercise.

(iii) The result will differ in form from that of the original procedure of 4.13 if (i) or (ii) is used instead.

(iv) If $Ax^2+Bx+C=0$ has real distinct roots say M_1, M_2 i.e.

$Ax^2+Bx+C \equiv A(x-M_1)(x-M_2)$ then the PF of $\frac{1}{Ax^2+Bx+C}$ reduces the integral to the form 4.8 with $r=1$. Note example 56 in the exercise.

4.14 INTEGRAL OF THE TYPE $\int \frac{f(x) dx}{\sqrt{(ax^4+2bx^3+cx^2+2bx+a)}}$,

WHERE $f(x)$ IS A RATIONAL FUNCTION OF x .

The denominator can be written as

$$x \sqrt{\left\{ a \left(x^2 + \frac{1}{x^2} \right) + 2b \left(x + \frac{1}{x} \right) + c \right\}}$$

and the numerator can be expressible in the form

$\left(x - \frac{1}{x} \right) \phi \left(x + \frac{1}{x} \right)$ or $\left(x + \frac{1}{x} \right) \phi \left(x - \frac{1}{x} \right)$. Then substitution $u = \left(x + \frac{1}{x} \right)$ or $\left(x - \frac{1}{x} \right)$ will reduce the integral to the form 4.8.

Examples

1. Integrate $\frac{1}{(2x^2-2x+1)\sqrt{3x^2-2x+1}}$.

This is of type 4.13.

Here the quadratic expressions have no real zeros and even no common imaginary zeros. Let us substitute $x = \frac{p+t}{t+1}$ or $dx = \frac{p-q}{(t+1)^2} dt$ where p and q are two distinct real roots of

$$(2 \cdot -1 - 3 \cdot -1) m^2 - (1 \cdot 3 - 2 \cdot 1) m + (-1 \cdot 1 - 1 \cdot -1) = 0$$

i.e. $m^2 - m = 0$ or $m = 0, 1$. Let $p = 1, q = 0$.

$$\therefore x = \frac{t}{t+1} \quad \text{or} \quad dx = \frac{1}{(t+1)^2} dt.$$

$$\text{Now, } I = \int \frac{dx}{(2x^2-2x+1)\sqrt{3x^2-2x+1}} = \int \frac{t+1}{(t^2+1)\sqrt{2t^2+1}} dt + c$$

$$\begin{aligned}
 &= \int \frac{t}{(t^2+1)\sqrt{2t^2+1}} dt + \int \frac{1}{(t^2+1)\sqrt{2t^2+1}} dt + c \\
 &= I_1 + I_2 \text{ (say)} + c
 \end{aligned}$$

For I_1 , let $2t^2+1=u^2$ or $t dt = \frac{1}{2} u du$

$$\begin{aligned}
 \therefore I_1 &= \frac{1}{2} \int \frac{u du}{\left(\frac{u^2-1}{2} + 1\right) \sqrt{u^2}} + k_1 \\
 &= \int \frac{du}{u^2+1} + k_1 = \tan^{-1} u + k_1 \\
 &= \tan^{-1} \left(\frac{\sqrt{3x^2-2x+1}}{1-x} \right) + k_1
 \end{aligned}$$

And I_2 is of type 4.9. So, let $\frac{2t^2+1}{t^2+1} = y^2$ or $t^2 = -\frac{1-y^2}{2-y^2}$

$$\text{or } t dt = \frac{y}{(2-y^2)^2} dy \quad \text{or } dt = \frac{y dy}{(y^2-2)^{3/2} (y^2-1)^{1/2}}$$

$$\begin{aligned}
 \therefore I_2 &= \int \frac{(2-y^2)\sqrt{2-y^2}}{\sqrt{y^2}} \cdot \sqrt{\frac{2-y^2}{y^2-1}} \cdot \frac{y}{(2-y^2)^2} dy + k_2 \\
 &= \int \frac{1}{\sqrt{y^2-1}} dy + k_2 = \cosh^{-1} y + k_2
 \end{aligned}$$

$$= \cosh^{-1} \sqrt{\frac{3x^2-2x+1}{2x^2-2x+1}} + k_2$$

$$\therefore I = \tan^{-1} \left(\frac{\sqrt{3x^2-2x+1}}{1-x} \right) + \cosh^{-1} \sqrt{\frac{3x^2-2x+1}{2x^2-2x+1}} + c$$

$$= \sin^{-1} \sqrt{\frac{3x^2-2x+1}{2(2x^2-2x+1)}} + \cosh^{-1} \sqrt{\frac{3x^2-2x+1}{2x^2-2x+1}} + c$$

Q.P. 2. Compute $\int \frac{dx}{\sqrt{1-x+x^2}}$
This is of the type 4.3. So,

$$\int \frac{dx}{\sqrt{1-x+x^2}} = \int \frac{dx}{\sqrt{\left(x-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}} + c \text{ which is of type 4.2 (i).}$$

$$\text{Let } x - \frac{1}{2} = \frac{\sqrt{3}}{2} \sinh \theta. \therefore dx = \frac{\sqrt{3}}{2} \cosh \theta d\theta$$

$$\begin{aligned} \therefore \int \frac{dx}{\sqrt{1-x+x^2}} &= \int \frac{\frac{\sqrt{3}}{2} \cosh \theta}{\frac{\sqrt{3}}{2} \sqrt{1 + \sinh^2 \theta}} d\theta + c = \int d\theta + c = \theta + c \\ &= \sinh^{-1} \frac{x - \frac{1}{2}}{\frac{\sqrt{3}}{2}} + c = \sinh^{-1} \frac{2x - 1}{\sqrt{3}} + c \end{aligned}$$

3. Find the integral $\int \frac{e^x dx}{\sqrt{(5e^{2x} - 12e^x + 4)}}$.

To identify the type let $e^x = u$ and $e^x dx = du$.

$$\text{So } \int \frac{e^x dx}{\sqrt{(5e^{2x} - 12e^x + 4)}}$$

$$= \int \frac{du}{\sqrt{(5u^2 - 12u + 4)}} + c \text{ which is of type 4.3}$$

$$= \frac{1}{\sqrt{5}} \int \frac{du}{\sqrt{\left(u^2 - \frac{12}{5}u + \frac{4}{5}\right)}} + c$$

$$= \frac{1}{\sqrt{5}} \int \frac{du}{\sqrt{\left\{\left(u - \frac{6}{5}\right)^2 - \left(\frac{4}{5}\right)^2\right\}}} \text{ which is of type 4.2. (ii)}$$

$$\text{So, let } u - \frac{6}{5} = \frac{4}{5} \cosh \theta \text{ or, } du = \frac{4}{5} \sinh \theta d\theta$$

$$\therefore \int \frac{e^x dx}{\sqrt{(5e^{2x} - 12e^x + 4)}} = \frac{1}{\sqrt{5}} \int \frac{\frac{4}{5} \sinh \theta d\theta}{\sqrt{\left(\frac{4}{5}\right)^2 [\cosh^2 \theta - 1]}} + c$$

$$\begin{aligned}
 &= \frac{1}{\sqrt{5}} \int d\theta + c \\
 &= \frac{1}{\sqrt{5}} \theta + c = \frac{1}{\sqrt{5}} \cosh^{-1} \frac{u-6/5}{4/5} + c \\
 &= \frac{1}{\sqrt{5}} \cosh^{-1} \frac{5e^x - 6}{4} + c
 \end{aligned}$$

4. Integrate $\frac{1}{\sqrt{(x-4)(6-x)}}$.

This is of type 4.4. So, letting $6-x=u^2$ or $dx = -2u du$.

Therefore,
$$\int \frac{dx}{\sqrt{(x-4)(6-x)}} = \int \frac{-2u du}{\sqrt{(6-u^2-4)u^2}} + c$$

$$= -2 \int \frac{du}{\sqrt{2-u^2}} + c \text{ which is of 4.2 (iii)}$$

Again, let $u = \sqrt{2} \sin \theta$ or $du = \sqrt{2} \cos \theta d\theta$. So,

$$\int \frac{dx}{\sqrt{(x-4)(6-x)}} = -2 \int \frac{\sqrt{2} \cos \theta d\theta}{\sqrt{2} \sqrt{(1-\sin^2 \theta)}} + c$$

$$= -2 \int d\theta + c = -2\theta + c$$

$$= -2 \sin^{-1} \frac{u}{\sqrt{2}} + c = -2 \sin^{-1} \left(\frac{\sqrt{6-x}}{\sqrt{2}} \right) + c$$

Note: If, instead of $(6-x)$, the substitution was made for $(x-4)$,

then the result will be $\left(2 \sin^{-1} \sqrt{\frac{x-4}{2}} + c \right)$

5. Compute $\int \frac{x+3}{\sqrt{4x^2-4x+3}} dx$

This is of type 4.6. So,

$$\int \frac{x+3}{\sqrt{4x^2-4x+3}} dx = \int \frac{\frac{1}{8} \frac{d}{dx} (4x^2-4x+3) + \frac{7}{2}}{\sqrt{4x^2-4x+3}} dx + c$$

$$= \frac{1}{8} \int \frac{\frac{d}{dx} (4x^2-4x+3)}{\sqrt{4x^2-4x+3}} + \frac{7}{2} \int \frac{1}{\sqrt{4x^2-4x+3}} dx + c$$

$$= \frac{1}{4} \sqrt{4x^2-4x+3} + \frac{7}{4} \int \frac{dx}{\sqrt{x^2-x+3/4}} + c.$$

$$\begin{aligned}
&= \frac{1}{4} \sqrt{4x^2 - 4x + 3} + \frac{7}{4} \int \frac{dx}{\sqrt{\left(x - \frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}} + c \\
&= \frac{1}{4} \sqrt{4x^2 - 4x + 3} + \frac{7}{4} \sinh^{-1} \left(\frac{x - \frac{1}{2}}{\frac{1}{\sqrt{2}}} \right) + c \quad \text{by 4.2 (i)} \\
&= \frac{1}{4} \sqrt{4x^2 - 4x + 3} + \frac{7}{4} \sinh^{-1} \sqrt{2} (2x - 1) + c
\end{aligned}$$

6. Compute $\int \frac{dx}{(x+3)\sqrt{x+2}}$.

This is of type 4.5. So, letting $x+2=u^2$ or $dx=2u du$. Then,

$$\begin{aligned}
\int \frac{dx}{(x+3)\sqrt{x+2}} &= \int \frac{2u du}{(u^2-2+3)u} + c = 2 \int \frac{du}{u^2+1} + c \\
&= 2 \tan^{-1} u + c \quad \text{by 2.17.} \\
&= 2 \tan^{-1} \sqrt{x+2} + c.
\end{aligned}$$

7. Integrate $\frac{1}{(1+x)\sqrt{1+x+x^2}}$

This is of type 4.8 with $r=1$. So, putting $x+1=\frac{1}{u}$ or, $dx=-\frac{1}{u^2} du$.

Therefore

$$\begin{aligned}
\int \frac{dx}{(1+x)\sqrt{1+x+x^2}} &= \int \frac{-\frac{1}{u^2} du}{\frac{1}{u} \sqrt{1+\frac{1}{u}-1+\left(\frac{1}{u}-1\right)^2}} + c \\
&= - \int \frac{du}{\sqrt{u^2+1-u^2+(1-u)^2}} + c = \int \frac{-du}{\sqrt{(1-u)^2+1}} + c \\
&= \sinh^{-1}(1-u) + c \quad \text{by 4.2 (i)} \\
&= \sinh^{-1} \left(1 - \frac{1}{x+1} \right) + c = \sinh^{-1} \left(\frac{x}{x+1} \right) + c
\end{aligned}$$

8. Integrate $\frac{1}{(1+x)^3 \sqrt{1+x+x^2}}$.

This is of type 4.8 with $r=3$. So, putting $x+1=\frac{1}{u}$ or, $dx=-\frac{1}{u^2} du$.

Therefore, let

$$\begin{aligned} I &= \int \frac{dx}{(1+x)^3 \sqrt{1+x+x^2}} = \int \frac{-\frac{1}{u^2} du}{\frac{1}{u^3} \sqrt{1+\frac{1}{u}-1+\left(\frac{1}{u}-1\right)^2}} \\ &= - \int \frac{u^2 du}{\sqrt{(1+u)^2+u}} + c = - \int \frac{u^2 du}{\sqrt{u^2-u+1}} + c \end{aligned}$$

which is of type 4.7 with $n=2$.

$$= - \int \frac{(u^2-u+1)+(u-1)}{\sqrt{u^2-u+1}} du + c$$

$$= - \int \sqrt{u^2-u+1} du - \int \frac{(u-1) du}{\sqrt{u^2-u+1}} + c.$$

$= -I_1 - I_2 + c$ in which I_1 is w.o. 1 of sec. 5.6 & I_2 is of type 4.6.

$$\text{So } I_1 = -\frac{2u-1}{4} \sqrt{u^2-u+1} - \frac{3}{8} \sinh^{-1} \frac{2u-1}{\sqrt{3}}.$$

$$\begin{aligned} \text{and } I_2 &= \frac{\frac{1}{2}(2u-1) - \frac{1}{2}}{\sqrt{u^2-u+1}} du = \sqrt{u^2-u+1} - \frac{1}{2} \int \frac{dx}{\sqrt{u^2-u+1}} \\ &= \sqrt{u^2-u+1} - \frac{1}{2} \sinh^{-1} \frac{2u-1}{\sqrt{3}} \text{ by w.o. 2 above.} \end{aligned}$$

$$\begin{aligned} \therefore I &= - \left(\frac{2u-1}{4} + 1 \right) \sqrt{u^2-u+1} + \left(\frac{1}{2} - \frac{3}{8} \right) \sinh^{-1} \frac{2u-1}{\sqrt{3}} \\ &= -\frac{2u+3}{4} \sqrt{u^2-u+1} + \frac{1}{8} \sinh^{-1} \frac{2u-1}{\sqrt{3}} \\ &= -\frac{3x+5}{4(x+1)^2} \sqrt{x^2+x+1} + \frac{1}{8} \sinh^{-1} \frac{1-x}{\sqrt{3}(1+x)}. \end{aligned}$$

9. Integrate $\frac{u^2}{\sqrt{u^2-2u+2}}$

$$\begin{aligned} \text{Let } I &= \int \frac{u^2 du}{\sqrt{u^2-2u+2}} \\ &= \left(\sum_{r=0}^{2-1} A_{2-r-1} u^r \right) \sqrt{u^2-2u+2} + A_2 \int \frac{du}{\sqrt{u^2-2u+2}} + k \end{aligned}$$

Differentiating w.r.t. $u, \frac{u^2}{\sqrt{u^2-2u+2}}$

$$= \frac{(A_1 + A_0 u)(u-1)}{\sqrt{u^2-2u+2}} + A_0 \sqrt{u^2-2u+2} + A_2 \frac{1}{\sqrt{u^2-2u+2}}$$

or $u^2 = (A_1 + A_0 u)(u-1) + A_0(u^2-2u+2) + A_2$.

Equating like powers of u , we get,

$$1 = 2A_0, \quad 0 = A_1 - 3A_0, \quad 0 = A_2 - A_1 + 2A_0$$

i.e. $A_0 = 1/2, \quad A_1 = 3/2, \quad A_2 = 1/2$

$\therefore$ (I) becomes,

$$\begin{aligned} \int \frac{u^2 du}{\sqrt{u^2-2u+2}} &= \frac{1}{2}(3+u)\sqrt{u^2-2u+2} + \frac{1}{2} \int \frac{du}{\sqrt{u^2-2u+2}} + k \\ &= \frac{1}{2}(3+u)\sqrt{u^2-2u+2} + \frac{1}{2} \int \frac{du}{\sqrt{(u-1)^2+1}} + k \\ &= \frac{1}{2}(3+u)\sqrt{u^2-2u+2} + \frac{1}{2} \sinh^{-1}(u-1) + k. \end{aligned}$$

9. Find $\int \frac{dx}{(x^2+1)\sqrt{x^2+5}}$.

This is of type 4.9.

Let $\frac{x^2+5}{x^2+1} = u^2$ or $x^2 = -\frac{u^2-5}{u^2-1}$ or $x dx = -\frac{4u}{(u^2-1)^2} du$

$\therefore \int \frac{dx}{(x^2+1)\sqrt{x^2+5}}$

$$= - \int \frac{1}{\left(1 - \frac{u^2-5}{u^2-1}\right) \sqrt{5 - \frac{u^2-5}{u^2-1}}} \cdot \frac{4u}{(u^2-1)^2} \cdot \sqrt{\frac{u^2-1}{5-u^2}} du + c$$

$$\begin{aligned}
 &= -\frac{1}{2} \int \frac{1}{\sqrt{5-u^2}} du + c = -\frac{1}{2} \sin^{-1} \frac{u}{\sqrt{5}} + c \\
 &= -\frac{1}{2} \sin^{-1} \left(\sqrt{\frac{x^2+5}{5(x^2+1)}} \right) + c
 \end{aligned}$$

* N.B. In expressing x in terms of u under type 4.9, always make x^2 +ve by subtracting the variable u^2 from the larger const. & take the square root.

Alternatively.

Let $x = \frac{1}{u}$ or $dx = -\frac{1}{u^2} du$. Therefore, let

$$\begin{aligned}
 I &= \int \frac{1}{(x^2+1)\sqrt{x^2+5}} = - \int \frac{1}{\left(\frac{1}{u^2}+1\right)\sqrt{\frac{1}{u^2}+5}} \cdot \frac{du}{u^2} + c \\
 &= - \int \frac{u}{(1+u^2)\sqrt{5u^2+1}} du + c.
 \end{aligned}$$

Again let $5u^2+1=p^2$ or $u du = \frac{p}{5} dp$.

$$\begin{aligned}
 \therefore I &= -\frac{1}{5} \int \frac{p dp}{\left(1+\frac{p-1}{5}\right) \cdot p} + c = - \int \frac{1}{p^2+4} dp + c \\
 &= -\frac{1}{2} \tan^{-1} \frac{p}{2} + c = -\frac{1}{2} \tan^{-1} \frac{\sqrt{5u^2+1}}{2} + c \\
 &= -\frac{1}{2} \tan^{-1} \frac{\sqrt{5+x^2}}{2x} + c = -\frac{1}{2} \sin^{-1} \left(\sqrt{\frac{x^2+5}{5(x^2+1)}} \right) + c
 \end{aligned}$$

Exercises 4

Compute the following

Ex 1. $\int \frac{dx}{\sqrt{1-25x^2}}$

Ex 2. $\int \frac{dx}{\sqrt{9x^2+16}}$

Ex 3. $\int \frac{dx}{\sqrt{9x^2-16}}$

4. $\int \frac{5x}{\sqrt{1-4x^2}} dx$

5. $\int \frac{5x}{\sqrt{4x^2+1}} dx$

6. $\int \frac{e^2 dx}{\sqrt{(a^2+b^2-4e^{2x})}}$

7. $\int \frac{e^x dx}{\sqrt{a^2 - b^2 + 4e^{2x}}}$

8. $\int \frac{\sin \theta d\theta}{\sqrt{4 - 9 \cos^2 \theta}}$

9. $\int \frac{e^x dx}{\sqrt{4e^{2x} - a^2 + b^2}}$

10. $\int \frac{dx}{\sqrt{2ax + x^2 + b^2}}$

11. $\int \frac{d\theta}{\cos^2 \theta \sqrt{4 \tan^2 \theta + 4 \tan \theta - 1}}$

12. $\int \frac{dt}{\sqrt{1-t^2} \sqrt{3 - 2 \sin^{-1} t + (\sin^{-1} t)^2}}$

13. $\int \frac{dx}{\sqrt{9x^2 + 12x + 8}}$

14. $\int \frac{dx}{\sqrt{1-x-x^2}}$

Q 15. $\int \frac{dx}{\sqrt{2-3x-4x^2}}$

Q 16. $\int \frac{e^x}{\sqrt{3+3e^x+e^{2x}}} dx$

17. $\int \frac{dt}{(1+t^2) \sqrt{(\tan^{-1} t)^2 + \tan^{-1} t - 2}}$

18. $\int \frac{dx}{\sqrt{(x-1)(x-2)}}$

19. $\int \frac{dx}{\sqrt{x(x-a)}}$

20. $\int \frac{dx}{\sqrt{2ax - a^2}}$

Q 21. $\int \frac{dx}{\sqrt{4x^2 - 12x + 7}}$

Q 22. $\int \frac{dx}{\sqrt{8+6x-9x^2}}$

Q 23. $\int \frac{x dx}{\sqrt{1-x^2}}$

Q 24. $\int \frac{x+3}{\sqrt{x^2+4}} dx$

Q 25. $\int \frac{x+2}{\sqrt{6x-x^2}} dx$

Q 26. $\int \frac{2x+1}{\sqrt{4-x^2}} dx$

Q 27. $\int \frac{5t-1}{\sqrt{3t^2-9}} dt$

28. $\int \sqrt{\frac{2+x}{2-x}} dx$

29. $\int \sqrt{\frac{2x+1}{3x+2}} dx$

$$30. \int \frac{x+1}{\sqrt{x^2-3x+5}} dx$$

$$32. \int \frac{2x+3}{\sqrt{4x^2+2x+1}} dx$$

$$34. \int \frac{dt}{(3t-1)\sqrt{2t-3}}$$

$$36. \int \frac{dx}{x\sqrt{x-2}}$$

$$38. \int \frac{dx}{(x-a)\sqrt{(x-a)(b-x)}}$$

$$40. \int \frac{dx}{x\sqrt{9x^2-a^2}}$$

$$42. \int \frac{dx}{x\sqrt{1+x^3}}$$

$$44. \int \frac{dx}{x\sqrt{2+x-x^2}}$$

$$46. \int \frac{x^4}{\sqrt{x^2+x+1}} dx$$

$$48. \frac{dx}{(x^2-1)\sqrt{x^2+1}}$$

$$50. \int \frac{x+2}{(x^2+3x+3)\sqrt{x+2}} dx$$

$$52. \int \frac{dx}{x\sqrt{x^n+1}}$$

$$54. \int \frac{dx}{(x^2+x+1)\sqrt{x^2-x+1}}$$

$$31. \int \frac{3x-2}{\sqrt{3+2x-4x^2}} dx$$

$$33. \int \frac{dx}{(1-x)\sqrt{1-x}}$$

$$35. \int \frac{dx}{(1-x)\sqrt{x}}$$

$$37. \int \frac{dx}{x\sqrt{9x^2+4x+1}}$$

$$39. \int \frac{dx}{x\sqrt{x^2+4}}$$

$$41. \int \frac{dt}{(t-1)\sqrt{3t-t^2-1}}$$

$$43. \int \frac{dx}{x\sqrt{x^2+1}}$$

$$45. \int \frac{x^3+x^2+x+1}{\sqrt{x^2+2x+3}} dx$$

$$47. \int \sin^2 \theta \sin 3\theta \sqrt{\cos 2\theta} d\theta$$

$$49. \int \frac{dx}{(x^2+3x+3)\sqrt{x+2}}$$

$$51. \int \frac{dx}{x\sqrt{x^2-1}}$$

$$53. \int \frac{(x+1) dx}{(2x^2-2x+1)\sqrt{3x^2-2x+1}}$$

$$55. \int \frac{dx}{(4x^2+4x+1)\sqrt{4x^2+4x+5}}$$

$$56. \int \frac{dx}{(2x^2 + 9x + 9)\sqrt{x^2 + 3x + 2}} \quad 57. \int \frac{dx}{(2x^2 + x - 1)\sqrt{x^2 + x + 1}}$$

$$58. \int \frac{dx}{(2x^2 - 2x + 1)\sqrt{3x^2 - 2x + 1}} \quad 59. \int \frac{(3x + 4) dx}{(5x^2 + 8x)\sqrt{4x^2 - 2x + 1}}$$

$$60. \int \frac{(x - 1) dx}{(2x^2 + 2x + 1)\sqrt{(3x^2 + 2x + 1)}}$$

$$61. \int \frac{dx}{(21x^2 + 34x + 9)\sqrt{(5x^2 + 2x - 7)}}$$

$$62. \int \frac{1 - x^2}{1 + x^2} \frac{dx}{\sqrt{x^4 + 1}}$$

$$63. \int \frac{1 + x^2}{1 - x^2} \frac{dx}{\sqrt{1 - 3x^2 + x^4}}$$

$$64. \int \frac{1 + x^2}{1 - x^2} \frac{dx}{\sqrt{1 - x^2 + x^4}}$$

$$65. \int \frac{1 - x^2}{1 + x^2} \frac{dx}{\sqrt{x^4 + x^2 + 1}}$$

$$66. \int \frac{1 - x}{1 + x} \frac{dx}{\sqrt{x + x^2 + x^3}}$$

Answers

$$1. \frac{1}{5} \sin^{-1} 5x + C$$

$$2. \frac{1}{3} \sinh^{-1} \frac{3x}{4} + C$$

$$3. \frac{1}{3} \cosh^{-1} \frac{3x}{4} + C$$

$$4. \frac{5}{4} \sin^{-1} (2x^2) + C$$

$$5. \frac{5}{4} \sinh^{-1} (2x^2) + C$$

$$6. \frac{1}{2} \sin^{-1} \left(\frac{2e^x}{\sqrt{a^2 + b^2}} \right) + C$$

$$7. \frac{1}{2} \sinh^{-1} \left(\frac{2e^x}{\sqrt{a^2 - b^2}} \right) + C$$

$$8. -\frac{1}{3} \sin^{-1} \left(\frac{3 \cos \theta}{2} \right) + C$$

$$9. \frac{1}{2} \log (2e^x + \sqrt{4e^{2x} - a^2 + b^2}) + C$$

$$10. \sinh^{-1} \left(\frac{x + a}{\sqrt{b^2 - a^2}} \right) + C$$

$$11. \frac{1}{2} \cosh^{-1} \left(\frac{2 \tan \theta + 1}{\sqrt{2}} \right) + C$$

$$12. \sinh^{-1} \left(\frac{\sin^{-1} t - 1}{\sqrt{2}} \right) + C$$

$$13. \frac{1}{3} \sinh^{-1} \left(\frac{3x + 2}{2} \right) + C$$

$$14. \sin^{-1} \frac{2x + 1}{\sqrt{5}} + C$$

$$15. \frac{1}{2} \sin^{-1} \left(\frac{8x + 3}{\sqrt{41}} \right) + C$$

$$16. \sinh^{-1} \left(\frac{2e^x + 3}{\sqrt{3}} \right) + C$$

$$17. \cosh^{-1} \left(\frac{2 \tan^{-1} t + 1}{3} \right) + C$$

$$18. 2 \log (\sqrt{x-1} + \sqrt{x-2}) + C$$

$$19. 2 \log (\sqrt{x} + \sqrt{x-a}) + C$$

$$20. \sin^{-1} \left(\frac{x-a}{a} \right) + C$$

$$21. \frac{1}{2} \cosh^{-1} \left(\frac{2x-3}{\sqrt{2}} \right) + C$$

$$22. \frac{1}{3} \sin^{-1} \left(\frac{3x-1}{3} \right) + C$$

$$23. C - \sqrt{1-x^2}$$

$$24. \sqrt{x^2+4} + 3 \sinh^{-1} \frac{x}{2} + C$$

$$25. 5 \sin^{-1} \left(\frac{x-3}{3} \right) - \sqrt{6x-x^2} + C.$$

$$26. \frac{1}{3} \sin^{-1} \frac{3x}{2} - \frac{2}{9} \sqrt{4-9x^2} + C$$

$$27. \frac{5}{3} \sqrt{3x^2-9} - \frac{1}{\sqrt{3}} \cosh^{-1} \frac{t}{\sqrt{3}} + C$$

$$28. \sin^{-1} \frac{x}{2} - \sqrt{4-x^2} + C$$

$$29. \frac{1}{3} \sqrt{(2x+1)(3x+2)} - \frac{1}{6\sqrt{6}} \cosh^{-1} (12x+7)$$

$$30. \sqrt{x^2-3x+5} + \frac{5}{2} \sinh^{-1} \left(\frac{2x-3}{\sqrt{11}} \right) + C$$

$$31. C - \frac{3}{4} \sqrt{3+2x-4x^2} - \frac{5}{8} \sin^{-1} \left(\frac{4x-1}{\sqrt{13}} \right)$$

$$32. \frac{1}{2} \sqrt{4x^2+2x+1} + \frac{5}{4} \sinh^{-1} \left(\frac{4x+1}{\sqrt{3}} \right) + C$$

$$33. \frac{2}{\sqrt{1-x}} + C$$

$$34. \frac{2}{\sqrt{21}} \tan^{-1} \sqrt{\frac{3(2t-3)}{7}} + C$$

$$35. \log \frac{1+\sqrt{x}}{1-\sqrt{x}} + C$$

$$36. \sqrt{2} \tan^{-1} \sqrt{\frac{x-2}{2}} + C$$

$$37. \log x - \log (1+2x+\sqrt{9x^2+4x+1}) + C$$

$$38. \frac{2}{a-b} \sqrt{\frac{b-x}{x-a}} + C$$

$$39. \frac{1}{2} \log x - \frac{1}{2} \log (2+\sqrt{4+x^2}) + C$$

$$40. C - \frac{1}{a} \sin^{-1} \frac{a}{3x}$$

$$41. C - \log (1+t+2\sqrt{3t-t^2-1}) + \log (t-1)$$

$$42. \frac{1}{3} \log \frac{\sqrt{1+x^3}-1}{\sqrt{1+x^3}+1} + C$$

$$43. \log x - \log (1+\sqrt{x^2+1}) + C$$

$$44. \frac{1}{\sqrt{2}} \left[\log x - \log (x+4+2\sqrt{2}) \right] \left[\sqrt{2+x-x^2} \right] + C$$

$$45. \frac{1}{3} (x^2-x) \sqrt{x^2+2x+3} + 2 \log (x+1+\sqrt{x^2+2x+3}) + C$$

$$46. \left(\frac{1}{4} x^3 - \frac{7}{24} x^2 - \frac{1}{96} x + \frac{115}{192} \right) \sqrt{x^2+x+1} + \frac{273}{384} \sinh^{-1} \left(\frac{2x+1}{\sqrt{3}} \right) + C$$

$$47. \text{If } \cos \theta = t, \text{ then } (1-t^2)(4t^2-1)\sqrt{2t^2-1}$$

$$= \frac{(1-t^2)(4t^2-1)(2t^2-1)}{\sqrt{2t^2+1}} \text{ and } \left(\frac{2}{3}t^5 - \frac{4}{3}t^3 + \frac{3}{4}t\right)\sqrt{2t^2-1}$$

$$- \frac{1}{4\sqrt{2}} \log(\sqrt{2}t + \sqrt{2t^2-1}) + C$$

$$48. \frac{1}{2\sqrt{2}} \log \frac{\sqrt{2}x - \sqrt{x^2+1}}{\sqrt{2}x + \sqrt{x^2+1}} + C$$

$$49. \frac{1}{\sqrt{3}} \log \frac{x + \sqrt{3(x+2)} + 3}{x - \sqrt{3(x+2)} + 3} + C$$

$$50. \frac{2}{\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3(x+1)}} + C$$

Hints. Put $x+2 = t^2, t^4 - t^2 + 1$
 $= (t^2 - \sqrt{3}t + 1), (t^2 + \sqrt{3}t + 1)$, use P.F.

$$51. \frac{2}{n} \sec^{-1}(x^{n/2}) + C$$

$$52. \frac{1}{n} \log \frac{\sqrt{1+x^2} - 1}{\sqrt{1+x^2} + 1} + C$$

$$53. \cosh^{-1} \sqrt{\frac{3x^2 - 2x + 1}{2x^2 - 2x + 1}} + 2 \cos^{-1} \sqrt{\frac{3x^2 - 2x + 1}{2(2x^2 - 2x + 1)}} + C$$

$$54. \frac{1}{4\sqrt{6}} \log \frac{\sqrt{3(x^2 - x + 1)} - \sqrt{2}(1-x)}{\sqrt{3(x^2 - x + 1)} + \sqrt{2}(1-x)} - \frac{5}{4\sqrt{3}} \cosh^{-1} \sqrt{\frac{3(x^2 - x + 1)}{x^2 + x + 1}} + C$$

$$55. -\frac{1}{8} \frac{\sqrt{4x^2 + 4x + 5}}{2x + 1} + C \text{ Hints : Use 4.8 or 4.9.}$$

$$56. \frac{2}{3} \sec^{-1}(2x+3) + \frac{1}{3\sqrt{2}} \cosh^{-1} \left(\frac{3x+5}{x+5} \right) + C$$

Hints : Factorise $2x^2 + 9x + 9$ and take PF of $\frac{1}{2x^2 + 9x + 9}$ then use 4.1 with $r = 1$.

$$57. \frac{1}{3} \log \frac{u+2}{u} + \frac{2}{3\sqrt{7}} \log \frac{2u-1-\sqrt{7}}{2u-1+\sqrt{7}}, \text{ Hints } u = x + \sqrt{x^2 + x + 1}$$

$$58. \cosh^{-1} \sqrt{\frac{3x^2 - 2x + 1}{2x^2 - 2x + 1}} + \sin^{-1} \sqrt{\frac{3x^2 - 2x + 1}{2(2x^2 - 2x + 1)}} + C$$

$$59. \frac{10}{19} \cosh^{-1} \left\{ \frac{4}{\sqrt{3}} \sqrt{\frac{4x^2 - 2x + 1}{5x^2 + 8x}} \right\} - \frac{9}{19} \sinh^{-1} \left\{ \sqrt{\frac{4x^2 - 2x + 1}{5x^2 + 8x}} \right\} + C$$

$$60. 2 \sin^{-1} \sqrt{\frac{3x^2 + 2x + 1}{2(2x^2 + 2x + 1)}} - \cosh^{-1} \sqrt{\frac{3x^2 + 2x + 1}{2x^2 + 2x + 1}} + C$$

$$61. \frac{7}{80} \tan^{-1} \sqrt{\frac{4(5x+7)}{(x-1)}} - \frac{3}{80} \tan^{-1} \sqrt{\frac{5x+7}{4(x-1)}} + C$$

Hints : $5x^2 + 2x - 7 = (5x+7)(x-1)$. Put $x-1 = u^2$. Use PF of $\frac{1}{(3u^2+4)(7u^2+16)}$ and use 4.9.

62. $\frac{1}{\sqrt{2}} \sin^{-1} \left(\frac{x\sqrt{2}}{1+x^2} \right) + C$ [Hints : Type 4.14, $f(x) = \frac{1-x^2}{1+x^2}$]

$$\therefore \int \frac{-x^2 [1 - 1/x^2] \cdot \frac{dx}{x^2}}{x^2 [1 + (1/x^2)] \cdot x \sqrt{[x^2 + (1/x^2)]}}$$

$$= - \int \frac{\left(1 - \frac{1}{x^2}\right)}{\left(x + \frac{1}{x}\right) \sqrt{\left[\left(x + \frac{1}{x}\right)^2 - 2\right]}} \cdot \frac{dx}{x^2} \quad \text{Put } x + \frac{1}{x} = u$$

63. $\sin^{-1} \frac{x}{x^2+1} + C$

64. $\sinh^{-1} \frac{x}{x^2-1} + C$

65. $\sin^{-1} \frac{x}{x^2+1} + C$

66. $-2 \tan^{-1} \sqrt{\frac{x^2+x+1}{x}}$

Hints : $\int \frac{-(x^2-1)}{(x+1)^2 \cdot x \sqrt{(x+\frac{1}{x}+1)}} \cdot \frac{dx}{x^2} = - \int \frac{\left(1 - \frac{1}{x^2}\right) \frac{dx}{x^2}}{\left(x + \frac{1}{x} + 2\right) \sqrt{x + \frac{1}{x} + 1}}$

Let $x + \frac{1}{x} = u$, then $u+1 = p^2$.

INTEGRATION BY PARTS

5.1 INTRODUCTION

In the previous chapters special integral of products and divisions are performed by substitutions. In all these cases one of the part could always be expressed as a differential coefficient of the other. But there are functions which are not so expressible. In that case if one part is integrable, then the method of integration by parts is useful. This method is almost a similar procedure as differentiation of the product of two functions. This method also helps to integrate some other elementary transcendental functions like logarithmic and inverse trigonometric functions which could *not be integrated* by the earlier procedures.

5.2 METHOD OF INTEGRATION BY PARTS

Let $f(x)$ and $g(x)$ be two differentiable functions. So, by differentiation of products,

$$\frac{d}{dx} \{f(x) g(x)\} = f(x) g'(x) + g(x) f'(x),$$

prime indicates derivative w.r.t. x .

Integration w.r.t. x ,

$$\int \frac{d}{dx} \{f(x) g(x)\} dx = \int f(x) g'(x) dx + \int g(x) f'(x) dx$$

or
$$f(x) g(x) = \int f(x) g'(x) dx + \int g(x) f'(x) dx$$

or
$$\begin{aligned} \int f(x) g'(x) dx &= f(x) g(x) - \int f'(x) g(x) dx \\ &= f(x) \int g'(x) dx - \int \{f'(x) \int g'(x) dx\} dx \end{aligned}$$

Now, replacing $f(x)$ by $u(x)$ and $g'(x)$ by $v(x)$,

We have

$$\int u(x) v(x) dx = u(x) \int v(x) dx - \int \left[\frac{d}{dx} u(x) \int v(x) dx \right] dx \quad (1)$$

It is convention to call $u(x)$ as the *first function* and $v(x)$ as *second function*. Then the above relation can be stated as

(Integral of the product of two functions) = (First function) $\times$ (Integral of the second) - Integral of the product of (differential coefficient of the first) and (Integral of second).

This method is known as *integration by parts* (I.b.p.) because of "splitting" the integrand into two parts - first and second parts (functions). The procedure continues as long as the last integral in (1) contains the product.

Indicating the order of differentiation by primes (..) or superscripts (..) and successive integration by subscripts the above formula (1) can be generalised as follows :

$$\begin{aligned}
 \int u(x) v(x) dx &= u(x) \int v(x) dx - u'(x) \int \left(\int v(x) dx \right) dx \\
 &\quad + \int \left\{ u''(x) \int \left(\int \int v(x) dx \right) dx \right\} dx \\
 &= u(x) v_1(x) + (-1)^1 u'(x) v_2(x) + (-1)^2 \int u''(x) v_3(x) dx \\
 &= u(x) v_1(x) + (-1)^1 u'(x) v_2(x) + (-1)^2 u''(x) v_3(x) dx + \\
 &\quad (-1)^3 \int u'''(x) v_4(x) dx \\
 &= u(x) v_1(x) - u'(x) v_2(x) + u''(x) v_3(x) - \dots \\
 &\quad + (-1)^{n-1} u^{(n-1)}(x) v_n(x) + (-1)^n \int u^{(n)}(x) v_{n+1}(x) dx + C \\
 &= \sum_{i=0}^n (-1)^i u^{(i)}(x) v_{i+1}(x) + C \quad (2)
 \end{aligned}$$

where $u(x)$ is a polynomial of degree n . This formula is called Liebnitz's formula for integration by parts.

5.3. SELECTION OF FIRST AND SECOND FUNCTIONS

From sec. 5.2 it is apparent that product will disappear soon if the first function can be chosen properly. A function which disappears with fewer number of successive differentiation, is to be selected as a *first function* provided the integration of the remaining one i.e. the *second function* is known. In practice it sometimes happens that this selected first function is to be integrated when the integration of the second is not known in elementary form. This will happen with logarithmic and trigonometric inverse functions whose integration is yet unknown. Any way, the following selection procedures may be adopted :

(i) **Unknown integral** : The function whose elementary integral is *not known*, is to be selected as a first function whatever may be the second one. Logarithmic and inverse trigonometric function are of this category. See example 1 below.

(ii) **Smaller integer exponent and polynomial** : The function whose exponent is *integer* and *smaller* than the other, is to be chosen as a first function. Because successive differentiation reduces the smaller exponent to zero earlier. Also polynomial is to be selected as first function in the integrals of the form $\int (\text{polynomials}) (\sin ax \text{ or } \cos ax) dx$ and $\int (\text{polynomials}) (e^{ax} \text{ or } \log x) dx$. See example 2 below.

(iii) **An important utility :** It was stated in sec. 5.1 that integration by parts helps to integrate some elementary functions like $\log x$, $\sin^{-1} x$ etc. which cannot be integrated by any other means. In such cases 1, the *unity*, is to be selected as the *second function*. Examples 3, 4 are to be noted. Similarly some standard integrals (Sec. 5.5 and 5.6) are integrated with this method.

(iv) **No bar in selection :** In the product of exponential and trigonometric sine or cosine functions — circular or hyperbolic, none will vanish on differentiation. But sine or cosine reappears after double differentiation. In such cases either the exponential or the trigonometric sine or cosine may be selected as a first function. The original integral will *reappear* after two successive operation of integration by parts. By the transposition of sides the required integral can be *solved*. See for examples 5 and the required integral can be *solved*. See for examples 5 and 6. But *once* the first function is selected, its differential coefficient must be the first function in the next by parts integration; otherwise zero will be equated to zero. This restriction is applicable to other forms also.

5.4 INTEGRAL OF THE TYPE $\int e^{ax} \{af(x) + f'(x)\} dx$

It is known that $\frac{d}{dx} \{e^{ax} f(x)\} = e^{ax} \{af(x) + f'(x)\}$

$$\therefore \int e^{ax} \{af(x) + f'(x)\} dx = e^{ax} f(x) + c.$$

This formula can also be established using I.b.p. as follows :

$$\begin{aligned} \int e^{ax} \{af(x) + f'(x)\} dx \\ &= a \int e^{ax} f(x) dx + \int e^{ax} f'(x) dx + c \\ &= e^{ax} f(x) - \int e^{ax} f'(x) dx + \int e^{ax} f'(x) dx + c \\ &= e^{ax} f(x) + c. \end{aligned}$$

Examples

1. Integrate $x^2 \tan^{-1} x$.

The integrand is a product of which integration of $\tan^{-1} x$ is *not known*. So, x^2 is to be selected as a *second function* although it vanishes after double differentiation. Therefore,

$$\int x^2 \tan^{-1} x dx = \tan^{-1} x \int x^2 dx - \int \left\{ \frac{d}{dx} (\tan^{-1} x) \int x^2 dx \right\} dx + c$$

$$= \frac{1}{3} x^3 \tan^{-1} x - \frac{1}{3} \int \frac{x^3}{1+x^2} dx + c$$

$$= \frac{1}{3} x^3 \tan^{-1} x - \frac{1}{3} \int \frac{x(x^2+1)-x}{1+x^2} dx + c,$$

$\therefore$ integrand is an improper fraction.

$$= \frac{1}{3} x^3 \tan^{-1} x - \frac{1}{3} \int x dx + \frac{1}{3} \int \frac{x}{x^2+1} dx + c$$

$$= \frac{1}{3} x^3 \tan^{-1} x - \frac{x^2}{6} + \frac{1}{3} \cdot \frac{1}{2} \int \frac{2x}{x^2+1} dx + c$$

$$= \frac{1}{3} x^3 \tan^{-1} x - \frac{x^2}{6} + \frac{1}{6} \log(x^2+1) + c \text{ by sec 2.4}$$

N.B. Liebnitz formula is not useful here, because $\tan^{-1} x$ is an infinite series.

2. Compute $\int x^2 \sin x dx$.

The integrand here is a product. Integration of both elementary functions are known. Yet power 2 of x^2 is less than that of $\sin x$. Because $\sin x$ is an infinite polynomial. So, x^2 is to be chosen or selected as a first function otherwise product will continue endlessly and creates a massacre. Therefore,

$$\begin{aligned} \int x^2 \sin x dx &= x^2 \int \sin x dx - \int \left\{ \frac{d}{dx} (x^2) \int \sin x dx \right\} dx + c \\ &= -x^2 \cos x + 2 \int x \cos x dx + c \\ &= -x^2 \cos x + 2x \int \cos x dx - 2 \int \left\{ \frac{d}{dx} (x) \int \cos x dx \right\} dx + c \\ &= -x^2 \cos x + 2x \sin x - 2 \int \sin x dx + c \\ &= -x^2 \cos x + 2x \sin x + 2 \cos x + c. \\ &= 2x \sin x + 2 \cos x - x^2 \cos x + c. \end{aligned}$$

OR. To use Liebnitz' formula. Let $u(x) = x^2$ and $v(x) = \sin x$.

$$\begin{aligned} \therefore \int x^2 \sin x dx &= \int u(x) v(x) dx = u(x) v_1(x) - u'(x) v_2(x) \\ &\quad + \int u''(x) v_2(x) \quad \because u(x) \text{ is quadratic,} \\ &= -x^2 \cos x + 2x \sin x - \int 2 \sin x dx + c \\ &= -x^2 \cos x + 2x \sin x + 2 \cos x + c \\ &= 2x \sin x + 2 \cos x - x^2 \cos x + c. \end{aligned}$$

3. Find the integral $\int \log_x dx$.

The integrand here is not a product, but an elementary function. Yet its integral is not known. So, we can split $\log x$ as $(1 \cdot \log x)$, a product of unity and $\log x$ itself. Since integration of unity and differentiation of $\log x$ are known. So using integration by parts,

$$\begin{aligned}\int \log_x dx &= (\log x) \int dx - \left\{ \frac{d}{dx} (\log x) \int dx \right\} dx + c \\ &= x \log x - \int \frac{1}{x} \cdot x dx + c \\ &= x \log x - \int dx + c = x \log x - x + c \\ &= x (\log x + 1) + c.\end{aligned}$$

$\therefore$ integration by parts gives the integral of $\log x$ as $x (\log x - 1) + c$.

4. Find $\int \cos^{-1} dx$.

Here also the similar case as in Ex. 3. So

$$\begin{aligned}\int \cos^{-1} x dx &= (\cos^{-1} x) \int dx - \int \left\{ \frac{d}{dx} (\cos^{-1} x) \int dx \right\} dx + c \\ &= x \cos^{-1} x + \int \frac{x}{\sqrt{1-x^2}} dx + c \\ &= x \cos^{-1} x + \left(-\frac{1}{2} \right) \int \frac{-2x}{\sqrt{1-x^2}} dx + c \\ &= x \cos^{-1} x - \frac{1}{2} \int \frac{du}{\sqrt{u}} + c \quad \text{when } 1-x^2 = u \\ &= x \cos^{-1} x - \sqrt{u} + c = x \cos^{-1} x - \sqrt{1-x^2} + c.\end{aligned}$$

5. Compute $\int e^{ax+c} \cos (bx+d) dx$

The product in this integrand will continue i.e. none will vanish on differentiation. Therefore, let

$$\begin{aligned}I &= \int e^{ax+c} \cos (bx+d) dx \\ &= \frac{1}{a} e^{ax+c} \cos (bx+d) + \frac{b}{a} \int e^{ax+c} \sin (bx+d) dx + K \\ &= \frac{1}{a} e^{ax+c} \cos (bx+d) + \frac{b}{a^2} e^{ax+c} \sin (bx+d) - \frac{b^2}{a^2} I + K\end{aligned}$$

$$\text{or} \quad \left(1 + \frac{b^2}{a^2} \right) I = e^{ax+c} \frac{(a \cos (bx+d) + b \sin (bx+d))}{a^2} + K$$

$$\text{or} \quad I = \frac{e^{ax+c} (a \cos (bx+d) + b \sin (bx+d))}{a^2 + b^2} + K'$$

N.B. Similar procedure gives

$$\int e^{ax+c} \sin (bx+d) dx = \frac{e^{ax+c} [a \sin (bx+d) - b \cos (bx+d)]}{a^2 + b^2}$$

The two integrals may be considered as elementary standard integrals and the results can be stated as follows :

"Differentiate the product with opposite sign for trigonometric circular functions and divide the expression so obtained by the sum of the squares of the coeffs. of x's."

6. Compute the integral $\int e^{ax+c} \sinh (bx+d) dx$.

As in example 5, the product in the integrand will never vanish on differentiation. So, let

$$\begin{aligned} I &= \int e^{ax+c} \sinh (bx+d) dx \\ &= \frac{1}{a} e^{ax+c} \sinh (bx+d) - \frac{b}{a} \int e^{ax+c} \cosh (bx+d) dx + K' \\ &= \frac{1}{a} e^{ax+c} \sinh (bx+d) - \frac{b}{a^2} e^{ax+c} \cosh (bx+d) + \frac{b^2}{a^2} I + K' \end{aligned}$$

$$\text{or } \left(1 - \frac{b^2}{a^2}\right) I = \frac{1}{a^2} e^{ax+c} [a \sinh (bx+d) - b \cosh (bx+d)] + K'$$

$$\text{or } I = \frac{e^{ax+c} [a \sinh (bx+d) - b \cosh (bx+d)]}{a^2 - b^2} + K, \quad a \neq b$$

N.B. 1. Similar procedure gives

$$\begin{aligned} \int e^{ax+c} \cosh (bx+d) dx \\ = \frac{e^{ax+c} [a \cosh (bx+d) - b \sinh (bx+d)]}{a^2 - b^2} + K, \quad a \neq b \end{aligned}$$

N.B. 2. These results of integration can be stated as in Ex. 5 —

"Differentiate the integrand with negative sign for the differential coefficient of hyperbolic function and divide the result so obtained by the difference of the square of the coefficient of x in hyperbolic from that in exponential if both are unequal."

N.B. 3. If $a=b$, integration by parts is not valid. In such case hyperbolic function is to be expressed in terms of exponential and integrate the exponential function simply. The exponential part of the result may again be expressed into hyperbolic. For example

$$\begin{aligned} \int e^x \cosh x dx &= \frac{1}{2} \int (e^{2x} + 1) dx = \frac{1}{4} e^{2x} + \frac{1}{2} x^2 + c \\ &= \frac{1}{4} (\cosh 2x + \sinh 2x) + \frac{1}{2} x^2 + c \end{aligned}$$

7. Find the integral $\int e^x \frac{2 - \sin 2x}{1 - \cos 2x} dx$.

$$\text{Let } I = \int e^x \cdot \frac{2 - \sin 2x}{2 \sin^2 x} dx + c = \int e^x [\operatorname{cosec}^2 x - \cot x] dx$$

which is of type 5.3 as $f(x) = -\cot x$ & $f'(x) = \operatorname{cosec}^2 x$.

$$\therefore I = \int e^x [f(x) + f'(x)] dx = e^x f(x) + c \\ = -e^x \cot x + c.$$

Exercises 5 (1)

Compute the following :

Q1. $\int x e^x dx$

Q2. $\int x \sec^2 x dx$

Q3. $\int x \sec x \tan x dx$

Q4. $\int x \log x dx$

Q5. $\int \frac{\log x}{x^2} dx$

Q6. $\int x^n \log x dx$

Q7. $\int x^2 \tan^{-1} x dx$

Q8. $\int x \sin^2 x dx$

Q9. $\int x \sin x \cos x dx$

Q10. $\int x^2 \sin 2x dx$

Q11. $\int x^2 \cos nx dx$

Q12. $\int x \tan^2 x dx$

Q13. $\int \frac{\sin^{-1} x}{(1-x^2)^{3/2}} dx$ (let $x = \sin \theta$)

Q14. $\int \cos^{-1} \frac{1-x^2}{1+x^2} dx$ (let $x = \tan \theta$)

Q15. $\int \tan^{-1} \frac{2x}{1-x^2} dx$

Q16. $\int \cot^{-1} x dx$

Q17. $\int \sin^{-1} (3x - 4x^3) dx$

Q18. $\int \sec^{-1} t dt$

Q19. $\int \operatorname{cosec}^{-1} x dx$

Q20. $\int \operatorname{cosec}^{-1} \sqrt{\frac{x+a}{x}} dx$

Q21. $\int \cos^{-1} \frac{1}{x} dx$

Q22. $\int x \sec^{-1} x dx$

Q23. $\int x^2 \sin^{-1} x dx$

Q24. $\int \tan^{-1} \sqrt{x} dx$

Q25. $\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$

Q26. $\int x^3 e^x dx$

Q27. $\int x^3 e^{x^2} dx$

Q28. $\int (\log x)^2 dx$

$$29. \int \frac{\log x}{(x+1)^2} dx$$

$$30. \int x^{-4} \sqrt{x^2+1} [\log(x^2+1) - 2 \log x] dx \quad [\text{Hints : Put } 1 + \frac{1}{x^2} = u]$$

$$31. \int \frac{\log(x+1)}{\sqrt{x+1}} dx$$

$$32. \int \log(x + \sqrt{x^2 + a^2}) dx$$

$$33. \int \log(x^2 - x - 1) dx$$

$$34. \int x \log\left(1 + \frac{1}{x}\right) dx$$

$$35. \int \frac{x}{1 + \cos x} dx$$

$$36. \int \{1 - (t+2) \cot t\} \operatorname{cosec} t dt$$

$$37. \int (2t + t^2 \tan t) \sec t dt$$

$$38. \int e^t (1 - \cot t) \operatorname{cosec} t dt$$

$$39. \int \frac{x e^x}{(x+1)^2} dx$$

$$40. \int e^t (1 + \tan t) \sec t dt$$

$$41. \int e^x \frac{1 + \sin x}{1 + \cos x} dx$$

$$42. \int e^{-t} \cos \pi t dt$$

$$43. (i) \int \cos(\log x) dx$$

$$(ii) \int \frac{x}{1 + \sin x} dx$$

$$44. \int e^x (\cos x + \sin x) dx$$

$$45. \int e^x \sin^2 x dx$$

$$46. \int e^x \sin 2x \cos 3x dx$$

$$47. \int \sec^3 \theta d\theta$$

$$48. \int \operatorname{cosec}^3 \theta d\theta$$

$$49. \int e^{4x} \cos(3x+5) dx$$

$$50. \int 3^x \cos x dx$$

$$51. \text{ Prove that } \int u \frac{d^2 v}{dx^2} dx = u \frac{dv}{dx} - v \frac{du}{dx} + \int v \frac{d^2 u}{dx^2} dx.$$

$$52. \text{ Integrating by parts } \int \frac{dx}{x^2+1}, \text{ find } \int \frac{dx}{(x^2+1)^2}. \text{ Hence determine}$$

$$\int \frac{x^4 dx}{(x^2+1)^2} \left[\text{Hints : Integrate by parts } \int \frac{dx}{x^2+1} \text{ using 1 as the second function \& solve and finally integrate } \frac{1}{x^2+1} \text{ as usual} \right]$$

$$53. \text{ If } u = \int e^{ax} \cos bx dx \text{ and } v = \int e^{ax} \sin bx dx \text{ then prove that}$$

$$(i) \tan^{-1} \frac{v}{u} + \tan^{-1} \frac{b}{a} = bx \text{ and } (ii) (a^2 + b^2)(u^2 + v^2) = e^{2ax}.$$

54. Show that $\int x^2 2^x \sin 2x \, dx = \frac{x^2 2^x}{r} \sin(2x - \alpha) - 2x \cdot \frac{2^x}{r^2} \sin 2(x - \alpha) + 2 \cdot \frac{2^x}{r^3} \sin(2x - 3\alpha)$ where $r = \sqrt{(\log 2)^2 + 4}$ and $\alpha = \tan^{-1}\left(\frac{2}{\log 2}\right)$.
55. Show that $\int \frac{\sin(\log x)}{x^3} \, dx = -[2 \sin(\log x) + \cos(\log x)] + c$
56. Prove that $\int \frac{x^2 \, dx}{(x \sin x + \cos x)^2} = -\frac{x \sec x}{x \sin x + \cos x} + \tan x + c$
[Hints : $-\int x \sec x \, d\left(\frac{1}{t}\right)$ where $t = x \sin x + \cos x$]
57. Prove that (i) $\int \frac{x^3}{(x \sin x + \cos x)^2} \, dx = -\frac{x \sec x}{x \sin x + \cos x} + \frac{1}{2}x^2 + \log(x \sin x + \cos x) + c$.
- (ii) $\int e^{2x} \sinh 2x \, dx = \frac{1}{8}(\cosh 4x + \sinh 4x) - \frac{1}{2}x + c$
- (iii) $\int e^{3x} \cosh 7x \, dx = -\frac{1}{40}e^{3x}(3 \cosh 7x - 7 \sinh 7x) + c$
- (iv) $\int \sin x \sin 2x \, dx = \frac{2}{3} \sin^3 x + c$ by "integration by parts"

Answers

- | | |
|--|---|
| 1. $e^x(x-1) + c$ | 2. $x \tan x + \log \cos x + c$ |
| 3. $x \sec x - \log(\sec x + \tan x) + c$ | 4. $\frac{1}{2}x^2 \left(\log x - \frac{1}{2} \right) + c$ |
| 5. $c - \frac{1}{x}(\log x + 1)$ | 6. $\frac{x^{n+1}}{n+1} \left(\log x - \frac{1}{n+1} \right) + c$ |
| 7. $\frac{1}{3}x^3 \tan^{-1} x - \frac{1}{6}x^2 + \frac{1}{6} \log(1+x^2) + c$ | 8. $\frac{1}{8}(2x^2 - 2x \sin 2x - \cos 2x) + c$ |
| 9. $\frac{1}{8} \sin 2x - \frac{1}{4} x \cos 2x + c$ | 10. $\frac{1}{4} \cos 2x + \frac{1}{2} x \sin 2x - \frac{1}{2} x^2 \cos 2x + c$ |
| 11. $\frac{x^2}{n} \sin nx + \frac{2x \cos nx}{n^2} - \frac{2}{n^3} \sin nx + c$ | 12. $x \tan x + \log \cos x - \frac{x^2}{2} + c$ |
| 13. $\frac{x \sin^{-1} x}{\sqrt{1-x^2}} + \frac{1}{2} \log(1-x^2) + c$ | 14. $2x \tan^{-1} x - \log(1+x^2) + c$ |

15. $2x \tan^{-1} x - \log(1+x^2) + c$
16. $x \cot^{-1} x + \frac{1}{2} \log(1+x^2) + c$
17. $3x \sin^{-1} x + 3\sqrt{1-x^2} + c$
18. $t \sec^{-1} t - \log(t + \sqrt{t^2-1}) + c$
19. $x \operatorname{cosec}^{-1} x + \log(x + \sqrt{x^2-1}) + c$
20. $(x+a) \tan^{-1} \left(\frac{x}{a} \right)^{1/2} - \sqrt{ax} + c$
21. $x \sec^{-1} x - \log(x + \sqrt{x^2-1}) + c$
22. $\frac{1}{2} x^2 \sec^{-1} x - \frac{1}{2} \sqrt{x^2-1} + c$
23. $\frac{1}{3} x^3 \sin^{-1} x + \frac{1}{9} (x^2+2) \sqrt{1-x^2} + c$
24. $(x+1) \tan^{-1} \sqrt{x} - \sqrt{x} + c$
25. $x - \sqrt{1-x^2} \sin^{-1} x + c$
26. $(x^3 - 3x^2 + 6x - 6) e^x + c$
27. $\frac{1}{2} (x^2-1) e^{x^2} + c$
28. $x(\log x)^2 - 2x \log x + 2x + c$
29. $\frac{x}{x+1} \log x - \log(x+1) + c$
30. $(x^2+1)^{3/2} \left[2 - 3 \log \left(1 + \frac{1}{x^2} \right) \right] / 9x^3 + c$
31. $2\sqrt{x+1} \{ \log(x+1) - 2 \} + c$
32. $x \log(x + \sqrt{x^2+a^2}) - \sqrt{x^2+a^2} + c$
33. $\left(x - \frac{1}{2} \right) \log(x^2-x-1) - 2x - \frac{\sqrt{5}}{2} \log \frac{2x-1-\sqrt{5}}{2x-1+\sqrt{5}} + c$
34. $\frac{1}{2} (x^2-1) \log(x+1) - \frac{1}{2} x^2 \log x + \frac{x}{2} + c$
35. $x \tan \frac{x}{2} + 2 \log \cos \frac{x}{2} + c$
36. $[(t+2) \operatorname{cosec} t + c]$
37. $t^2 \sec t + c$
38. $e^t \operatorname{cosec} t + c$
39. $\frac{e^x}{1+x} + c$
40. $e^t \sec t + c$
41. $e^x \tan \frac{x}{2} + c$
42. $\frac{e^{-t} (\pi \sin \pi t - \cos \pi t)}{\pi^2 + 1} + c$
43. (i) $\frac{x}{2} \{ \cos(\log x) + \sin(\log x) \} + c$
- (ii) $-x \tan \left(\frac{\pi}{4} - \frac{x}{2} \right) - 2 \log \sec \left(\frac{\pi}{4} - \frac{x}{2} \right) + c$
44. $e^x \sin x + c$
45. $\frac{1}{2} e^x \left\{ 1 - \frac{1}{5} (\cos 2x + 2 \sin 2x) \right\} + c$
46. $\frac{1}{2} e^x \left\{ \frac{1}{\sqrt{25}} \sin(5x - \tan^{-1} 5) - \frac{1}{2} (\sin x - \cos x) \right\} + c$
47. $\frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \log(\sec \theta + \tan \theta) + c$
48. $\frac{1}{2} \left(\log \tan \frac{\theta}{2} - \operatorname{cosec} \theta \cot \theta \right) + c$
49. $\frac{1}{5} e^{4x} \cos \left(3x + 5 - \tan^{-1} \frac{3}{4} \right) + c$
50. $3^x (\sin x + (\log 3) \cos x) / (1 + (\log 3)^2) + c$
52. $\frac{1}{2} \left(\frac{x}{x^2+1} + \tan^{-1} x \right) + c$ and $x - \frac{3}{2} \tan^{-1} x + \frac{x}{2(x^2+1)} + c$

5.5 STANDARD INTEGRALS OF THE TYPES $\int \sqrt{x^2 \pm a^2} dx$

AND $\int \sqrt{a^2 - x^2} dx$.

These types are the reciprocals of the types in sec. 4.1. The substitutions used in sec. 4.1 are still valid here but integration by parts is essential except the third one if circular trigonometric functions are used. Let us consider them one after another.

$$(A) \int \sqrt{x^2 + a^2} dx = \frac{1}{2} \left[x \sqrt{x^2 + a^2} + a^2 \sinh^{-1} \frac{x}{a} \right] + c$$

Proof. To prove the result, let 1, the unity, be the second function & let

$$\begin{aligned} I = \int \sqrt{x^2 + a^2} dx &= \sqrt{x^2 + a^2} \int dx - \int \left\{ \frac{d}{dx} \sqrt{x^2 + a^2} \int dx \right\} dx + c' \\ &= x \sqrt{x^2 + a^2} - \int \frac{2x}{2 \sqrt{x^2 + a^2}} x dx + c' \\ &= x \sqrt{x^2 + a^2} - \int \frac{(x^2 + a^2) - a^2}{\sqrt{x^2 + a^2}} dx + c' \\ &= x \sqrt{x^2 + a^2} - I + a^2 \int \frac{1}{\sqrt{x^2 + a^2}} dx + c' \end{aligned}$$

$$\text{or} \quad 2I = x \sqrt{x^2 + a^2} + a^2 \sinh^{-1} \frac{x}{a} + c' \text{ by 4.1 (A)}$$

$$\text{or} \quad I = \frac{1}{2} \left[x \sqrt{x^2 + a^2} + a^2 \sinh^{-1} \frac{x}{a} \right] + c.$$

$$(B) \int \sqrt{x^2 - a^2} dx = \frac{1}{2} \left[x \sqrt{x^2 - a^2} - a^2 \cosh^{-1} \frac{x}{a} \right] + c.$$

Proof. As in (A) Let 1, the unity, be the second function & let

$$\begin{aligned} I = \int \sqrt{x^2 - a^2} dx &= \sqrt{x^2 - a^2} \int dx - \int \left\{ \frac{d}{dx} \sqrt{x^2 - a^2} \int dx \right\} dx + c' \\ &= x \sqrt{x^2 - a^2} - \int \frac{x^2}{\sqrt{x^2 - a^2}} dx + c' \\ &= x \sqrt{x^2 - a^2} - \int \frac{(x^2 - a^2) + a^2}{\sqrt{x^2 - a^2}} dx + c' \\ &= x \sqrt{x^2 - a^2} - I - a^2 \int \frac{1}{\sqrt{x^2 - a^2}} dx + c' \end{aligned}$$

$$\text{or} \quad 2I = x \sqrt{x^2 - a^2} - a^2 \cosh^{-1} \frac{x}{a} + c'$$

$$\text{or } I = \frac{1}{2} \left[x \sqrt{x^2 - a^2} - a^2 \cosh^{-1} \frac{x}{a} \right] + c.$$

N.B. If $x = a \tan \theta$ is substituted in (A) then

$$\begin{aligned} I &= \int \sqrt{x^2 + a^2} dx = \int \sqrt{a^2 (1 + \tan^2 \theta)} \cdot a \sec^2 \theta d\theta + c' = a^2 \int \sec^3 \theta d\theta + c' \\ &= a^2 \int \sec \theta \cdot \sec^2 \theta d\theta + c' \\ &= a^2 [\sec \theta \int \sec^2 \theta d\theta - \int \sec \theta \tan^2 \theta d\theta] + c' \\ &= a^2 [\sec \theta \tan \theta - \int \sec \theta (\sec^2 \theta - 1) d\theta] + c' \\ &= a^2 \sec \theta \tan \theta - I + a^2 \int \sec \theta d\theta + c' \end{aligned}$$

$$\text{or } 2I = a^2 \sec \theta \tan \theta + a^2 \log (\sec \theta + \tan \theta) + c'$$

$$\text{or } = a \sqrt{1 + \tan^2 \theta} \cdot a \tan \theta + a^2 \log (\sqrt{1 + \tan^2 \theta} + \tan \theta) + c'$$

$$= x a \sqrt{1 + \frac{x^2}{a^2}} + a^2 \log \left(\frac{x}{a} + \sqrt{1 + \frac{x^2}{a^2}} \right) + c'$$

$$\text{or } I = \frac{1}{2} \left[x \sqrt{x^2 + a^2} + a^2 \log (x + \sqrt{x^2 + a^2}) \right] + c.$$

$$= \frac{1}{2} \left[x \sqrt{x^2 + a^2} + a^2 \sinh^{-1} \frac{x}{a} \right] + c.$$

Similarly, substitution of $x = a \sec \theta$ in (B) gives

$$\begin{aligned} \int \sqrt{x^2 - a^2} dx &= \frac{1}{2} \left[x \sqrt{x^2 - a^2} - a^2 \log (x + \sqrt{x^2 - a^2}) \right] + c \\ &= \frac{1}{2} \left[x \sqrt{x^2 - a^2} - a^2 \cosh^{-1} \frac{x}{a} \right] + c. \end{aligned}$$

$$(C) \int \sqrt{a^2 - x^2} dx = \frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{x}{2} \sqrt{a^2 - x^2} + c$$

Proof.

$$\begin{aligned} I &= \int \sqrt{a^2 - x^2} dx = \sqrt{a^2 - x^2} \int dx - \int \left\{ \frac{-2x}{2\sqrt{a^2 - x^2}} \cdot \int dx \right\} dx + c' \\ &= x \sqrt{a^2 - x^2} + \int \frac{x^2}{\sqrt{a^2 - x^2}} dx + c' \\ &= x \sqrt{a^2 - x^2} + \int \frac{-(a^2 - x^2) + a^2}{\sqrt{a^2 - x^2}} dx + c' \\ &= x \sqrt{a^2 - x^2} - I + a^2 \int \frac{1}{\sqrt{a^2 - x^2}} dx + c' \end{aligned}$$

$$\text{or} \quad 2I = x\sqrt{a^2 - x^2} + a^2 \sin^{-1} \frac{x}{a} + c'$$

$$\text{or} \quad I = \frac{1}{2} \left[x\sqrt{a^2 - x^2} + a^2 \sin^{-1} \frac{x}{a} \right] + c.$$

N.B. $x = a \sin \theta$ will produce the same result without using integration by parts.

5.6 INTEGRAL OF TYPES $\int \sqrt{ax^2 + bx + c} dx$ AND

$$\int (px + q) \sqrt{ax^2 + bx + c} dx.$$

General procedure for these integrals remain same as in sec. 4.3 and 4.6.

$$(A) \int \sqrt{ax^2 + bx + c} dx = \sqrt{a} \int \sqrt{\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}} dx + c$$

which is of type 5.5 (A) or (B) according as $4ac > b^2$ or $4ac < b^2$ and of type (C) if $a < 0$.

(B) $\int (px + q) \sqrt{ax^2 + bx + c} dx$ is decomposed into two integrals by expressing $(px + q) = \frac{p}{2a} \frac{d}{dx} (ax^2 + bx + c) + \left(q - \frac{pb}{2a}\right)$. Therefore

$$\begin{aligned} \int (px + q) \sqrt{ax^2 + bx + c} dx &= \frac{p}{2a} \int \sqrt{ax^2 + bx + c} \frac{d}{dx} (ax^2 + bx + c) dx + \\ &\quad \left(q - \frac{pb}{2a}\right) \int \sqrt{ax^2 + bx + c} dx + k \\ &= \frac{p}{3a} (ax^2 + bx + c)^{3/2} + \left(q - \frac{pb}{2a}\right) \int \sqrt{ax^2 + bx + c} dx + k \end{aligned}$$

The remaining integral on the right is of type 5.6 (A) above.

N.B. For integrals of sec. 5.5., the following substitutions are also useful and valid :

Integrals	Substitutions
$\int \sqrt{a^2 - x^2} dx$	$x = a \sin \theta, a \cos \theta, a \tanh \theta, a \operatorname{sech} \theta$
$\int \sqrt{a^2 + x^2} dx$	$x = a \tan \theta, a \cot \theta, a \sinh \theta, a \operatorname{cosech} \theta$
$\int \sqrt{x^2 - a^2} dx$	$x = a \sec \theta, a \operatorname{cosec} \theta, a \cosh \theta, a \coth \theta$

Examples

1. Integrate $(2x + 1) \sqrt{x^2 - x + 1}$.

This is of type 5.6 (B). Therefore, let

$$\begin{aligned}
 I &= \int (2x+1)\sqrt{x^2-x+1} \, dx \\
 &= \int \left\{ \frac{d}{dx}(x^2-x+1) + 2 \right\} \sqrt{x^2-x+1} \, dx + c \\
 &= \int \sqrt{x^2-x+1} \frac{d}{dx}(x^2-x+1) \, dx + 2 \int \sqrt{x^2-x+1} \, dx + c \\
 &= I_1 + 2I_2 \quad (\text{say}).
 \end{aligned}$$

For I_1 let $x^2-x+1=u$. $\therefore \frac{d}{dx}(x^2-x+1) = \frac{du}{dx}$

$$\begin{aligned}
 \therefore I_1 &= \int \sqrt{u} \frac{du}{dx} \, dx + k_1 = \int \sqrt{u} \, du + k_1 = \frac{2}{3} u^{3/2} + k_1 \\
 &= \frac{2}{3} (x^2-x+1)^{3/2} + k_1
 \end{aligned}$$

$$\begin{aligned}
 \text{And } I_2 &= \int \sqrt{\left(x-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \, dx \\
 &= \frac{1}{2} \left\{ \left(x-\frac{1}{2}\right) \sqrt{x^2-x+1} + \frac{3}{4} \sinh^{-1} \frac{x-\frac{1}{2}}{\sqrt{3}/2} \right\} + c \\
 &= \frac{1}{2} \left\{ \frac{2x-1}{2} \sqrt{x^2-x+1} + \frac{3}{4} \sinh^{-1} \frac{2x-1}{\sqrt{3}} \right\} + c \\
 \therefore I &= \frac{2}{3} (x^2-x+1)^{3/2} + \frac{2x-1}{2} \sqrt{x^2-x+1} + \frac{3}{4} \sinh^{-1} \frac{2x-1}{\sqrt{3}} + c
 \end{aligned}$$

Exercises 5(2)

Compute the following :

- | | |
|--|--|
| 1. $\int \sqrt{1-9x^2} \, dx$ | 2. $\int \sqrt{1-x^2} \, dx$ |
| 3. $\int \sqrt{36-25x^2} \, dx$ | 4. $\int \sqrt{1-x^2} \, dx$ |
| 5. $\int \sqrt{t^2-2t+5} \, dx$ | 6. $\int \sqrt{s^2-2s+10} \, ds$ |
| 7. $\int \sqrt{3-2u-u^2} \, du$ | 8. $\int \sqrt{2ax-x^2} \, dx$ |
| 9. $\int \sqrt{(2-x)(x-4)} \, dx$ | 10. $\int (x+1)\sqrt{x^2-1} \, dx$ |
| 11. $\int (2x-3)\sqrt{4x^2-4x-1} \, dx$ | 12. $\int (x-3)\sqrt{\frac{2x^3-3x+1}{x-1}} \, dx$ |
| 13. $\int (x-2)\sqrt{\frac{x+1}{x-2}} \, dx$ | 14. $\int \frac{x^2+x+1}{\sqrt{x^2+2x+3}} \, dx$ |

$$15. \int \frac{x^2 + x + 1}{\sqrt{x^2 + 1}} dx$$

$$16. \int \frac{x^2 + 4x + 3}{\sqrt{x^2 + 4x + 5}} dx$$

$$17. \int \frac{x^3 + 6x^2 + 9x - 1}{\sqrt{x^2 + 4x + 1}} dx$$

18. If $P_n(x)$ be a polynomial of degree n , then show that

$$\int P_n(x) e^{kx} dx = e^{kx} \left[\frac{P_n(x)}{k} - \frac{P_n'(x)}{k^2} + \frac{P_n''(x)}{k^3} - \dots + (-1)^{n-1} \frac{P_n^{(n-1)}(x)}{k^n} - \frac{(-1)^{(n-1)}}{k^{n+1}} P_n^{(n)}(x) \right] + C.$$

19. Prove that

$$\int (2x^3 + 3x^2 - 8x + 1) \sqrt{2x + 6} dx = \frac{\sqrt{2x+6}}{5.79} (2x+6) (70x^3 - 45x^2 - 396x + 897) + C.$$

20. Prove that

$$\int (x^3 - 2x + 3) \log x dx = \left(\frac{x^3}{3} - x^2 + 3x \right) \log x - \frac{x^3}{9} + \frac{x^2}{2} - 3x + C$$

21. Show that

$$\int \frac{x^2 - 7x + 1}{(2x + 1)^{1/3}} dx = \frac{3}{4} (x^2 - 7x + 1) (2x + 1)^{2/3} - \frac{9}{40} (2x - 7) (2x + 1)^{5/3} + C.$$

22. Prove that

$$(i) \int \frac{dx}{x \sqrt{(a^2 - x^2)}} = -\frac{1}{a} \operatorname{sech}^{-1} \frac{x}{a} + C$$

$$(ii) \int \frac{dx}{x \sqrt{x^2 + a^2}} = -\frac{1}{a} \operatorname{cosech}^{-1} \frac{x}{a} + C$$

$$(iii) \int \sinh^{-1} x dx = x \sinh^{-1} x - \sqrt{1 + x^2} + C$$

$$(iv) \int \operatorname{sech}^3 x dx = \frac{1}{2} \operatorname{sech} x \tanh x + \frac{1}{2} \tanh^{-1} (\sinh x) + C$$

$$(v) \int x^2 \cosh x dx = (x^2 + 2) \sinh x + 2x \cosh x + C$$

23. Prove that

$$\int x^2 \sqrt{a^2 + x^2} dx = \frac{1}{8} x (a^2 + 2x^2) \sqrt{a^2 + x^2} - \frac{a^4}{8} \sinh^{-1} \frac{x}{a} + C$$

24. Show that

$$\int \frac{x^2}{\sqrt{a^2+x^2}} dx = -\frac{a^2}{2} \sinh^{-1} \frac{x}{a} + \frac{x\sqrt{a^2+x^2}}{2} + C$$

25. Prove that

$$\int x^2 \sqrt{x^2 - a^2} dx = \frac{x}{8} (2x^2 - a^2) \sqrt{x^2 - a^2} - \frac{a^4}{8} \cosh^{-1} \frac{x}{a} + C$$

Answers

1. $\frac{x}{2} \sqrt{1-9x^2} + \frac{1}{6} \sin^{-1} 3x + C$
2. $\frac{x}{2} \sqrt{1+4x^2} + \frac{1}{4} \sinh^{-1} 2x + C$
3. $\frac{x}{2} \sqrt{36-25x^2} + \frac{18}{5} \sin^{-1} \frac{5x}{6} + C$
4. $\frac{x}{2} \sqrt{9x^2+4} + \frac{4}{3} \sinh^{-1} \left(\frac{3x}{2} \right) + C$
5. $\frac{t-1}{2} \sqrt{t^2-2t+5} + 2 \sinh^{-1} \left(\frac{t-1}{2} \right) + C$
6. $\frac{s-1}{2} \sqrt{s^2-2s+10} + \frac{9}{2} \sinh^{-1} \left(\frac{s-1}{3} \right) + C$
7. $\frac{u+1}{2} \sqrt{3-2u-u^2} + 2 \sin^{-1} \frac{u+1}{2} + C$
8. $\frac{x-a}{2} \sqrt{2ax-x^2} + \frac{a^2}{2} \sin^{-1} \frac{x-a}{2} + C$
9. $\frac{x-3}{2} \sqrt{(2-x)(x-4)} + \frac{1}{2} \sin^{-1} (x-3) + C$
10. $\frac{1}{3} (x^2-1)^{3/2} + \frac{x}{2} \sqrt{x^2-1} - \frac{1}{2} \cosh^{-1} x + C$
11. $\frac{1}{6} (4x^2-4x-1)^{3/2} - \frac{1}{2} (2x-1) \sqrt{4x^2-4x-1} + \cosh^{-1} \left(\frac{2x-1}{2} \right) + C$
12. $\frac{1}{6} (2x^2+2x-1)^{3/2} - \frac{7}{\sqrt{2}} \left(\frac{2x+1}{4} \cdot \sqrt{x^2+x-(1/2)} \right) + \frac{21}{8\sqrt{2}} \cosh^{-1} \left(\frac{2x+1}{\sqrt{3}} \right) + C$
13. $\frac{2x-1}{4} \sqrt{x^2-x-2} - \frac{9}{8} \cosh^{-1} \left(\frac{2x-1}{3} \right) + C$
14. $\frac{x-1}{2} \sqrt{x^2+2x+3} + C$
15. $\left(\frac{x}{2} + 1 \right) \sqrt{1+x^2} + \frac{1}{2} \sinh^{-1} x + C$
16. $\frac{1}{2} (x+2) \sqrt{x^2+4x+5} - \frac{3}{2} \sinh^{-1} (x+2) + C$
17. $\frac{1}{3} (x^2+4x+1)^{3/2} - 3 \cosh^{-1} \left(\frac{x+2}{\sqrt{3}} \right) + C$

5.7 INTEGRATION BY ELIMINATING SURDS

Integration of some special irrational function by substitution has already discussed in chapter 4 and secs. 2.8 (C), (D), 5.5 and 5.6. There are some more functions which need new types of substitutions. Some of them are the following :

(i) INTEGRAL OF THE TYPE

$$\int R(x, x^{p_1/q_1}, x^{p_2/q_2}, \dots, x^{p_k/q_k}) dx$$

i.e. integrand is a rational function of x and rational exponent of x . No previous substitution is valid here. The surds of the variables can be removed by one substitutions, $x = t^m$, where m is the L.C.M. of $q_1, q_2, \dots, q_n$. Now the integrand can be integrated.

(ii) INTEGRAL OF THE TYPE

$$\int R \left\{ x, \left(\frac{ax+b}{cx+d} \right)^{p_1/q_1}, \left(\frac{ax+b}{cx+d} \right)^{p_2/q_2}, \dots, \left(\frac{ax+b}{cx+d} \right)^{p_k/q_k} \right\} dx$$

i.e. integrand is a rational function of x and rational exponent of $\frac{ax+b}{cx+d}$. Here also no previous substitution is valid. As in (i) remove the surd by assuming $t^m = \frac{ax+b}{cx+d}$, where m is the L.C.M. of $q_1, q_2, \dots, q_k$.

(iii) INTEGRAL OF THE TYPE $\int R(x, \sqrt{ax+b}, \sqrt{a_1x+b_1}) dx$

i.e., integrand is a rational function of $x, \sqrt{ax+b}$ and $\sqrt{a_1x+b_1}$. Here an extraordinary substitution is made. Let $x = ly + m$ where l and m are such that $\sqrt{ax+b} = \sqrt{al} \sqrt{y+2}$ and $\sqrt{a_1x+b_1} = \sqrt{a_1l} \sqrt{y-2}$.

Then $l = \frac{a_1b - ab_1}{4aa_1}$ and $m = -\frac{a_1b + ab_1}{2aa_1}$. Therefore the given integral

reduces to the form $R_1(y, \sqrt{y+2}, \sqrt{y-2})$. Again on using $y = t^2 + \frac{1}{t^2}$, the original integral

$$\int R(x, \sqrt{ax+b}, \sqrt{a_1x+b_1}) dx = 2 \int R_1 \left(t^2 + \frac{1}{t^2}, t + \frac{1}{t}, t - \frac{1}{t} \right) \cdot \left(t - \frac{1}{t^3} \right) dt.$$

This can be integrated by P.F. or by expansion.

(iv) INTEGRAL OF THE TYPE $\int R(x, \sqrt[n]{ax+b}) dx, n > 0$ integer.

i.e., integrand is a rational function of x and $\sqrt[n]{ax+b}$; $n > 0$ integer. Here substitution of $ax+b = t^n$ simplifies to rational function of t and can be integrated by P.F. or expansion.

(v) INTEGRAL OF THE TYPE $\int \sqrt{\frac{x-\alpha}{x-\beta}} dx$ or

$$\int \sqrt{(x-\alpha)(x-\beta)} dx \text{ AND } \int \sqrt{\frac{x-a}{\beta-x}} dx \text{ or } \int \sqrt{(x-\alpha)(\beta-x)} dx.$$

The integrals $\int \sqrt{(x-\alpha)(x-\beta)} dx$ as well as $\int \sqrt{(x-\alpha)(\beta-x)} dx$ were treated in sec. 4.4 and used substitutions like $x-\alpha=u^2$ or $x-\beta=u^2$ or $\beta-x=u^2$. But these substitution did not remove the surd wholly, yet the integrand was made integrable. Here we discuss the substitution for total removal of surds.

(a) The irrational integrand $\sqrt{(x-\alpha)(x-\beta)} = (x-\alpha) \sqrt{\frac{x-\beta}{x-\alpha}}$

or $(x-\beta) \sqrt{\frac{x-\alpha}{x-\beta}}$.

Hence substitution of either $\frac{x-\alpha}{x-\beta} = t^2$ or $\frac{x-\beta}{x-\alpha} = t^2$ completely removes the surd i.e. the irrationality. Other substitutions for the same purpose are :

(1) $x = \alpha \sec^2 \theta - \beta \tan^2 \theta$ with $dx = 2(\alpha - \beta) \sec^2 \theta \tan \theta d\theta$

(2) $x = \alpha \cosh^2 \theta - \beta \sinh^2 \theta$ with $dx = 2(\alpha - \beta) \sinh \theta \cosh \theta d\theta$

(b) The irrational integrand $\sqrt{(x-\alpha)(\beta-x)} = (x-\alpha) \sqrt{\frac{\beta-x}{x-\alpha}}$

or $(\beta-x) \sqrt{\frac{x-\alpha}{\beta-x}}$.

Hence substitution of either $\frac{\beta-x}{x-\alpha} = t^2$ or $\frac{x-\alpha}{\beta-x} = t^2$

completely removes the surd i.e. the irrationality. Other substitution serving the same purpose is

$x = \alpha \cos^2 \theta + \beta \sin^2 \theta$ with $dx = 2(\beta - \alpha) \sin \theta \cos \theta d\theta$ only.

(iv) INTEGRAL OF THE TYPE $\int R(x, \sqrt{ax^2 + bx + c}) dx$

The integrand is a rational function of x and $\sqrt{ax^2 + bx + c}$, an irrational quadratic expression. This can be integrated by using any one of the following three *Euler* substitutions.

(1) $\sqrt{ax^2 + bx + c} = t \pm x \sqrt{a}$ if $a > 0$

(2) $\sqrt{ax^2 + bx + c} = tx \pm \sqrt{c}$ if $c > 0$

(3) $\sqrt{ax^2 + bx + c} = (x-\alpha)$ if $ax^2 + bx + c \equiv a(x-\alpha)(x-\beta)$

i.e. α is a real root of the quadratic equation.

Simpler types of this integral have been already discussed in secs. 4.3, 4.6, 4.7, 4.8, 4.9, 4.13, 5.5 & 5.6. Other than those simple types should be considered by Euler substitution. This procedure lead often to cumbersome calculations and apparently gives two results for two different signs. But further calculation gives identical results. So any one of the double signs may be considered without bothering for result given in the book.

Advantage of this substitution is that the integrand becomes a rational function of t after simplification and the integration can be performed using PF or other types of expansion (Ref. Ex. 2 and 3).

Examples

1. Integrate $\frac{(x+1)}{(x+2)(x+3)^{3/2}}$.

The integrand is an irrational function of $(x+3)^{3/2}$ and x and is of the type 5.7 (iv). So, let us substitute $x+3 = t^2$ or $dx = 2t dt$. Therefore,

$$\begin{aligned}\int \frac{(x+1) dx}{(x+2)(x+3)^{3/2}} &= \int \frac{(t^2-2)}{(t^2-1)t^3} 2t dt + C = 2 \int \frac{t^2-2}{t^2(t^2-1)} dt + C \\ &= 2 \left(\frac{2}{t^2} - \frac{1}{t^2-1} \right) dt + C \quad \because \text{powers of } t \text{ are even. So, by PF} \\ &= -\frac{4}{t} - \log \frac{t+1}{t-1} + C = -\frac{4}{\sqrt{x+3}} - \log \frac{\sqrt{x+3}-1}{\sqrt{x+3}+1} + C.\end{aligned}$$

2. Compute $\int \frac{dx}{1+\sqrt{x^2+2x+2}}$.

This is of type 5.7 (vi) (1) as $a=1>0$. Therefore the Euler substitution is

$$\sqrt{x^2+2x+2} = t+x \text{ (say) taking } +\text{ve sign.}$$

Squaring both sides and simplifying

$$x = \frac{t^2-2}{2(1-t)} \text{ or } dx = \frac{2t-t^2-2}{2(1-t)^2} dt$$

$$\text{Also } 1+\sqrt{x^2+2x+2} = (1+t) + \frac{t^2-2}{2(1-t)} = \frac{-t^2}{2(1-t)}$$

$$\begin{aligned}\therefore \int \frac{dx}{1+\sqrt{x^2+2x+2}} &= - \int \frac{2(1-t)}{t^2} \cdot \frac{2t-t^2-2}{2(1-t)^2} dt + C \\ &= \int \frac{t^2-2t+2}{t^2(1-t)} dt + C = \int \left(\frac{1}{1-t} + \frac{2}{t^2} \right) dt + C \text{ by PF} \\ &= -\log(1-t) - 2/t + C \\ &= -\log(-\sqrt{x^2+2x+2} + x+1) - \frac{2}{(\sqrt{x^2+2x+2}-x)} + C \\ &= \log(x+1+\sqrt{x^2+2x+2}) - \frac{2}{\sqrt{x^2+2x+2}-x} + C_1\end{aligned}$$

Note : If -ve sign is considered in substitution, result seems to be different. But further simplification gives same result.

3. Find $\int \frac{x dx}{(\sqrt{7x-10-x^2})^3}$

Here both $a = -1 < 0$ and $c = -10 < 0$ and roots $\alpha = 2, \beta = 5$ are real. So, it is of type 5.7 (vi) (3). Therefore the Euler substitution is

$$\sqrt{7x-10-x^2} = \sqrt{(x-2)(5-x)} = (x-2)t$$

Squaring and cancelling $(x-2)$, $5-x = (x-2)t^2$

$$\therefore x = \frac{5+2t^2}{1+t^2}, \quad dx = -\frac{6t dt}{(1+t^2)^2}$$

$$\begin{aligned} \therefore \int \frac{x dx}{(\sqrt{7x-10-x^2})^3} &= -\frac{6}{27} \int \frac{5+2t^2}{t^2} dt + C = -\frac{2}{9} \int \left(\frac{5}{t^2} + 2 \right) dt + C \\ &= -\frac{2}{9} \left(-\frac{5}{t} + 2t \right) + C \text{ where } t = \frac{\sqrt{7x-10-x^2}}{x-2} \end{aligned}$$

4. Find the integral $\int \frac{x+1}{1+\sqrt{x+1}+\sqrt{x-1}} dx$.

The integrand is of type 5.7 (iii). So, let $x = ly + m$ where $\sqrt{x+1} = \sqrt{l} \sqrt{y+2}$ and $\sqrt{x-1} = \sqrt{l} \sqrt{y-2}$ l, m are such that

i.e. $x+1 = l(y+2)$ and $x-1 = l(y-2)$.

or $ly+m+1 = l(y+2)$ and $ly+m-1 = l(y-2)$ $\therefore x = ly+m$.

or $m+1 = 2l$ and $m-1 = -2l$

$$\therefore m = 0, l = \frac{1}{2}$$

$$\therefore x = \frac{1}{2}y \text{ and } dx = \frac{1}{2}dy$$

$$\begin{aligned} \therefore I &= \int \frac{x+1}{1+\sqrt{x+1}+\sqrt{x-1}} dx \\ &= \frac{1}{2} \int \frac{\frac{1}{2}y+1}{1+\sqrt{(1/2)y+1}+\sqrt{(1/2)y-1}} dy + C \end{aligned}$$

Now, let $y = t^2 + \frac{1}{t^2}$, $\therefore dy = 2 \left(t - \frac{1}{t^3} \right) dt + C$

$$\therefore I = \frac{1}{2} \int \frac{\frac{1}{2} \left(t^2 + \frac{1}{t^2} \right) + 1}{1 + \sqrt{\frac{1}{2} \left(t^2 + \frac{1}{t^2} \right) + 1} + \sqrt{\frac{1}{2} \left(t^2 + \frac{1}{t^2} \right) - 1}} \cdot 2 \left(t - \frac{1}{t^3} \right) dt + C$$

$$\begin{aligned}
&= \frac{1}{2} \int \frac{\left(t + \frac{1}{t}\right)^2 \left(t - \frac{1}{t^3}\right)}{1 + \sqrt{\frac{1}{2} \left(t + \frac{1}{t}\right)^2} + \sqrt{\frac{1}{2} \left(t - \frac{1}{t}\right)^2}} dt + C \\
&= \frac{1}{2} \int \frac{\left(t + \frac{1}{t}\right)^2 \left(t - \frac{1}{t^3}\right)}{1 + \frac{1}{\sqrt{2}} \left(t + \frac{1}{t}\right) + \frac{1}{\sqrt{2}} \left(t - \frac{1}{t}\right)} dt + C \\
&= \frac{1}{2} \int \frac{\left(t + \frac{1}{t}\right)^2 \left(t - \frac{1}{t^3}\right)}{1 + \frac{2t}{\sqrt{2}}} dt + C = \int \frac{\left(t + \frac{1}{t}\right)^2 \left(t - \frac{1}{t^3}\right)}{2(1 + \sqrt{2}t)} dt + C \\
&= \int \frac{1}{2} \left(\frac{t^2}{\sqrt{2}} - \frac{t}{2} + \frac{5}{2\sqrt{2}} - \frac{5}{2\sqrt{2}(1 + \sqrt{2}t)} - \frac{2}{t^3(1 + \sqrt{2}t)} - \frac{1}{t^5(1 + \sqrt{2}t)} \right) dt + C \\
&= \frac{1}{2} \left[\frac{t^3}{3\sqrt{2}} - \frac{1}{4} + \frac{5t}{2\sqrt{2}} - 8 \log t - \frac{4\sqrt{2}}{t} + 2 \right. \\
&\quad \left. - \frac{\sqrt{2}}{3t^3} + \frac{1}{4t^4} + \frac{27}{4} \log(1 + \sqrt{2}t) \right] + C_1 \text{ using P.F.} \\
&= \frac{1}{2} \left[\frac{1}{12} (\sqrt{x+1} + \sqrt{x-1})^3 - \frac{x + \sqrt{x^2-1}}{4} + \frac{5}{4} (\sqrt{x+1} + \sqrt{x-1}) \right. \\
&\quad \left. + \frac{27}{4} \log(1 + \sqrt{x+1} + \sqrt{x-1}) - 8 \log(\sqrt{x+1} + \sqrt{x-1}) \right. \\
&\quad \left. - \frac{4}{3(\sqrt{x+1} + \sqrt{x-1})^3} + \frac{1}{4(x + \sqrt{x^2-1})^2} - \frac{8}{\sqrt{x+1} + \sqrt{x-1}} + \frac{2}{x + \sqrt{x^2-1}} \right] + C_2 \\
&\left[\because x = \frac{1}{2}y = \frac{1}{2} \left(t^2 + \frac{1}{t^2} \right) = \frac{t^4 + 1}{2t^2} \therefore \frac{x+1}{x-1} = \frac{(t^2+1)^2}{(t^2-1)^2} \right. \\
&\text{or } \frac{\sqrt{x+1}}{\sqrt{x-1}} = \frac{t^2+1}{t^2-1} \text{ or } t^2 = \frac{\sqrt{x+1} + \sqrt{x-1}}{\sqrt{x+1} - \sqrt{x-1}} = \frac{(\sqrt{x+1} + \sqrt{x-1})^2}{2} \\
&\left. \therefore t = \frac{\sqrt{x+1} + \sqrt{x-1}}{\sqrt{2}} \right]
\end{aligned}$$

Exercises 5 (3)

1. $\int \frac{x + \sqrt[3]{x^2} + \sqrt[6]{x}}{x(1 + \sqrt[3]{x})} dx$
2. $\int \frac{(2x-3)^{1/2}}{(2x-3)^{1/3} + 1} dx$
3. $\int \frac{2}{(2-x)^2} \sqrt[3]{\frac{2-x}{2+x}} dx$
4. $\int \frac{dx}{\sqrt[4]{(x-1)^3(x+2)^5}}$
5. $\int \frac{dx}{x^{1/2} - x^{1/4}}$
6. $\int \frac{x dx}{\sqrt{x+1} + \sqrt{x-1}}$
7. $\int \frac{x dx}{(1+x)^{1/3} - (1+x)^{1/2}}$
8. $\int \frac{dx}{x - \sqrt{x^2 + 2x + 4}}$
9. $\int \frac{dx}{(1+x)\sqrt{1+x-x^2}}$
10. $\int \frac{dx}{x + \sqrt{x^2 - x + 1}}$
11. $\int \frac{x dx}{(x^2 - 3x + 2)\sqrt{x^2 - 4x + 3}}$
12. $\int \frac{dx}{(x+1)^3 \sqrt{x^2 + 3x + 2}}$

Answers

1. $m=6, x=t^6, \frac{3}{2}x^{2/3} + 6 \tan^{-1}(\sqrt[6]{x}) + C$
2. $m=6, (2x-3)=t^6, 3 \left\{ \frac{1}{7}(2x-3)^{7/6} - \frac{1}{5}(2x-3)^{5/6} + \frac{1}{3}(2x-3)^{1/2} - (2x-3)^{1/6} + \tan^{-1}(2x-3)^{1/6} \right\} + C$
3. $\frac{2-x}{2+x}=t^3, \frac{3}{4} \sqrt[3]{\left(\frac{2+x}{2-x}\right)^2} + C$
4. Write as $(x-1)(x+2) \sqrt[4]{\frac{x+2}{x-1}}$, put $\frac{x+2}{x-1}=t^4, \frac{4}{3} \sqrt[4]{\frac{x-1}{x+2}} + C$
5. $2x^{1/2} + 4x^{1/4} + 4 \log(x^{1/4} - 1) + C$
6. $\frac{(\sqrt{x+1} + \sqrt{x-1})^3}{24} + \frac{2}{5(\sqrt{x+1} + \sqrt{x-1})^5} + C$
7. $-6 \left\{ \frac{(1+x)^{3/2}}{9} + \frac{(1+x)^{4/3}}{8} + \frac{(1+x)^{7/6}}{7} + \frac{1+x}{6} + \frac{(1+x)^{5/6}}{5} + \frac{(1+x)^{2/3}}{4} \right\} + C$
8. $2 \log \left(\sqrt{x^2 + 2x + 4} - x \right) - \frac{3}{2 \left(\sqrt{x^2 + 2x + 4} - x - 1 \right)} - \frac{3}{2} \log \left(\sqrt{x^2 + 2x + 4} - x - 1 \right) + C$
9. $-2 \tan^{-1} \left(\frac{\sqrt{1+x-x^2}}{x} + 1 \right) + C$
10. $2 \log t - \frac{1}{2} \log(t-1) + \frac{3}{t+1} - \frac{3}{2} \log(t+1) + C, t = \frac{\sqrt{x^2 - x + 1} + 1}{x}$
11. $-\frac{\sqrt{x^2 - 4x + 3}}{x-1} - 2 \sin^{-1} \left(\frac{1}{x-2} \right) + C$
12. $-\frac{2}{15} \sqrt{\frac{x+2}{x+1}} \cdot \frac{8x^2 + 12x + 7}{(x+1)^2} + C$

Hints: 6. w.o. Ex. 4, 7. $1+x=t^6$, 8. $\sqrt{x^2 + 2x + 4} = t + x$, 9., 11., 12. w.o. 3.

SUCCESSIVE REDUCTIONS

6.1 INTRODUCTION

The process relating one integral to one or more integrals of the same type but simpler is called a reduction and the relation is called a reduction formula. The process arises when the integrand has higher order exponents. These exponents are reduced to a simplest one by this process step by step and so the process is successive reductions.

Methods of derivation : The reduction formula is derived by two different methods. One is *integration by parts* and the other is *differentiation of a suitable function*, say, $P(x)$. This $P(x)$ is so chosen that $\frac{dP}{dx}$ has at least one function which is the integrand of the given integral whose reduction formula is needed. The second method consists of three steps :

- (i) selection of one $P(x)$,
- (ii) differentiation of $P(x)$,
- (iii) integration of step (ii) above

This second method is illustrated in deriving the reduction formula 6.14 and suggested in 6.15 and in some problems in exercises.

6.2 REDUCTION FORMULAE FOR $\int \sin^m x \cos^n x dx$

Some *particular* cases have already described in sec. 2.11. Now *general* case for m and n is given below through its reduction formulae. These are

$$(A) \quad I_{m,n} = \int \sin^m x \cos^n x dx \\ = \frac{\cos^{n-1} x \sin^{m+1} x}{m+n} + \frac{n-1}{m+n} I_{m,n-2}, \quad m+n \neq 0$$

$$(B) \quad I_{m,n} = \int \sin^m x \cos^n x dx \\ = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} I_{m-2,n}, \quad m+n \neq 0$$

Proof of (A) :

This formula *reduces* the power or exponent m of *cosine* in each successive steps by 2. To establish the relation, one cosine is separated from the product so that $(\sin^m x \cos x)$ can be integrated and can be treated as a second function of the integration by parts. Therefore, I.b.p. gives

$$\begin{aligned}
 I_{m,n} &= \cos^{n-1} x \int \sin^m x \cos x \, dx - \int \left[\frac{d}{dx} (\cos^{n-1} x) \int \sin^m x \cos x \, dx \right] dx \\
 &= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \cos^{n-2} x \sin^{m+2} x \, dx \\
 &= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \cos^{n-2} x \sin^m x (1 - \cos^2 x) \, dx \\
 &= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2} - \frac{n-1}{m+1} I_{m,n} \\
 \text{or } \left(1 + \frac{n-1}{m+1} \right) I_{m,n} &= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2} \\
 \text{or } I_{m,n} &= \frac{\cos^{n-1} x \sin^{m+1} x}{m+n} + \frac{n-1}{m+n} I_{m,n-2}, \quad m+n \neq 0.
 \end{aligned}$$

Using this relation successively the product will be reduced to the form $\int \sin^p x \, dx$ which can be integrated completely by using reduction formula in sec. 6.3 below.

Proof of (B)

This formula *reduces* the power or exponent of *sine* in each successive step by 2. To establish the formula, one sine is separated from the product, so that $(\cos^n x \sin x)$ can be integrated and can be treated as a second function of the integration by parts. Therefore following the above procedure (A). The required reduction formula comes as

$$I_{m,n} = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} I_{m-2,n}, \quad m+n \neq 0$$

Just as (A) with interchanges of m, n and sine, cosine and placing a (-ve) sign in front of the first term. After complete reduction of the product, use 6.3 for the rest.

6.3 REDUCTION FORMULAE FOR $\int \sin^m x \, dx$ and $\int \cos^n x \, dx$.

(A) In sec. 6.2, put $m=0$ in (A), so

$$I_n = \int \cos^n x \, dx = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n} I_{n-2}, \quad n \neq 0$$

The process of separating of one $\cos x$ and integration by parts also give the same result.

(B) Similarly, in sec. 6.2, put $n \neq 0$ in (B), so

$$I_m = \int \sin^m x \, dx = -\frac{\sin^{m-1} x \cos x}{m} + \frac{m-1}{m} I_{m-2}, \quad m \neq 0$$

6.4 REDUCTION FORMULAE FOR

$$\int \frac{dx}{\sin^m x \cos^n x} = \int \operatorname{cosec}^m x \sec^n x dx.$$

This type occurs when both m and n are $-ve$ in $\int \sin^m x \cos^n x dx$.

Let us replace n in 6.2 (A) above by $(n+2)$.

$$\therefore I_{m, n+2} = \int \sin^m x \cos^{n+2} x = \frac{\sin^{m+1} x \cos^{n+1} x}{m+n+2} + \frac{n+1}{m+n+2} I_{m, n}$$

In this relation, let us replace m by $(-m)$ and n by $(-n)$, we get,

$$\begin{aligned} I_{-m, -n+2} &= \int \frac{dx}{\sin^m x \cos^{n-2} x} \\ &= -\frac{1}{m+n-2} \cdot \frac{1}{\sin^{m-1} x \cos^{n-1} x} + \frac{n-1}{m+n-2} I_{-m, -n} \end{aligned}$$

$$\begin{aligned} \text{or } I_{-m, -n} &= \int \frac{dx}{\sin^m x \cos^n x} \\ &= \frac{1}{(n-1) \sin^{m-1} x \cos^{n-1} x} + \frac{m+n-2}{n-1} I_{-m, -n+2}, n \neq 1 \quad (C) \end{aligned}$$

Similarly, from 6.2 (B), we get

$$\begin{aligned} I_{-m, -n} &= \int \frac{dx}{\sin^m x \cos^n x} \\ &= -\frac{1}{(m-1) \sin^{m-1} x \cos^{n-1} x} + \frac{m+n-2}{m-1} I_{-m+2, -n}, m \neq 1 \quad (D) \end{aligned}$$

It is evident from (C) and (D) that powers of cosine and sine are respectively reduced by 2 in each step of reduction.

Putting $m=0$ in (C) and $n=0$ in (D) it is easy to obtain

$$I_{-n} = \int \frac{1}{\cos^n x} dx = \int \sec^n x dx = \frac{\sin x}{(n-1) \cos^{n-1} x} + \frac{n-2}{n-1} I_{-n+2}, n \neq 1 \quad (E)$$

$$\begin{aligned} \text{and } I_{-m} &= \int \frac{1}{\sin^m x} dx = \int \operatorname{cosec}^m x dx \\ &= -\frac{\cos x}{(m-1) \sin^{m-1} x} + \frac{m-2}{m-1} I_{-m+2}, m \neq 1 \quad (F) \end{aligned}$$

Equivalent forms of (C), (D), (E) and (F) are

$$I_{m, n} = \int \operatorname{cosec}^m x \sec^n x dx = \frac{\operatorname{cosec}^{m-1} x \sec^{n-1} x}{n-1} + \frac{m+n-2}{n-1} I_{m, n-2} \quad (C_1)$$

$$I_{m, n} = \int \operatorname{cosec}^m x \sec^n x dx = -\frac{\operatorname{cosec}^{m-1} x \sec^{n-1} x}{m-1} + \frac{m+n-2}{m-1} I_{m-2, n} \quad (D_1)$$

$$I_n = \int \sec^n x dx = \frac{\tan x \sec^{n-2} x}{n-1} + \frac{n-2}{n-1} I_{n-2} \quad (E_1)$$

$$I_m = \int \operatorname{cosec}^m x dx = -\frac{\cot x \operatorname{cosec}^{m-2} x}{m-1} + \frac{m-2}{m-1} I_{m-2} \quad (F_1)$$

The formulae (C), (C₁) & (E₁) are valid when $n \neq 1$ and those of (D), (D₁) and (F₁) are valid when $m \neq 1$,

6.5 REDUCTION FORMULAE FOR

$$\int \frac{\sin^n x dx}{\cos^n x} dx = \int \sin^n x \sec^n x dx \text{ AND}$$

$$\int \frac{\cos^n x}{\sin^n x} dx = \int \cos^n x \operatorname{cosec}^n x dx.$$

These types of integrals occur when any one of m or n is -ve in $\int \sin^m x \cos^n x dx$. Let

$$\begin{aligned} I_{m, -n} &= \int \frac{\sin^m x}{\cos^n x} dx = \int \frac{\sin^{m-1} x \sin x}{\cos^n x} dx \\ &= \frac{\sin^{m-1} x}{(n-1) \cos^{n-1} x} - \frac{m-1}{n-1} \int \frac{\sin^{m-2} x}{\cos^{n-2} x} dx, \text{ (I.b.p.)} \\ &= \frac{\sin^{m-1} x}{(n-1) \cos^{n-1} x} - \frac{m-1}{n-1} I_{m-2, -n+2}, n \neq 1 \end{aligned} \quad (1)$$

Similarly,

$$I_{-m, n} = \int \frac{\cos^n x}{\sin^m x} dx = -\frac{\cos^{n-1} x}{(m-1) \sin^{m-1} x} - \frac{n-1}{m-1} I_{-m+2, n-2}, m \neq 1 \quad (2)$$

In the above two cases powers of the numerator have been reduced. In the first case a sine and in the second a cosine of the numerator has been separated and integrated respectively. As a result the powers of both numerator and denominator have been reduced by two in both cases.

Equivalent formulae for (1) and (2) are

$$I_{m, n} = \int \sin^m x \sec^n x dx = \frac{\sin^{m-1} x \sec^{n-1} x}{n-1} - \frac{m-1}{n-1} I_{m-2, n-2}, n \neq 1$$

$$\text{and } I_{m, n} = \int \operatorname{cosec}^m x \cos^n x dx$$

$$= -\frac{\operatorname{cosec}^{m-1} x \cos^{n-1} x}{m-1} - \frac{n-1}{m-1} I_{m-2, n-2}, m \neq 1$$

6.6 REDUCTION FORMULAE FOR $\int \tan^n x dx$ AND $\int \cot^n x dx$.

These can easily be obtained from 6.5 above by setting $m=n$. Other procedures are shown in sec. 2.13.

6.7 REDUCTION FORMULAE FOR

$$\int \tanh^n x \, dx \text{ AND } \int \coth^n x \, dx$$

$$(A) \text{ Let } I_n = \int \tanh^n x \, dx = \int \tanh^{n-2} x \tanh^2 x \, dx$$

$$= \int \tanh^{n-2} x (1 - \operatorname{sech}^2 x) \, dx = \int \tanh^{n-2} x \, dx - \frac{\tanh^{n-1} x}{n-1}$$

$$= I_{n-2} - \frac{\tanh^{n-1} x}{n-1}, \quad n \neq 1.$$

$$(B) \text{ Similarly, } I_n = \int \coth^n x \, dx = I_{n-2} - \frac{\coth^{n-1} x}{n-1}, \quad n \neq 1.$$

6.8 REDUCTION FORMULAE FOR $\int \sinh^m x \cosh^n x \, dx$

Reduction formulae for this integral are identical to those of 6.2 and can be derived in the same way as follows :

$$I_{m,n} = \int \sinh^m x \cosh^n x \, dx = \int \sinh^{m-1} x (\cosh^n x \sinh x) \, dx$$

$$= \sinh^{m-1} x \frac{\cosh^{n+1} x}{n+1} - \frac{m-1}{n+1} \int \sinh^{m-2} x \cosh^{n+2} x \, dx, \quad (\text{l.b.p.})$$

$$= \frac{\sinh^{m-1} x \cosh^{n+1} x}{n+1} - \frac{m-1}{n+1} \int \sinh^{m-2} x \cosh^n x (1 + \sinh^2 x) \, dx$$

$$= \frac{\sinh^{m-1} x \cosh^{n+1} x}{n+1} - \frac{m-1}{n+1} I_{m-2,n} - \frac{m-1}{n+1} I_{m,n}$$

$$\text{or } \left(1 + \frac{m-1}{n+1}\right) I_{m,n} = \frac{\sinh^{m-1} x \cosh^{n+1} x}{n+1} - \frac{m-1}{n+1} I_{m-2,n}$$

$$\text{or } I_{m,n} = \frac{\sinh^{m-1} x \cosh^{n+1} x}{m+n} - \frac{m-1}{m+n} I_{m-2,n}, \quad m+n \neq 0 \quad (A)$$

which reduces power of $\sinh x$ by two in each successive steps. Similarly, the formula below reduces power of $\cosh x$ by two in each successive step :

$$I_{m,n} = \frac{\sinh^{m+1} x \cosh^{n-1} x}{m+n} + \frac{n-1}{m+n} I_{m,n-2}, \quad m+n \neq 0 \quad (B)$$

If $n=0$ in (A) and $m=0$ in (B) then

$$I_m = \int \sinh^m x \, dx = \frac{\sinh^{m-1} x \cosh x}{m} - \frac{m-1}{m} I_{m-2}, \quad m \neq 0 \quad (C)$$

$$\text{and } I_n = \int \cosh^n x \, dx = \frac{\cosh^{n-1} x \sinh x}{n} + \frac{n-1}{n} I_{n-2}, \quad n \neq 0 \quad (D)$$

6.9 REDUCTION FORMULAE FOR $\int \tan^m x \sec^n x dx$ AND $\int \cot^m x \operatorname{cosec}^n x dx$.

(A) To reduce the power of $\tan x$, write the integral as

$$\begin{aligned} I_{m,n} &= \int \tan^m x \sec^n x dx = \int \tan^{m-1} x (\sec^{n-1} x \sec x \tan x) dx \\ &= \tan^{m-1} x \frac{\sec^n x}{n} - \frac{m-1}{n} \int \tan^{m-2} x \sec^2 x \sec^n x dx \\ &= \frac{\tan^{m-1} x \sec^n x}{n} - \frac{m-1}{n} \int \tan^{m-2} x (1 + \tan^2 x) \sec^n x dx \\ &= \frac{\tan^{m-1} x \sec^n x}{n} - \frac{m-1}{n} I_{m-2,n} - \frac{m-1}{n} I_{m,n} \end{aligned}$$

or
$$I_{m,n} = \frac{\tan^{m-1} x \sec^n x}{m+n-1} - \frac{m-1}{m+n-1} I_{m-2,n}, m+n \neq 1$$

(B) Similarly, the power of $\sec x$ is reduced by two after integrating $\int \tan^m x \sec^2 x dx$ by parts and the formula stands as

$$I_{m,n} = \int \tan^m x \sec^n x dx = \frac{\tan^{m+1} x \sec^{n-2} x}{m+n-1} + \frac{n-2}{m+n-1} I_{m,n-2}, m+n \neq 1.$$

The formulae

$$\begin{aligned} \text{(C)} \quad I_{m,n} &= \int \cot^m x \operatorname{cosec}^n x dx \\ &= \frac{\cot^{m-1} x \operatorname{cosec}^n x}{m-n+1} - \frac{m-1}{m-n+1} I_{m-2,n}, m+1 \neq n \text{ and} \end{aligned}$$

$$\begin{aligned} \text{(D)} \quad I_{m,n} &= \int \cot^m x \operatorname{cosec}^n x dx \\ &= \frac{\cot^{m+1} x \operatorname{cosec}^{n-2} x}{n-m+1} - \frac{n-2}{n-m+1} I_{m,n-2}, n+1 \neq m \end{aligned}$$

can easily be verified.

6.10 REDUCTION FORMULAE FOR $\int \cos^m x \cos nx dx$,

$$\int \cos^m x \sin nx dx, \int \sin^m x \cos nx dx \text{ AND } \int \sin^m x \sin nx dx.$$

(A) Let $I_{m,n} = \int \cos^m x \cos nx dx$.

$$= \cos^m x \frac{\sin nx}{n} + \frac{m}{n} \int \cos^{m-1} x \sin x \sin nx dx, \text{ (I.b.p.)} \quad (1)$$

$$= \frac{\cos^m x \sin nx}{n} + \frac{m}{n} \int \cos^{m-1} x [\cos(n-1)x - \cos nx \cos x] dx$$

$$\because \cos(n-1)x = \cos nx \cos x + \sin nx \sin x$$

$$= \frac{\cos^m x \sin nx}{n} + \frac{m}{n} I_{m-1, n-1} - \frac{m}{n} I_{m, n}$$

$$\text{or } \left(1 + \frac{m}{n}\right) I_{m, n} = \frac{\cos^m x \sin nx}{n} + \frac{m}{n} I_{m-1, n-1}$$

$$\text{or } I_{m, n} = \frac{\cos^m x \sin nx}{m+n} + \frac{m}{m+n} I_{m-1, n-1}, m+n \neq 0$$

Here both power and angle reduce by unity. To reduce the power only without affecting the angle, let us integrate by parts the R.H. integral of (1) in (A) above. Therefore.

$$\begin{aligned} I_{m, n} &= \frac{\cos^m x \sin nx}{n} + \frac{m}{n} \left\{ \cos^{m-1} x \sin x \left(\frac{-\cos nx}{n} \right) \right. \\ &\quad \left. + \int [\cos^m x - (m-1) \cos^{m-2} x \sin^2 x] \frac{\cos nx}{n} dx \right\} \\ &= \frac{\cos^m x \sin nx}{n} - \frac{m}{n^2} \cos^{m-1} x \sin x \cos nx \\ &\quad + \frac{m}{n^2} \int [\cos^m x - (m-1) \cos^{m-2} x (1 - \cos^2 x)] \cos nx dx \\ &= \frac{\cos^m x \sin nx}{n} - \frac{m}{n^2} \cos^{m-1} x \sin x \cos nx \\ &\quad + \frac{m^2}{n^2} I_{m, n} - \frac{m(m-1)}{n^2} I_{m-2, n} \end{aligned}$$

$$\text{or } \left(1 - \frac{m^2}{n^2}\right) I_{m, n} = \frac{\cos^m x \sin nx}{n} - \frac{m}{n^2} \cos^{m-1} x \sin x \cos nx - \frac{m(m-1)}{n^2} I_{m-2, n}$$

$$\begin{aligned} \text{or } I_{m, n} &= -\frac{n \cos^m x \sin nx}{m^2 - n^2} + \frac{m}{m^2 - n^2} \cos^{m-1} x \sin x \cos nx \\ &\quad + \frac{m(m-1)}{m^2 - n^2} I_{m-2, n} \\ &= -\frac{\cos^{m-1} x}{m^2 - n^2} [n \cos x \sin nx - m \sin x \cos nx] \\ &\quad + \frac{m(m-1)}{m^2 - n^2} I_{m-2, n} \\ &= -\frac{\cos^2 nx}{m^2 - n^2} \frac{d}{dx} \left(\frac{\cos^m x}{\cos nx} \right) + \frac{m(m-1)}{m^2 - n^2} I_{m-2, n} \end{aligned}$$

Here the power of cosine is reduced by two only.

$$(B) \text{ Let } I_{m, n} = \int \cos^m x \sin nx dx$$

$$\begin{aligned}
 &= \cos^m x \left(-\frac{\cos nx}{n} \right) - \int m \cos^{m-1} x (-\sin x) \left(-\frac{\cos nx}{n} \right) dx \\
 &= -\frac{\cos^m x \cos nx}{n} - \frac{m}{n} \int \cos^{m-1} x \sin x \cos nx \, dx \quad (1) \\
 &= -\frac{\cos^m x \cos nx}{n} - \frac{m}{n} \int \cos^{m-1} x (\sin nx \cos x - \sin(n-1)x) \, dx \\
 &\quad \because \sin(n-1)x = \sin nx \cos x - \cos nx \sin x. \\
 &= -\frac{\cos^m x \cos nx}{n} - \frac{m}{n} I_{m,n} + \frac{m}{n} I_{m-1,n-1}
 \end{aligned}$$

$$\text{or} \quad \left(1 + \frac{m}{n} \right) I_{m,n} = -\frac{\cos^m x \cos nx}{n} + \frac{m}{n} I_{m-1,n-1},$$

$$\text{or} \quad I_{m,n} = -\frac{\cos^m x \cos nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1}; \quad m+n \neq 0.$$

Again, integrating by parts the R.H. integral of (1) in (B), we get,

$$\begin{aligned}
 I_{m,n} &= -\frac{\cos^m x \cos nx}{n} - \frac{m}{n} \left[(\cos^{m-1} x \sin x) \frac{\sin nx}{n} \right. \\
 &\quad \left. - \int \{(m-1) \cos^{m-2} x (-\sin^2 x) + \cos^m x\} \frac{\sin nx}{n} \, dx \right] \\
 &= -\frac{\cos^m x \cos nx}{n} - \frac{m}{n^2} \cos^{m-1} x \sin x \sin nx \\
 &\quad + \frac{m}{n^2} \int \{-(m-1) \cos^{m-2} x (1 - \cos^2 x) + \cos^m x\} \sin nx \, dx \\
 &= -\frac{\cos^m x \cos nx}{n} - \frac{m}{n^2} \cos^{m-1} x \sin x \sin nx - \\
 &\quad \frac{m(m-1)}{n^2} I_{m-2,n} + \frac{m^2}{n^2} I_{m,n}
 \end{aligned}$$

$$\text{or} \quad \left(1 - \frac{m^2}{n^2} \right) I_{m,n} = -\frac{\cos^m x \cos nx}{n} - \frac{m}{n^2} \cos^{m-1} x \sin x \sin nx - \frac{m(m-1)}{n^2} I_{m-2,n}$$

$$\text{or} \quad I_{m,n} = \frac{n \cos^m x \cos nx}{m^2 - n^2} + \frac{m \cos^{m-1} x \sin x \sin nx}{m^2 - n^2} + \frac{m(m-1)}{m^2 - n^2} I_{m-2,n}$$

$$\begin{aligned}
 \text{(C) Let } I_{m,n} &= \int \sin^m x \cos nx \, dx \\
 &= \sin^m x \frac{\sin nx}{n} - \int m \sin^{m-1} x \cos x \frac{\sin nx}{n} \, dx \\
 &= \frac{\sin^m x \sin nx}{n} - \frac{m}{n} \int \sin^{m-1} x \cos x \sin nx \, dx
 \end{aligned}$$

Note that addition formula $\sin(n-1)x = (\sin nx \cos x - \cos nx \sin x)$ brings no symmetry with $I_{m,n}$. So, instead of using this, let us proceed with integration by parts once more. Therefore,

$$\begin{aligned} I_{m,n} &= \frac{\sin^m x \sin nx}{n} - \frac{m}{n} \left[(\sin^{m-1} x \cos x) \left(\frac{-\cos nx}{n} \right) \right. \\ &\quad \left. - \int [(m-1) \sin^{m-2} x \cos^2 x - \sin^m x] \left(\frac{-\cos nx}{n} \right) dx \right] \\ &= \frac{\sin^m x \sin nx}{n} + \frac{m}{n^2} \sin^{m-1} x \cos x \cos nx \\ &\quad - \frac{m}{n^2} \int [(m-1) \sin^{m-2} x (1 - \sin^2 x) - \sin^m x] \cos nx \, dx \\ &= \frac{\sin^m x \sin nx}{n} + \frac{m}{n^2} \sin^{m-1} x \cos x \cos nx - \\ &\quad \frac{m(m-1)}{n^2} I_{m-2,n} + \frac{m^2}{n^2} I_{m,n} \end{aligned}$$

$$\text{or } I_{m,n} = -\frac{n \sin^m x \sin nx}{m^2 - n^2} - \frac{m \sin^{m-1} x \cos x \cos nx}{m^2 - n^2} + \frac{m(m-1)}{m^2 - n^2} I_{m-2,n}$$

$$(D) \text{ Let } I_{m,n} = \int \sin^m x \sin nx \, dx$$

Proceeding as in (C) it can be seen that

$$I_{m,n} = \frac{n \sin^m x \cos nx}{m^2 - n^2} - \frac{m \sin^{m-1} x \cos x \sin nx}{m^2 - n^2} + \frac{m(m-1)}{m^2 - n^2} I_{m-2,n}$$

6.11 REDUCTION FORMULAE FOR $\int e^{ax} \cos^n x \, dx$ AND

$$\int e^{ax} \sin^n x \, dx.$$

$$\begin{aligned} (A) \text{ Let } I_n &= \int e^{ax} \cos^n x \, dx = \frac{e^{ax}}{a} \cos^n x + \frac{n}{a} \int e^{ax} \cos^{n-1} x \sin x \, dx \\ &= \frac{e^{ax}}{a} \cos^n x + \frac{n}{a} \left\{ \frac{e^{ax}}{a} \cos^{n-1} x \sin x \right. \\ &\quad \left. - \int \frac{e^{ax}}{a} [\cos^n x - (n-1) \cos^{n-2} x \sin^2 x] \, dx \right\} \\ &= \frac{e^{ax}}{a} \cos^n x + \frac{n}{a^2} e^{ax} \cos^{n-1} x \sin x - \frac{n}{a^2} I_n \\ &\quad + \frac{n(n-1)}{a^2} \int e^{ax} \cos^{n-2} x (1 - \cos^2 x) \, dx \\ &= \frac{e^{ax}}{a} \cos^n x + \frac{n}{a^2} e^{ax} \cos^{n-1} x \sin x - \frac{n}{a^2} I_n \\ &\quad + \frac{n(n-1)}{a^2} I_{n-2} - \frac{n(n-1)}{a^2} I_n \end{aligned}$$

$$\text{or } \left(1 + \frac{n}{a^2} + \frac{n(n-1)}{a^2}\right) I_n = e^{ax} \cos^{n-1} x \frac{[a \cos x + n \sin x]}{a^2} + \frac{n(n-1)}{a^2} I_{n-2}$$

$$\text{or } I_n = \frac{e^{ax} \cos^{n-1} x [a \cos x + n \sin x]}{n^2 + a^2} + \frac{n(n-1)}{n^2 + a^2} I_{n-2}.$$

(B) Similarly, it can be shown that

$$I_n = \int e^{ax} \sin^n x dx = \frac{e^{ax} \sin^{n-1} x}{n^2 + a^2} [a \sin x - n \cos x] + \frac{n(n-1)}{n^2 + a^2} I_{n-2}.$$

6.12 Reduction formulae for $\int (\sin^{-1} x)^n dx$ AND $\int (\cos^{-1} x)^n dx$

$$\begin{aligned} \text{Let } I_n &= \int (\cos^{-1} x)^n dx = x (\cos^{-1} x)^n - \int n (\cos^{-1} x)^{n-1} \left(-\frac{1}{\sqrt{1-x^2}}\right) x dx \\ &= x (\cos^{-1} x)^n + n \int (\cos^{-1} x)^{n-1} \frac{x}{\sqrt{1-x^2}} dx \\ &= x (\cos^{-1} x)^n - n (\cos^{-1} x)^{n-1} \sqrt{1-x^2} \\ &\quad - n \int (n-1) (\cos^{-1} x)^{n-2} \left(-\frac{1}{\sqrt{1-x^2}}\right) \sqrt{1-x^2} dx \\ &= x (\cos^{-1} x)^n - n (\cos^{-1} x)^{n-1} \sqrt{1-x^2} + n(n-1) I_{n-2} \end{aligned}$$

Similarly,

$$I_n = \int (\sin^{-1} x)^n dx = x (\sin^{-1} x)^n + n (\sin^{-1} x)^{n-1} \sqrt{1-x^2} - n(n-1) I_{n-2}.$$

6.13 REDUCTION FORMULA FOR $\int \cos nx \operatorname{cosec} x dx$ AND

$$\int \sin nx \sec x dx.$$

$$(A) \text{ Let } I_n = \int \cos nx \operatorname{cosec} x dx$$

$$\begin{aligned} &= \frac{\sin nx}{n} \operatorname{cosec} x + \int \frac{\sin nx}{n} \operatorname{cosec} x \cot x dx \\ &= \frac{\sin nx}{n} \operatorname{cosec} x + \int \frac{1}{n} \frac{\sin nx \cos x}{\sin^2 x} dx \\ &= \frac{\sin nx \operatorname{cosec} x}{n} + \frac{1}{n} \int \frac{\sin (n-1)x + \cos nx \sin x}{\sin^2 x} dx \\ &\quad \because \sin (n-1)x = \sin nx \cos x - \cos nx \sin x \\ &= \frac{\sin nx \operatorname{cosec} x}{n} + \frac{1}{n} \int \sin (n-1)x \operatorname{cosec}^2 x dx + \frac{1}{n} I_n \end{aligned}$$

$$\begin{aligned} \text{or } \left(1 - \frac{1}{n}\right) I_n &= \frac{\sin nx \operatorname{cosec} x}{n} + \frac{1}{n} [\sin (n-1)x (-\cot x) - \\ &\quad (n-1) \int \cos (n-1)x (-\cot x) dx] \end{aligned}$$

$$\text{or } \frac{n-1}{n} I_n = \frac{\sin nx \operatorname{cosec} x}{n} - \frac{\sin(n-1)x \cot x}{n} + \frac{(n-1)}{n} \int \frac{\cos(n-1)x \cos x}{\sin x} dx$$

$$\begin{aligned} \text{or } I_n &= \frac{\sin nx \operatorname{cosec} x}{n-1} - \frac{\sin(n-1)x \cot x}{n-1} + \int \frac{\cos(n-2)x - \sin(n-1)x \sin x}{\sin x} dx \\ &= \frac{\sin nx \operatorname{cosec} x}{n-1} - \frac{\sin(n-1)x \cot x}{n-1} + I_{n-2} - \int \sin(n-1)x dx \\ &= \frac{\sin nx \operatorname{cosec} x}{n-1} - \frac{\sin(n-1)x \cot x}{n-1} + I_{n-2} + \frac{1}{n-1} \cos(n-1)x \\ \text{or } I_n - I_{n-2} &= \frac{1}{n-1} \left[\frac{\sin nx}{\sin x} - \frac{\sin(n-1)x \cos x}{\sin x} + \cos(n-1)x \right] \\ &= \frac{1}{n-1} \left[\frac{\sin nx - \sin(n-1)x \cos x}{\sin x} + \cos(n-1)x \right] \\ &= \frac{1}{n-1} \left[\frac{\sin(n-1+1)x - \sin(n-1)x \cos x}{\sin x} + \cos(n-1)x \right] \\ &= \frac{2 \cos(n-1)x}{n-1} \end{aligned}$$

(B) Similarly,

$$I_n = \int \sin nx \sec x dx = -\frac{2 \cos(n-1)x}{n-1} - I_{n-2}$$

$$\text{or } I_n + I_{n-2} = -\frac{2 \cos(n-1)x}{n-1}$$

6.14 REDUCTION FORMULAE FOR $\int \frac{dx}{(a+b \cos x)^n}$ AND

$$\int \frac{dx}{(a+b \sin x)^n}$$

(A) Let $I_n = \int \frac{dx}{(a+b \cos x)^n}$. To use differentiation method for the derivation of the reduction formula,

$$\text{let } P(x) = \frac{\sin x}{(a+b \cos x)^{n-1}} = \frac{\sin x}{t^{n-1}} \text{ where } t = a+b \cos x$$

and so $\cos x = \frac{t-a}{b}$. Now, differentiating w.r.t. x ,

$$\frac{dP(x)}{dx} = \frac{\cos x}{t^{n-1}} - (n-1) \frac{\sin x}{t^n} \frac{dt}{dx} = \frac{\cos x}{t^{n-1}} - (n-1) \frac{\sin x}{t^n} (-b \sin x)$$

$$\begin{aligned}
&= \frac{t-a}{b t^{n-1}} + \frac{b(n-1)}{t^n} (1 - \cos^2 x) \\
&= \frac{t-a}{b t^{n-1}} + \frac{b(n-1)}{t^n} \left(1 - \left(\frac{t-a}{b} \right)^2 \right) \\
&= \frac{t-a}{b t^{n-1}} + \frac{b(n-1)}{t^n} \left[1 - \frac{t^2 - 2at + a^2}{b^2} \right] \\
&= \frac{(n-1)(b^2 - a^2)}{b t^n} + \frac{a(2n-3)}{b t^{n-1}} - \frac{n-2}{b t^{n-2}} \\
&= \frac{(n-1)(b^2 - a^2)}{b} \cdot \frac{1}{(a + b \cos x)^n} + \frac{a(2n-3)}{b} \cdot \frac{1}{(a + b \cos x)^{n-1}} \\
&\quad - \frac{n-2}{b} \cdot \frac{1}{(a + b \cos x)^{n-2}}
\end{aligned}$$

Now, integrating w.r.t. x

$$P(x) = \frac{(n-1)(b^2 - a^2)}{b} I_n + \frac{a(2n-3)}{b} I_{n-1} - \frac{n-2}{b} I_{n-2}$$

or
$$I_n = \frac{b \sin x}{(n-1)(b^2 - a^2)(a + b \cos x)^{n-1}} - \frac{a(2n-3)}{(n-1)(b^2 - a^2)} I_{n-1} + \frac{(n-2)}{(n-1)(b^2 - a^2)} I_{n-2}$$

(B) Similarly selecting $P(x) = \frac{\cos x}{(a + b \sin x)^{n-1}}$,

$$\begin{aligned}
I_n &= \int \frac{dx}{(a + b \sin x)^n} = \frac{b}{(n-1)(a^2 - b^2)} \cdot \frac{\cos x}{(a + b \sin x)^{n-1}} \\
&\quad + \frac{(2n-3)a}{(n-1)(a^2 - b^2)} I_{n-1} - \frac{n-2}{(n-1)(a^2 - b^2)} I_{n-2}
\end{aligned}$$

6.15 REDUCTION FORMULA FOR $\int \frac{x^n}{\sqrt{ax^2 + bx + c}} dx$.

$$\begin{aligned}
\text{Let } I_n &= \int \frac{x^n}{\sqrt{ax^2 + bx + c}} dx = \frac{1}{2a} \int \frac{[(2ax + b) - b] x^{n-1}}{\sqrt{ax^2 + bx + c}} dx \\
&= \frac{1}{2a} \int \frac{(2ax + b) x^{n-1}}{\sqrt{ax^2 + bx + c}} dx - \frac{b}{2a} \int \frac{x^{n-1}}{\sqrt{ax^2 + bx + c}} dx \\
&= \frac{1}{2a} \left[x^{n-1} \cdot 2\sqrt{ax^2 + bx + c} - \int (n-1) x^{n-2} \cdot 2\sqrt{ax^2 + bx + c} dx \right] \\
&\quad - \frac{b}{2a} I_{n-1}, \text{ (I.b.p.)}
\end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{a} x^{n-1} \sqrt{ax^2 + bx + c} - \frac{(n-1)}{a} \int \frac{x^{n-2} (ax^2 + bx + c)}{\sqrt{ax^2 + bx + c}} dx - \frac{b}{2a} I_{n-1} \\
 &= \frac{1}{a} x^{n-1} \sqrt{ax^2 + bx + c} - (n-1) I_n - \frac{b(n-1)}{a} I_{n-1} \\
 &\quad - \frac{c(n-1)}{a} I_{n-2} - \frac{b}{2a} I_{n-1} \\
 \therefore I_n &= \frac{1}{na} x^{n-1} \sqrt{ax^2 + bx + c} - \frac{b(2n-1)}{2an} I_{n-1} - \frac{c(n-1)}{an} I_{n-2}
 \end{aligned}$$

This reduction formula can also be derived using differentiation of $P(x) = x^{n-1} \sqrt{ax^2 + bx + c}$.

6.16 REDUCTION FORMULA FOR $\int \frac{px+q}{(ax^2+bx+c)^n} dx$

$$\begin{aligned}
 \text{Let } I_n &= \int \frac{px+q}{(ax^2+bx+c)^n} dx = \frac{p}{2a} \int \frac{2ax+b}{(ax^2+bx+c)^n} dx + \\
 &\quad \left(q - \frac{bp}{2a} \right) \cdot \int \frac{dx}{(ax^2+bx+c)^n} \\
 &= -\frac{p}{2a(n-1)} \cdot \frac{1}{(ax^2+bx+c)^{n-1}} \\
 &\quad + \left(q - \frac{pb}{2a} \right) \cdot \frac{1}{a^n} \int \frac{dx}{\left\{ \left(x + \frac{b}{2a} \right)^2 + \frac{4ac-b^2}{4a^2} \right\}^n} \\
 &= -\frac{p}{2a(n-1)} \cdot \frac{1}{(ax^2+bx+c)^{n-1}} + \frac{2qa-pb}{2a^{n+1}} \int \frac{dt}{(t^2+\lambda)^n} \quad (1) \\
 &\quad \text{where } t = x + \frac{b}{2a} \text{ and } \lambda = \frac{4ac-b^2}{4a^2}
 \end{aligned}$$

Now integrating by parts the R.H. integral of (1), we get,

$$\begin{aligned}
 P_n &= \int \frac{dt}{(t^2+\lambda)^n} = \frac{t}{(t^2+\lambda)^n} + n \int \frac{2t^2}{(t^2+\lambda)^{n+1}} dt \\
 &= \frac{t}{(t^2+\lambda)^n} + 2n \int \frac{(t^2+\lambda) - \lambda}{(t^2+\lambda)^{n+1}} dt \\
 &= \frac{t}{(t^2+\lambda)^n} + 2n \int \frac{1}{(t^2+\lambda)^n} - 2n\lambda \int \frac{dt}{(t^2+\lambda)^{n+1}}
 \end{aligned}$$

Replacing n by $(n-1)$ in this relation and solving for $\int \frac{dt}{(t^2 + \lambda)^n}$, we have,

$$\int \frac{dt}{(t^2 + \lambda)^n} = \frac{1}{2(n-1)\lambda} \cdot \frac{t}{(t^2 + \lambda)^{n-1}} + \frac{2n-3}{2(n-1)\lambda} \int \frac{dt}{(t^2 + \lambda)^{n-1}} \quad (2)$$

Using (2) in (1) we have the required reduction formula,

$$\begin{aligned} I_n &= -\frac{p}{2a(n-1)} \cdot \frac{1}{(ax^2 + bx + c)^{n-1}} + \\ &\quad \frac{2aq - pb}{2a^{n+1}} \left[\frac{1}{2(n-1)\lambda} \cdot \frac{t}{(t^2 + \lambda)^{n-1}} + \frac{(2n-3)}{2(n-1)\lambda} P_{n-1} \right] \\ &= -\frac{p}{2a(n-1)} \cdot \frac{1}{(ax^2 + bx + c)^{n-1}} + \\ &\quad \frac{2aq - pb}{4(n-1)\lambda} \cdot \left[\frac{t}{(t^2 + \lambda)^{n-1}} + (2n-3) P_{n-1} \right] \\ &\quad \text{where } P_n = \int \frac{dt}{(t^2 + \lambda)^n}, \quad t = x + \frac{b}{2a} \text{ and } \lambda = \frac{4ac - b^2}{4a^2} \end{aligned}$$

6.17 BINOMIAL DIFFERENTIAL AND ITS REDUCTION FORMULAE

The expression $x^m (a + bx^n)^p dx$ is called a *binomial differential* when m, n, p, a and b are all constant.

The integral $\int x^m (a + bx^n)^p dx$ of the binomial differential can be reduced to elementary integrals on substitution if m, n and p are rational. Then there arises *three* cases as follows:

- (i) p is an integer +ve, -ve or zero,
- (ii) $\frac{m+1}{n}$ is an integer +ve, -ve or zero,
- (iii) $\frac{m+1}{n} + p$ is an integer +ve, -ve or zero.

Let us consider each of the cases.

Case (i) If $p = 0$, the integral reduces to $\int x^m dx = \frac{1}{m} x^{m+1} + c$. If

$p > 0$, then $(a + bx^n)^p$ can be expanded binomially and can be integrated easily. If $p < 0$, let $p = -k$, k is any +ve integer and $k \neq 0$. Then the integral becomes,

$$\int \frac{x^m}{(a + bx^n)^k} dx.$$

As m and n are rational & $k > 0$ is an integer, the substitution $x = t^r$, r is l. c. m. of the denominator of m and n , reduces the integrand to a rational function of t and can be easily integrated.

Case (ii). Let $I = \int x^m (a + bx^n)^p dx$, Since p is rational. Let $p = \frac{r}{k}$.

So the substitution $a + bx^n = u^k$ i.e. $x^n = \frac{u^k - a}{b}$

or $nx^{n-1} dx = \frac{u}{b} u^{k-1} du$ or $dx = \frac{k}{nb^{1/n}} u^{k-1} (u^k - a)^{\frac{1}{n}-1} du$

reduces the integral to

$$I = \left(\frac{u^k - a}{b} \right)^{\frac{m}{n}} u^r \frac{k}{nb^{1/n}} u^{k-1} (u^k - a)^{(1-n)/n} du.$$

$$= \frac{k}{nb^{(m+1)/n}} \int u^{r+k-1} (u^k - a)^{(m-n+1)/n} du$$

If $\frac{m+1}{n} = 0$, then $I = \frac{k}{n} \int \frac{u^{r+k-1}}{u^k - a} du$.

This can be integrated by dividing the integrand.

If $\frac{m+1}{n} > 0$ integer, then $(u^k - a)^{(m-n+1)/n}$ equals unity if $\frac{m+1}{n} = 1$ or admits of a binomial expansion making the integrand integrable.

If $\frac{m+1}{n} < 0$ integer, Let $\frac{m+1}{n} - 1 = -s, s > 0$, an integer, then

$$I = \frac{k}{nb^{-s+1}} \int \frac{u^{k-1}}{(u^k - a)^s} du = \frac{1}{nb^{-s+1}} \frac{1}{(s-1)} \frac{1}{(u^k - a)^{s-1}} + c.$$

Case (iii). For this case, let us substitute $x = 1/y$, so that

$$I = \int x^m (a + bx^n)^p dx = \int \frac{1}{y^m} \left(a + \frac{b}{y^n} \right)^p \left(-\frac{1}{y^2} \right) dy$$

$$= - \int y^{-np-m-2} (ay^n + b)^p dy$$

$$= - \int y^{-n \left(\frac{m+1}{n} + p \right) - 1} (ay^n + b)^p dy$$

Now, if $\frac{m+1}{n} + p = 0$ then $I = - \int \frac{a(y^n + b)^p}{y} dy$ can be integrated using 5.9 as n and p are rational.

If $\left(\frac{m+1}{n} + p \right) > 0$ or $\left(\frac{m+1}{n} + p \right) < 0$ then 5.9 can be used and it can be integrated.

REDUCTION FORMULAE :

The reduction formulae for $I_{m,p} = \int x^m (a + bx^n)^p dx$ are the following :

$$(A) \quad I_{m,p} = \frac{x^{m-n+1} (a + bx^n)^{p+1}}{(np + m + 1)b} - \frac{(m - n + 1)a}{(np + m + 1)b} I_{m-n,p}$$

$$(B) \quad I_{m,p} = \frac{x^{m+1} (a + bx^n)^p}{np + m + 1} - \frac{anp}{np + m + 1} I_{m,p-1}$$

$$(C) \quad I_{m,p} = \frac{x^{m+1} (a + bx^n)^{p+1}}{(m+1)a} - \frac{(np + n + m + 1)b}{(m+1)a} I_{m+n,p}$$

$$(D) \quad I_{m,p} = -\frac{x^{m+1} (a + bx^n)^{p+1}}{n(p+1)a} + \frac{(np + n + m + 1)}{n(p+1)a} I_{m,p+1}$$

The formulae (A) and (B) are valid when $np + m + 1 \neq 0$, (C) and (D) are valid if $m + 1 \neq 0$ and $n, p + 1 \neq 0$ respectively.

Proof (A). This can be established using integration by parts. So, the integral can be written as

$$\begin{aligned} I_{m,p} &= \int x^m (a + bx^n)^p dx = \frac{1}{nb} \int x^{m-n+1} \{(a + bx^n)^p nbx^{n-1}\} dx \\ &= \frac{x^{m-n+1}}{nb(p+1)} (a + bx^n)^{p+1} - \frac{1}{nb} \int (m-n+1) x^{m-n} \frac{(a + bx^n)^{p+1}}{p+1} dx \\ &= \frac{x^{m-n+1} (a + bx^n)^{p+1}}{nb(p+1)} - \frac{m-n+1}{nb(p+1)} \int x^{m-n} (a + bx^n)^p (a + bx^n) dx \\ &= \frac{x^{m-n+1} (a + bx^n)^{p+1}}{nb(p+1)} - \frac{(m-n+1)a}{nb(p+1)} I_{m-n,p} - \frac{m-n+1}{n(p+1)} I_{m,p} \\ \text{or } \left[1 + \frac{m-n+1}{n(p+1)} \right] I_{m,p} &= \frac{x^{m-n+1} (a + bx^n)^{p+1}}{nb(p+1)} - \frac{(m-n+1)a}{nb(p+1)} I_{m-n,p} \\ \text{or } I_{m,p} &= \frac{x^{m-n+1} (a + bx^n)^{p+1}}{(np+m+1)b} - \frac{(m-n+1)a}{(np+m+1)b} I_{m-n,p} \end{aligned}$$

Here the power m is reduced by n to $m-n$ and the formula exists if $np + m + 1 \neq 0$.

(B) To establish (B), separate a factor $(a + bx^n)$ from the integrand so that

$$\begin{aligned} I_{m,p} &= \int x^m (a + bx^n)^{p-1} (a + bx^n) dx \\ &= a \int x^m (a + bx^n)^{p-1} dx + b \int x^{m+n} (a + bx^n)^{p-1} dx \\ &= a I_{m,p-1} + b I_{m+n,p-1} \end{aligned} \quad (1)$$

Now, replacing m by $(m+n)$ and p by $(p-1)$ in (A) we have,

$$I_{m+n,p-1} = \frac{x^{m+1} (a + bx^n)^p}{(np+m+1)b} - \frac{(m+1)a}{(np+m+1)a} I_{m,p-1}$$

with this result (1) becomes

$$\begin{aligned}
 I_{m,p} &= a I_{m,p-1} + \frac{x^{m+1} (a + bx^n)^p}{np + m + 1} - \frac{a(m+1)}{np + m + 1} I_{m,p-1} \\
 &= \frac{x^{m+1} (a + bx^n)^p}{np + m + 1} + \frac{anp}{np + m + 1} I_{m,p-1}
 \end{aligned}$$

in which power p is reduced by unity to $(p-1)$ and formula exists if $np + m + 1 \neq 0$.

(C) Solving for $I_{m-n,p}$ from (A) and replacing m by $(m+n)$ we have (C) and is valid for $m+1 \neq 0$.

(D) Solving for $I_{m,p-1}$ from (B) and replacing p by $(p+1)$, we have (D) and is valid for $n, p+1 \neq 0$.

Examples

1. Use reduction formulae to integrate $\sin^{1/2} x \cos^{7/2} x$.

This is of the type 6.2 with $m = \frac{1}{2}$, $n = \frac{7}{2}$. Let us reduce the power of cosine. So we use formula (A) of 6.2.

$$\therefore I_{1/2, 7/2} = \int \sin^{1/2} x \cos^{7/2} x dx = \frac{\cos^{5/2} x \sin^{3/2} x}{4} + \frac{5}{8} I_{1/2, 3/2}$$

$$\text{Also } I_{1/2, 3/2} = \frac{\cos^{1/2} x \sin^{3/2} x}{2} + \frac{1}{4} I_{1/2, -1/2}$$

$$\therefore I_{1/2, 3/2} = \frac{\cos^{5/2} x \sin^{3/2} x}{4} + \frac{5}{16} \cos^{1/2} x \sin^{3/2} x + \frac{5}{32} I_{1/2, -1/2}$$

Further reduction is not possible as $m+n = \frac{1}{2} - \frac{1}{2} = 0$.

$$\begin{aligned}
 \text{But } I_{1/2, -1/2} &= \int \frac{\sqrt{\sin x}}{\sqrt{\cos x}} dx = \int \sqrt{\tan x} dx \\
 &= \frac{1}{2} \tan^{-1} \left(\frac{\tan x - 1}{\sqrt{2 \tan x}} \right) + \frac{1}{2\sqrt{2}} \log \frac{\tan x - \sqrt{2 \tan x}}{\tan x + \sqrt{2 \tan x}} + c \\
 &\quad \text{(cf. sec. 2.23 Ex. 2)}
 \end{aligned}$$

$$\begin{aligned}
 \therefore I_{1/2, 7/2} &= \frac{1}{4} \sqrt{\cos x} \sin^{3/2} x \left(\cos^2 x + \frac{5}{4} \right) \\
 &\quad + \frac{5}{32} \left[\frac{1}{2} \tan^{-1} \left(\frac{\tan x - 1}{\sqrt{2 \tan x}} \right) + \frac{1}{2\sqrt{2}} \log \frac{\tan x - \sqrt{2 \tan x}}{\tan x + \sqrt{2 \tan x}} \right] + c.
 \end{aligned}$$

2. Use reduction formulae to compute $\int \sin^4 x \cos^5 x dx$.

This is of the type 6.2 with $m=4$, $n=5$. Let us reduce the power of sine using 6.2 (B.).

$$I_{4,5} = \int \sin^4 x \cos^5 x dx = -\frac{\sin^3 x \cos^6 x}{9} + \frac{1}{3} I_{2,5}.$$

$$I_{2,5} = \int \sin^2 x \cos^5 x \, dx = -\frac{\sin x \cos^6 x}{7} + \frac{1}{7} I_{0,5}$$

$$I_{0,5} = I_5 = \int \cos^5 x \, dx \text{ which is of type 6.3 (A)}$$

$$= \frac{\sin x \cos^4 x}{5} + \frac{4}{5} I_3$$

$$I_3 = \int \cos^3 x \, dx = \frac{\sin x \cos^2 x}{3} + \frac{2}{3} I_1$$

$$I_1 = \int \cos x \, dx = \frac{\sin x \cos^0 x}{1} + \frac{0}{1} I_0 = \sin x$$

$$\begin{aligned} \therefore I_{4,5} &= -\frac{\sin^3 x \cos^6 x}{9} - \frac{1}{21} \sin x \cos^6 x + \frac{1}{105} \sin x \cos^4 x \\ &\quad + \frac{4}{315} \sin x \cos^2 x + \frac{8}{315} \sin x + c \\ &= \frac{\sin x}{315} [35 \cos^8 x - 50 \cos^6 x + 3 \cos^4 x + 4 \cos^2 x + 8] + c. \end{aligned}$$

3. Use reduction formulae for integrating $\int \cos^4 x \cos 3x \, dx$.

This is of type 6.10 (A).

$$\begin{aligned} I_{4,3} &= \int \cos^4 x \cos 3x \, dx = -\frac{\cos^2 3x}{7} \frac{d}{dx} \left(\frac{\cos^4 x}{\cos 3x} \right) + \frac{4(4-1)}{7} I_{2,3} \\ &= \frac{4}{7} \cos 3x \cos^3 x \sin x - \frac{3}{7} \cos^4 x \sin 3x + \frac{12}{7} I_{2,3} \end{aligned}$$

$$\begin{aligned} I_{2,3} &= \int \cos^2 x \cos 3x \, dx = -\frac{\cos^2 3x}{5} \frac{d}{dx} \left(\frac{\cos^2 x}{\cos 3x} \right) + \frac{2(2-1)}{-5} I_{0,3} \\ &= -\frac{2}{5} \cos 3x \sin x \cos x + \frac{3}{5} \cos^3 x \sin 3x - \frac{2}{5} I_{0,3} \end{aligned}$$

$$I_{0,3} = \int \cos^0 x \cos 3x \, dx = \int \cos 3x \, dx = \frac{1}{3} \sin 3x + c$$

$$\begin{aligned} \therefore I_{4,3} &= \frac{4}{7} \cos 3x \cos^3 x \sin x - \frac{3}{7} \cos^4 x \sin 3x \\ &\quad - \frac{24}{35} \cos 3x \sin x \cos x + \frac{36}{35} \cos^2 x \sin 3x - \frac{8}{35} \sin 3x + c \\ &= \frac{4}{35} \cos 3x \cos x \sin x (5 \cos^2 x - 6) - \frac{1}{35} \sin 3x (15 \cos^4 x \\ &\quad - 36 \cos^2 x + 8) + c \end{aligned}$$

4. Compute the integral $\int \frac{1-7x^2+14x^4-8x^6}{\sqrt{2x^2-1}} \, dx$.

The reduction formula of the type 6.15 is applicable.

Let $I_n = \int \frac{x^n}{\sqrt{2x^2-1}}$ Therefore

$$I = \int \frac{1-7x^2+14x^4-8x^6}{\sqrt{2x^2-1}} = I_0 - 7I_2 + 14I_4 - 8I_6$$

From 6.15, $I_n = \frac{1}{2n} x^{n-1} \sqrt{2x^2-1} - \frac{0(2n-1)}{4n} I_{n-1} + \frac{(n-1)}{2n} I_{n-2}$

$\therefore a=2, b=0, c=-1$

$$\therefore I_6 = \frac{1}{12} x^5 \sqrt{2x^2-1} + \frac{5}{12} I_4$$

$$I_4 = \frac{1}{8} x^3 \sqrt{2x^2-1} + \frac{3}{8} I_2$$

$$I_2 = \frac{1}{4} x \sqrt{2x^2-1} + \frac{1}{4} I_0$$

$$I_0 = \int \frac{1}{\sqrt{2x^2-1}} dx = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{x^2-(1/2)}} dx = \frac{1}{\sqrt{2}} \cosh^{-1}(\sqrt{2}x)$$

$$\begin{aligned} \therefore I &= I_0 - 7I_2 + 14I_4 - \frac{2}{3} x^5 \sqrt{2x^2-1} - \frac{10}{3} I_4 \\ &= I_0 - 7I_2 + \frac{32}{3} \left(\frac{1}{8} x^3 \sqrt{2x^2-1} + \frac{3}{8} I_2 \right) - \frac{2}{3} x^5 \sqrt{2x^2-1} \\ &= I_0 - 3 \left(\frac{1}{4} x \sqrt{2x^2-1} + \frac{1}{4} I_0 \right) + \frac{4}{3} x^3 \sqrt{2x^2-1} - \frac{2}{3} x^5 \sqrt{2x^2-1} \\ &= \frac{1}{4} I_0 - \frac{3}{4} x \sqrt{2x^2-1} + \frac{4}{3} x^3 \sqrt{2x^2-1} - \frac{2}{3} x^5 \sqrt{2x^2-1} \\ &= \frac{1}{4\sqrt{2}} \cosh^{-1}(\sqrt{2}x) - \left(\frac{3}{4} x - \frac{4}{3} x^3 + \frac{2}{3} x^5 \right) \sqrt{2x^2-1} + c \end{aligned}$$

5. Perform the integration $\int \frac{dx}{(2+\cos x)^5}$

The reduction formula 6.14 (A) can be used.

$$\begin{aligned} \text{Let } I_5 &= \int \frac{dx}{(2+\cos x)^5} \\ &= -\frac{\sin x}{4 \cdot 3 (2+\cos x)^4} + \frac{2(10-3)}{4 \cdot 3} \cdot I_4 - \frac{3}{4 \cdot 3} I_3 \end{aligned}$$

$$\text{Also } I_4 = \int \frac{dx}{(2+\cos x)^4} = -\frac{\sin x}{3 \cdot 3} \cdot \frac{1}{(2+\cos x)^3} + \frac{2(8-3)}{3 \cdot 3} I_3 - \frac{2}{3 \cdot 3} I_2$$

$$I_3 = \int \frac{dx}{(2 + \cos x)^3} = -\frac{\sin x}{2 \cdot 3} \cdot \frac{1}{(2 + \cos x)^2} + \frac{2(6-3)}{2 \cdot 3} I_2 - \frac{1}{2 \cdot 3} I_1$$

$$I_2 = \int \frac{dx}{(2 + \cos x)^2} = -\frac{\sin x}{1 \cdot 3} \cdot \frac{1}{(2 + \cos x)} + \frac{2(4-3)}{1 \cdot 3} I_1$$

$$I_1 = \int \frac{dx}{(2 + \cos x)} = 2 \tan^{-1} \left(\frac{1}{\sqrt{3}} \tan \frac{x}{2} \right) + c_1$$

$$\therefore I_5 = -\frac{1}{12} \cdot \frac{\sin x}{(2 + \cos x)^4} + \frac{7}{6} \left[-\frac{1}{9} \cdot \frac{\sin x}{(2 + \cos x)^3} + \frac{10}{9} I_3 - \frac{2}{9} I_2 \right] - \frac{1}{4} I_3$$

$$= -\frac{1}{12} \cdot \frac{\sin x}{(2 + \cos x)^4} - \frac{7}{54} \cdot \frac{\sin x}{(2 + \cos x)^3} + \frac{113}{108} I_3 - \frac{7}{27} I_2$$

$$= -\frac{1}{12} \cdot \frac{\sin x}{(2 + \cos x)^4} - \frac{7}{54} \cdot \frac{\sin x}{(2 + \cos x)^3} + \frac{113}{108} \left[-\frac{1}{6} \cdot \frac{\sin x}{(2 + 3 \cos x)^2} + I_2 - \frac{1}{6} I_1 \right] - \frac{7}{27} I_2$$

$$= -\frac{1}{12} \cdot \frac{\sin x}{(2 + \cos x)^4} - \frac{7}{54} \cdot \frac{\sin x}{(2 + \cos x)^3} - \frac{113}{660} \cdot \frac{\sin x}{(2 + 3 \cos x)^2}$$

$$+ \frac{20}{27} \cdot \left[-\frac{1}{3} \cdot \frac{\sin x}{2 + \cos x} + \frac{2}{3} I_1 - \frac{1}{3} \cdot I_0 \right] - \frac{1}{6} I_1$$

$$= -\frac{1}{12} \cdot \frac{\sin x}{(2 + \cos x)^4} - \frac{7}{54} \cdot \frac{\sin x}{(2 + \cos x)^3} - \frac{113}{648} \cdot \frac{\sin x}{(2 + \cos x)^2} + \frac{85}{324} \cdot \frac{\sin x}{(2 + \cos x)} + \frac{170}{324} I_1 - \frac{113}{648} I_0$$

$$= -\frac{1}{12} \cdot \frac{\sin x}{(2 + \cos x)^4} - \frac{7}{54} \cdot \frac{\sin x}{(2 + \cos x)^3} - \frac{113}{648} \cdot \frac{\sin x}{(2 + \cos x)^2} - \frac{85}{324} \cdot \frac{\sin x}{(2 + \cos x)} + \frac{227}{324} \tan^{-1} \left(\frac{1}{\sqrt{3}} \tan \frac{x}{2} \right) \quad \text{where } c = c_1 + c_2$$

6. Perform the integration $\int \frac{x^3}{\sqrt{1-x^2}} dx$.

Reduction formula 6.17 (A) may be used. Here $m = 3, n = 2$,

$p = -\frac{1}{2}, a = 1, b = -1$. Let

$$I_{3, -1/2} = \int x^3 (1 - x^2)^{-1/2} dx$$

$$\begin{aligned}
&= \frac{x^{3-2+1} (1-x^2)^{-\frac{1}{2}+1}}{(2 \cdot (-1/2) + 3 + 1)(-1)} - \frac{(3-2+1) \cdot 1}{-1(2 \cdot (-1/2) + 3 + 1)} I_{1, -\frac{1}{2}} \\
&= -\frac{1}{3} x^2 (1-x^2)^{1/2} + \frac{2}{3} I_{1, -\frac{1}{2}} \\
&= -\frac{1}{3} x^2 (1-x^2)^{1/2} + \frac{2}{3} \int \frac{x}{\sqrt{1-x^2}} dx \\
&= -\frac{x^2}{3} \sqrt{1-x^2} - \frac{2}{3} \sqrt{1-x^2} + c \\
&= -\frac{1}{3} (x^2 + 2) \sqrt{1-x^2} + c.
\end{aligned}$$

Exercises 6

- Derive the reduction formula 6.5 (B). Hence perform the integration $\int \cos^3 x \operatorname{cosec}^2 x dx$
- Derive the reduction formula 6.4 (D), 6.5 (B), 6.7 (B), 6.9 (B), 6.9 (C), 6.9 (D), 6.11 (B), 6.13 (B), 6.14 (B).
- Use reduction formula to the following :-

$$(i) \int \sqrt{\cos \theta} \sin^3 \theta d\theta \quad (ii) \int \frac{d\theta}{\cos^3 \theta \sin \theta}$$

$$(iii) \int \frac{x^4}{\sqrt{1-x^2}} dx \quad (iv) \int \frac{x^4}{(a^2+x^2)^2} dx$$

$$(v) \int (1+x^2)^{3/2} dx \quad (vi) \int \frac{x^2 dx}{\sqrt{2ax-x^2}}$$

- Derive the following reduction formulae :

$$(i) I_n = \int x \sin^n x dx = -\frac{x \sin^{n-1} x \cos x}{n} + \frac{\sin^n x}{n^2} + \frac{n-1}{n} I_{n-2}$$

[Hints : Choose first function = $x \sin^{n-1} x$]

$$(ii) I_n = \int x \cos^n x dx = \frac{x \cos^{n-1} x \sin x}{n} + \frac{\cos^n x}{n^2} + \frac{n-1}{n} I_{n-2}$$

$$(iii) I_n = \int x^n \sin mx dx = -\frac{x^n \cos mx}{m} + \frac{nx^{n-1} \sin mx}{m^2} - \frac{n(n-1)}{m^2} I_{n-2}$$

$$(iv) I_n = \int x^n \cos mx dx = \frac{x^n \sin mx}{m} + \frac{n}{m^2} x^{n-1} \cos mx - \frac{n(n-1)}{m^2} I_{n-2}$$

$$(v) I_n = \int x^n e^{-x} dx = -x^n e^{-x} + n I_{n-1}$$

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$$(vi) I_{m,n} = \int \frac{x^m}{(\log x)^n} dx = -\frac{x^{m+1}}{(n-1)(\log x)^{n-1}} + \frac{m+1}{n-1} I_{m,n-1}$$

[Hints: select second function = $\frac{1}{x(\log x)^n}$]

5. Prove the following using reduction formulae:

$$(i) \int \frac{x+1}{(x^2+1)^3} dx = \frac{x-1}{4(x^2+1)^2} + \frac{3x}{8(x^2+1)} + \frac{3}{8} \tan^{-1} x + c$$

$$(ii) \int \frac{x^2-a^2}{(x^2+a^2)^3} dx = -\frac{x}{4a^2(x^2+a^2)} - \frac{x}{2(x^2+a^2)^2} - \frac{1}{4a^3} \tan^{-1} \frac{x}{a} + c$$

$$(iii) \int \frac{dx}{(x^2-x+1)^2} = \frac{2x-1}{3(x^2-x+1)} + \frac{4}{3\sqrt{3}} \tan^{-1} \left(\frac{2x-1}{\sqrt{3}} \right) + c$$

6. Find the integral of the following binomial differentials.

$$(i) \int \frac{x^5}{(a+bx^2)^4} dx$$

$$(ii) \int \frac{dx}{x^{1/2}(1+x^2)^{5/4}}$$

$$(iii) \int \frac{x^3 dx}{\sqrt{4x-x^2}}$$

7. (i) If $I_n = \int \frac{dx}{(x^2+a^2)^n}$, then prove that

$$I_{n+1} = \frac{1}{2na^2} \cdot \frac{x}{(x^2+a^2)^n} + \frac{2n-1}{2n} \cdot \frac{1}{a^2} I_n$$

(ii) If $I_n = \int (a^2-x^2)^n dx$, then show that

$$I_n = \frac{x(a^2-n^2)^n}{2n+1} + \frac{2na^2}{2n+1} I_{n-1}$$

(iii) If $I_n = \int (\log x)^n dx$, then show that $I_n = x(\log x)^n - n I_{n-1}$.

(iv) If $I_n = \int x^n e^x dx$, then show that $I_n = x^n e^x - n I_{n-1}$.

(v) If $I_n = \int e^{\alpha x} \sin^n x dx$, then prove that

$$I_n = \frac{e^{\alpha x}}{\alpha^2+n^2} \sin^{n-1} x (\alpha \sin x - n \cos x) + \frac{n(n-1)}{\alpha^2+n^2} I_{n-2}$$

8. Perform integration $\int \frac{x^3+5x^2-3x+4}{\sqrt{x^2+x+1}} dx$.

9. Use the formula 4 (iv) above to compute $\int x^3 \cos 3x dx$.

10. Prove that $\int x^3 e^{ax} dx = \frac{e^{ax}}{a^4} (a^3 x^3 - 3a^2 x^2 + 6ax - 6) + c$.

11. Establish the reduction formula

$$I_{k+1} = \frac{x}{2ka^2 (x^2 + a^2)^k} + \frac{2k-1}{2ka^2} I_k \quad \text{where } I_{k+1} = \int \frac{dx}{(x^2 + a^2)^{k+1}}$$

[Hints : Integrate I_k by parts using $\frac{1}{(x^2 + a^2)^k}$ as the first function and 1 as second function].

Answers

1. $-\frac{1}{3} \sin x [\cot^2 x + 2 \cos^2 x + 4] + c$.

3 (i) $\frac{2}{7} \cos^{7/2} \theta - \frac{2}{3} \cos^{3/2} \theta + c$.

(ii) $\frac{1}{2} \tan^2 \theta + \log \tan \theta + c$

(iii) $\frac{1.3}{2.4} \sin^{-1} x - \frac{x \sqrt{1-x^2}}{8} (3+2x^2) + c$

(iv) $c - \frac{x^3}{2(a^2+x^2)} + \frac{3}{2} \left(x - a \tan^{-1} \frac{x}{a} \right)$

(v) $\frac{\sin \theta}{4 \cos^4 \theta} + \frac{3 \sin \theta}{8 \cos^2 \theta} + \frac{3}{8} \log (\sec \theta + \tan \theta) + c, x = \tan \theta$

(vi) $c - \sqrt{2ax-x^2} \left(\frac{x}{2} + \frac{3a}{2} \right) + 3a^2 \sin^{-1} \frac{x-a}{a}$

6. (i) $\frac{1}{6a} \cdot \frac{x^4}{(a+bx^2)^3} + \frac{1}{12a^2} \cdot \frac{x^4}{(a+bx^2)^2} + c$

(ii) $2\sqrt{x} (1+x^2)^{-1/4} + c$

(iii) $c - \frac{1}{3} (x^2 + 5x + 30) \sqrt{4x-x^2} + 10 \cos^{-1} \left(1 - \frac{x-2}{2} \right)$

8. $\left(\frac{1}{3} x^3 + \frac{25}{12} x - \frac{163}{24} \right) \sqrt{x^2+x+1} + \frac{85}{16} \sinh^{-1} \left(\frac{2x+1}{\sqrt{3}} \right) + c$

9. $\frac{1}{27} (9x^2-2) \cos 3x + \frac{1}{9} (3x^3-2x) \sin 3x + c$.

DIFFERENTIATION OF INDEFINITE INTEGRALS

7.1 INTRODUCTION

Most of the standard integrals contains at least one parameter as seen in the previous chapters. So, the integrand as well as integral is a function of the parameter or parameters and the variable of integration i.e. $\int f(x, a) dx = F(x, a) + c$ for a single parameter 'a'. One thing must be clear at this point that here 'a' is parameter and 'c' is an arbitrary constant. Parameter has a dual character – sometimes acts as a variable and sometimes acts as a simple constant whereas arbitrary constant is always a constant. In *general* integrand is a continuous function of the variable and the parameters (sec. 1.1) and hence the integral becomes a continuous function of the variable and the parameters. Therefore, both the integrand and the integral are differentiable functions of the parameters. Now the aim of this chapter is to show how the differentiation w.r.t. parameter under the integral sign produces new integrand with new integrals without further pains. This introductory chapter will be further extended in chapters 8 and 9 and will be found very useful for future study of Advanced Mathematics and mathematical properties.

7.2 DIFFERENTIATION UNDER THE INTEGRAL SIGN

Consider the case of a single parameter. Let

$$\int f(x, a) dx = F(x, a) + c \quad (1)$$

By def. (1.1), $\frac{\partial}{\partial x} [F(x, a) + c] = f(x, a)$

or $\frac{\partial}{\partial x} F(x, a) = f(x, a) \quad (2)$

[∵ c is an arbitrary const.]

Let $F(x, a)$ be a continuous function of two variables x and a . Therefore from differential calculus

$$\frac{\partial^2 F}{\partial x \partial a} = \frac{\partial^2 F}{\partial a \partial x} \quad (3)$$

or $\frac{\partial}{\partial x} \left(\frac{\partial F}{\partial a} \right) = \frac{\partial}{\partial a} \left(\frac{\partial F}{\partial x} \right) = \frac{\partial}{\partial a} f(x, a) \quad \text{by (2)}$

i.e.
$$\frac{\partial}{\partial x} \left(\frac{\partial F}{\partial a} \right) = \frac{\partial}{\partial a} f(x, a)$$

$\therefore$ integrating both sides w.r.t. x

$$\begin{aligned} \int \frac{\partial}{\partial a} f(x, a) dx &= \int \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial a} \right) dx \\ &= \frac{\partial F}{\partial a} \quad \text{by def. (1.1)} \end{aligned} \quad (4)$$

This result is known as *differentiation under the integral sign*. Because in the left side integrand is differentiated under the integral sign. This result (4) shows that the partial differentiation of the integrand $f(x, a)$ w.r.t. the parameter ' a ' results into the partial differentiation of $F(x, a)$ in the right side with respect the same parameter ' a '. Thus this procedure may be used to deduce many new integrations when one has been performed.

Examples

1. Integrate $\frac{1}{\sqrt{(a^2 - x^2)^3}}$.

We know that
$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a} + c$$

Diff. w.r.t. ' a ',
$$\int \frac{-2a}{2(\sqrt{a^2 - x^2})^3} dx = \frac{1}{\sqrt{1 - \frac{x^2}{a^2}}} \left(-\frac{x}{a^2} \right) + c$$

or
$$\int \frac{1}{(\sqrt{a^2 - x^2})^3} dx = \frac{1}{a^2} \frac{x}{\sqrt{a^2 - x^2}} + c.$$

Alter. This can be integrated by the substitution $x = a \sin \theta$.

Note. Each successive differentiation will raise the power of $\sqrt{a^2 - x^2}$ by two.

2. Compute $\int \frac{1}{(a + \cos x)^3} dx, a < 1$.

If we drop the power, then it is easy to compute the integral

$\int \frac{1}{a + \cos x} dx$ i.e.

$$\int \frac{1}{a + \cos x} dx = \frac{2}{\sqrt{a^2 - 1}} \tan^{-1} \left(\sqrt{\frac{a-1}{a+1}} \tan \frac{x}{2} \right) + c.$$

Now differentiate twice successively w.r.t. ' a '.

$$\begin{aligned} \therefore \int \frac{-1}{(a + \cos x)^2} dx &= 2 \cdot -\frac{1}{2} \cdot \frac{2a}{(\sqrt{a^2-1})^3} \tan^{-1} \left(\sqrt{\frac{a-1}{a+1}} \tan \frac{x}{2} \right) \\ &\quad + \frac{2}{\sqrt{a^2-1}} \cdot \frac{1}{1 + \frac{a-1}{a+1} \tan^2 \frac{x}{2}} \cdot \frac{1}{(a+1)\sqrt{a^2-1}} \tan \frac{x}{2} + c \\ \text{or } \int \frac{1}{(a + \cos x)^2} dx &= \frac{2a}{(\sqrt{a^2-1})^3} \tan^{-1} \left(\sqrt{\frac{a-1}{a+1}} \tan \frac{x}{2} \right) \\ &\quad - \frac{\sin x}{(a^2-1)(a + \cos x)} + c. \end{aligned}$$

Diff. once more, w.r.t. 'a'

$$\begin{aligned} -2 \int \frac{1}{(a + \cos x)^3} dx &= \left[\frac{2}{(\sqrt{a^2-1})^3} + \frac{2a \cdot 3 \cdot 2a}{2(\sqrt{a^2-1})^5} \right] \cdot \tan^{-1} \left(\sqrt{\frac{a-1}{a+1}} \tan \frac{x}{2} \right) \\ &\quad + \frac{2a}{(\sqrt{a^2-1})^3} \cdot \frac{\sin x}{2(a + \cos x)\sqrt{a^2-1}} \\ &\quad - \sin x \left[-\frac{2a}{(a^2-1)^2} \left(\frac{1}{a + \cos x} \right) + \frac{1}{a^2-1} \cdot \frac{-1}{(a + \cos x)^2} \right] + c \\ \text{or } \int \frac{1}{(a + \cos x)^3} dx &= \frac{2a^2-1}{(\sqrt{a^2-1})^5} \tan^{-1} \left(\sqrt{\frac{a-1}{a+1}} \tan \frac{x}{2} \right) \\ &\quad - \frac{(\sin x)(3a^2 + a \cos x - 1)}{2(a^2-1)^2(a + \cos x)^2}. \end{aligned}$$

Exercises 7

1. Show. that

$$\int \frac{1}{(x^2 - a^2)^3} dx = \frac{1}{8a^4} \log \frac{x-a}{x+a} - \frac{1}{16a^4} \cdot \frac{a+4}{x^2 - a^2} + \frac{1}{a(x^2 - a^2)^2}.$$

2. If $I = \int \frac{1}{x^2 + a^2} dx$, then prove that

$$\frac{\partial^2 I}{\partial a^2} = \frac{2}{a^3} \tan^{-1} \frac{x}{a} + \frac{2x^2}{a^2(x^2 + a^2)} + \frac{2x}{(x^2 + a^2)^2}.$$

3. If $\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$, then show that

$$\int \frac{1}{(x^2 + a^2)^2} dx = -\frac{1}{2a^3} \tan^{-1} \frac{x}{a} - \frac{x}{2a^2(a^2 + x^2)} + c.$$

4. If $I = \int e^{ax} dx$ then show that

$$\frac{\partial^2 I}{\partial a^2} = \left(x - \frac{1}{a}\right) \frac{\partial I}{\partial a} + \frac{1}{a^2} I.$$

5. If $I = \int \frac{dx}{x^2 + a^2}$ then prove that

$$\int \frac{dx}{(x^2 + a^2)^3} = \frac{1}{8a^3} \left[-\frac{\partial I}{\partial a} + \frac{2}{a} I + \frac{2x(2a^2 + x^2)}{(x^2 + a^2)^2} \right].$$

6. Prove that $\int \frac{1}{(x+c)\sqrt{x-a}} dx = \frac{2}{\sqrt{c+a}} \tan^{-1} \sqrt{\frac{x-a}{c+a}}, (c+a) > 0.$

Hence deduce that

$$(i) \int \frac{1}{(x+c)^{n+1} \sqrt{x-a}} = \frac{2(-1)^n}{n!} \cdot \frac{\partial^n}{\partial c^n} \left\{ \frac{1}{\sqrt{c+a}} \tan^{-1} \sqrt{\frac{x-a}{c+a}} \right\}$$

$$(ii) \int \frac{1}{(x+c)(x-a)^{(2n+1)/2}} dx \\ = \frac{2^{n+1}}{1.3.5 \dots (2n-1)} \cdot \frac{\partial^n}{\partial a^n} \left\{ \frac{1}{\sqrt{c+a}} \tan^{-1} \sqrt{\frac{x-a}{c+a}} \right\}$$

$$(iii) \int \frac{dx}{(x+c)^{p+1} (x-a)^{(2n+1)/2}} \cdot \frac{2^{n+1}}{p! 1.3 \dots (2n-1)} \cdot \frac{\partial^{n+p}}{\partial c^p \partial a^n} \\ \left\{ \frac{1}{\sqrt{c+a}} \tan^{-1} \sqrt{\frac{x-a}{c+a}} \right\}$$

7. Perform the integration $\int \frac{dx}{(1+2\cos x)^3}$

8. Given that $\int x^n dx = \frac{x^{n+1}}{n+1}, n \neq -1$. Prove that

$$\int x^n (\log x)^n dx = \frac{x^{n+1}}{n+1} \left[\left(\log x - \frac{1}{n+1} \right)^3 + \frac{3 \log x}{(n+1)^2} - \frac{5}{(n+1)^3} \right]$$

9. Compute $\int x(x-1)a^{x-2} dx$.

10. Perform the integration $\int x^3 \cos ax dx$. Hence find $\int x^3 \cos 2x dx$.

Answers

$$7. \frac{-\frac{2}{3} \sin x \cos x}{(1+2 \cos x)^2} - \frac{1}{9} \sqrt{3} \log \left\{ \frac{\cos \left(\frac{1}{2} x + \frac{\pi}{6} \right)}{\cos \left(\frac{x}{2} - \frac{\pi}{6} \right)} \right\}$$

$$[\text{Hints : Diff. twice } \int \frac{dx}{a+b \cos x} = \frac{1}{\sqrt{b^2-a^2}} \log \frac{\sqrt{b+a} + \sqrt{b-a} \tan \frac{x}{2}}{\sqrt{b+a} - \sqrt{b-a} \tan \frac{x}{2}}]$$

if $a < b$, w.r.t. 'a' and put $a = 1, b = 2$]

$$9. \frac{a^x - 1}{\log a} \left[\left(x - \frac{1}{\log a} \right)^2 + \frac{1}{a (\log a)^2} \right]$$

$$[\text{Hints : Start with } \int a^x dx = \frac{a^x}{\log a}]$$

$$10. (a^2 x^2 - 6) \frac{x \cos ax}{a^3} + (x^2 a^2 - 2) \frac{3 \cos ax}{a^4};$$

$$(2x^2 - 3) \frac{x \sin 2x}{4} + (2x^2 - 1) \frac{3 \cos 2x}{8}.$$

DEFINITE INTEGRAL AND THEORY OF INTEGRATION

8.1. INTRODUCTION

Definite integral *definitizes* the arbitrary constants of the indefinite integral from many valued-ness (sec. 1.1) to uniqueness. This is carried out through some limiting algebraic sum. The study of this limiting process leading to the definite integral advances in two distinct but equivalent approaches—(i) *Geometric* (sec. 8.2) and (ii) *Analytic* (sec. 8.3). The first approach has a wide range of practical utility since 1850 B.C. (sec. 11.1) and since then it has been used in determining area bounded by curves, length of curves, volume, work, velocity, moment of inertia and even in biologic and economic problems. This approach is less suitable for the development of the theory. On the other hand the second approach is most suitable for theoretical development and provides with properties of definite integral (sec. 8.7) and condition of integrability of functions due to German mathematician G.F.B. Riemann (1826—1866) who published the definition “definite integral as a sum” in his paper in 1850 and so it is better known as *Riemannian integral*. The functions under this integral are known as Riemann integrable and functions non-integrable in this sense are brought to the cases of more general Lebesgue integral based on measure theory due to Henri Lebesgue (1875—1941) and Stieltjes (or Riemann—Stieltjes) integral due to Dutch mathematician T.J. Stieltjes (1856—1894). But the bounded continuous functions which are treated here, are integrable in all these *three* cases. These approaches for the evaluation of definite integral are laborious and time consuming. But the Fundamental theorem of integral calculus (sec. 8.6) provides with a powerful tool as well as time saving procedure for the evaluation of definite integral and this is being used in all applications.

Before going to the definition of definite integral, let us recollect two definitions below :

BOUNDED FUNCTIONS. Let $f(x)$ be defined in the closed interval $[a, b]$. Let m and M be two finite numbers such that $m \leq f(x) \leq M$ for all x in $[a, b]$. Then m is called the *lower bound*, M is called the *upper bound* of $f(x)$ and $f(x)$ is called a bounded function. If bounds are equal in magnitude, M but opposite in sign then it is expressed by $|f(x)| \leq M$.

LEAST UPPER BOUND (l.u.b.) and **GREATEST LOWER BOUND (g.l.b.)** are minimum of all the upper bounds and the maximum of all the lower bounds of $f(x)$ respectively.

8.2. THE GEOMETRICAL APPROACH (Area Concept)

The definition of definite integral in geometrical approach is based on the concept of area bounded by a continuous and positive function $f(x)$, the x -axis and the two ordinates. [$f(x) > 0$ i.e. positive means, the graph of $y = f(x)$ is above the x -axis.]

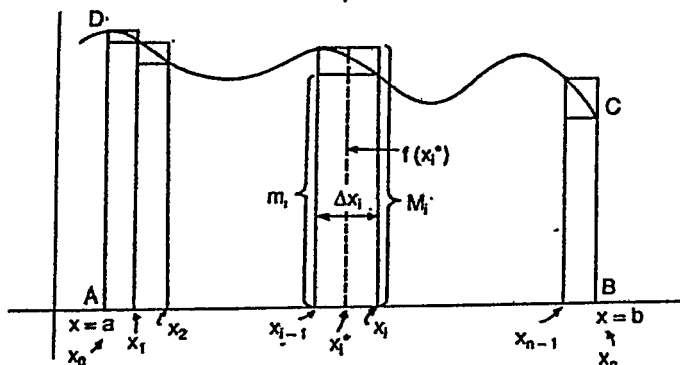


Fig. 2

Let $f(x)$ be a bounded continuous +ve function defined in the closed interval $a \leq x \leq b$. Let $ABCD$ be the area enclosed by $f(x)$, x -axis, the ordinates at $x = a$ and $x = b$. Let us divide the given interval by the points $a = x_0, x_1, x_2, \dots, x_n = b$ into n subintervals of unequal lengths $\Delta x_i = x_i - x_{i-1}$ such that $x_{i-1} < x_i, i = 1, 2, \dots, n$. Let m_i and M_i be g.l.b. and l.u.b. of $f(x)$ in the subintervals $[x_{i-1}, x_i]$. Let $x = x_i^*$ be an arbitrary point in $[x_{i-1}, x_i]$ such that $x_{i-1} \leq x_i^* \leq x_i$. Now let us erect ordinates at those points and complete the rectangles of heights m_i and M_i with length Δx_i both inscribing and circumscribing the curve $y = f(x)$ (fig. 2). Then evidently the ordinate $f(x_i^*)$ lies between the ordinates m_i and M_i i.e. the three ordinates hold the inequality $m_i \leq f(x_i^*) \leq M_i$. Multiplying by the length Δx_i , we have the inequality of areas, $\Delta x_i m_i \leq \Delta x_i f(x_i^*) \leq \Delta x_i M_i$ summing over $i = 1$ to n , we get,

$$\sum_{i=1}^n \Delta x_i m_i \leq \sum_{i=1}^n \Delta x_i f(x_i^*) \leq \sum_{i=1}^n \Delta x_i M_i \quad (1)$$

As the area $ABCD$ inscribes the area $\sum_{i=1}^n \Delta x_i m_i$ and the area

$\sum_{i=1}^n \Delta x_i M_i$ circumscribes the area $ABCD$, then the three areas hold the inequality

$$\sum_{i=1}^n \Delta x_i m_i \leq ABCD \leq \sum_{i=1}^n \Delta x_i M_i \quad (2)$$

Now in the limit when greatest $\Delta x_i \rightarrow 0$ i.e. number of divisions increases i.e. $n \rightarrow \infty$ then both inscribed and circumscribed area coincide on the curve $y = f(x)$ and in the limit

$$\text{area } ABCD = \lim_{n \rightarrow \infty} \sum_{i=1}^n \Delta x_i f(x_i^*)$$

The right hand limiting sum is symbolised as $\int_a^b f(x) dx$ and is called

the *definite integral* of $f(x)$ from 'a' to 'b'. 'a' is called the *lower limit* and 'b' is called the *upper limit* of the definite integral. These "limits" do not have any identity with the operation limit. Therefore, the *definite integral is a process of summation* and so to say areas can be sliced into smaller areas and can be determined by the process of summation.

The same reasoning is applicable to -ve functions only reckoning the convention of -ve area below the x -axis. So following the above discussion the definition of definite integral can be summarised as :

DEFINITION OF INTEGRAL AS THE LIMIT OF A SUM.

Let $f(x)$ be a bounded continuous function defined in the interval $a \leq x \leq b$. Let the interval be subdivided by the points $a = x_0 < x_1 < \dots < x_n = b$ with unequal lengths $\Delta x_i = (x_i - x_{i-1})$ and $x = x_i^*$ be an arbitrary point in $[x_{i-1}, x_i]$, $i = 1, 2, 3, \dots, n$, then

$\lim_{n \rightarrow \infty} \left[\sum_{i=1}^n \Delta x_i f(x_i^*) \right]$ is called the **DEFINITE INTEGRAL** of $f(x)$ and

is denoted by $\int_a^b f(x) dx$, where x is a *dummy variable* and 'a' & 'b' are

lower and upper limits respectively.

8.2.1. THE WORKABLE DEFINITION (A modification)

In applications, the definition of sec. 8.2, is not very suitable for every calculation. So, to make the calculation easy and handy the definition of sec. 8.2 requires following modifications :

(i) Sub-intervals are taken to be of equal lengths instead of arbitrary lengths;

(ii) arbitrary point $x = x_i^*$ in the subintervals is considered at either end points of the subintervals instead of anywhere in the subintervals.

On the basis of these modifications, let the interval $[a, b]$ be subdivided by the points $a = a, a + h, a + 2h, \dots, a + rh, \dots, a + nh = b$, so that $nh = (b - a)$ i.e. equal length of subintervals, $h = (b - a)/n$. Now erecting ordinates at the left end of each subintervals and completing the rectangles irrespective of inscribing and circumscribing the curve $y = f(x)$, the total area of the rectangles is

$$h [f(a) + f(a + h) + \dots + f(a + rh) + \dots + f(a + (n - 1)h)] \\ = h \sum_{r=0}^{n-1} f(a + rh) \quad (1)$$

Similar construction of rectangles with ordinates at the right of each subintervals will give the total area of the rectangles as

$$h [f(a + h) + f(a + 2h) + \dots + f(a + r + 1)h + \dots + f(a + nh)] \\ = h \sum_{r=1}^n f(a + rh) \quad (2)$$

Taking limit when $h \rightarrow 0$ i.e. making the length of subintervals sufficiently small, from (1) and (2)

$$\lim_{h \rightarrow 0} h \sum_{r=0}^{n-1} f(a + rh) = \text{Area } ABCD = \lim_{h \rightarrow 0} \sum_{r=1}^n f(a + rh) \\ \text{i.e. } \lim_{h \rightarrow 0} h \sum_{r=0}^{n-1} f(a + rh) = \int_a^b f(x) dx = \lim_{h \rightarrow 0} \sum_{r=1}^n f(a + rh)$$

where $nh = b - a$. So, in practice any one of the above limiting values may be considered.

8.3. ANALYTICAL APPROACH : (The Riemann integral)

Let the function $f(x)$ be defined in the closed interval $[a, b]$. Let us divide the interval $[a, b]$ into n -subintervals $[a = x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n = b]$ such that $a = x_0 < x_1 < x_2, \dots, x_{n-1} < x_n = b$. Let m_i and M_i be the lower and the upper bounds of $f(x)$ in subintervals $[x_{i-1}, x_i]$, $i = 1, 2, \dots, n$ (fig. 2). Let S and s denote the following sum of the products :

$$S = M_1 (x_1 - x_0) + M_2 (x_2 - x_1) + \dots + M_n (x_n - x_{n-1})$$

$$\text{and } s = m_1 (x_1 - x_0) + m_2 (x_2 - x_1) + \dots + m_n (x_n - x_{n-1}).$$

The sum S is called **UPPER SUM** as it is constructed using the upper bounds M_i 's and the sum s is called **LOWER SUM** as it is constructed using the lower bounds m_i 's. Then evidently $s \leq \int_a^b f(x) dx \leq S$.

The sets of numbers S and s can be constructed from all possible regular or irregular (i.e. equal or unequal lengths) modes of sub divisions of the interval $[a, b]$. Then the

$$g.l.b \text{ of } S = J = \int_a^b f(x) dx$$

and

$$l.u.b. \text{ of } s = I = \int_a^b f(x) dx$$

are respectively called **UPPER INTEGRAL SUM** and **LOWER INTEGRAL SUM**. If either $I = J$ or $J - I = \epsilon$, for some pre-assigned $\epsilon > 0$, then $f(x)$ is said to be *Riemann integrable* and their common value is called the *definite integral* of $f(x)$ from 'a' to 'b' and is denoted by $\int_a^b f(x) dx$.

DEFINITION OF DEFINITE INTEGRAL.

If $f(x)$ is bounded and either $I = J$ or $J - I = \epsilon$ for some +ve pre-assigned small number ϵ , then $f(x)$ is said to be integrable and the common value of the lower and the upper integral sum is called the **DEFINITE INTEGRAL** of $f(x)$ over $[a, b]$ and is denoted by $\int_a^b f(x) dx$.

8.4. SOME IMPORTANT THEOREMS

Statements of the properties of lower and upper integral sum, integrable functions and necessary and sufficient conditions are given below. For proofs see Integral Calculus of Ferrar, Oxford 1985 and Calculus with Analytic Geometry of W.K. Smith, Macmillan, 1969.

(a) *Properties of upper and lower integral sum :*

(1) If $f(x)$ is bounded in closed interval $[a, b]$ and J is the upper integral sum of $f(x)$ over $[a, b]$ then for small $\epsilon > 0$ there exists a number ζ such that

$$J \leq S < J + \epsilon$$

where S is the upper sum of subintervals of length less than ζ .

(2) If $f(x)$ is bounded in closed interval $[a, b]$ and I is the lower integral sum of $f(x)$ over $[a, b]$ then for small $\varepsilon > 0$ there exists a number η such that $I - \varepsilon < s \leq I$

where s is the lower sum of subintervals of length less than η .

(b) *Integrable functions :*

(1) Continuous functions defined in a closed interval $[a, b]$ is integrable in $[a, b]$.

(2) Bounded monotonic functions $f(x)$ in closed interval $[a, b]$ are integrable. Monotonic function means either monotonic increasing i.e. $f(x_1) \leq f(x_2)$ for $x_1 < x_2$ etc. or monotonic decreasing i.e. $f(x_1) \geq f(x_2)$ for $x_1 < x_2$ etc. for all x_1, x_2 in $[a, b]$.

(3) If $f(x)$ and $g(x)$ are bounded integrable in $[a, b]$ then

(i) $cf(x)$, c is a constant

(ii) $f(x) + g(x)$

(iii) $f(x)g(x)$

are all bounded integrable in $[a, b]$.

(4) If a function $f(x)$ is either piecewise continuous or piecewise monotonic in $[a, b]$ then $f(x)$ is integrable in $[a, b]$. A piecewise continuous or piecewise monotonic function is continuous or monotonic by pieces or subintervals when $[a, b]$ is divided into a finite number of subintervals.

(c) *Necessary and sufficient condition for integrability :*

Let S and s denote the upper and the lower sums of $f(x)$ in the closed interval $[a, b]$. Then the necessary and sufficient condition for integrability of $f(x)$ in $[a, b]$ is that for at least one subinterval $S - s < \varepsilon$ where ε is some small +ve number.

8-5. MEAN VALUE THEOREM FOR INTEGRAL

If $f(x)$ is a continuous function in the closed interval $a \leq x \leq b$ i.e. in $[a, b]$ then there is at least one value x_1 of x in $a \leq x_1 \leq b$ such that

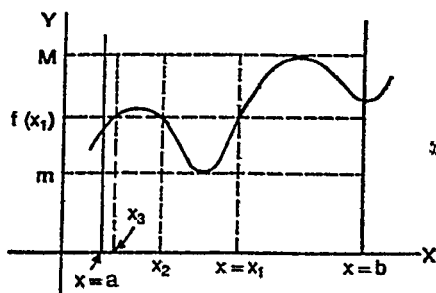
$$\int_a^b f(x) dx = (b - a) f(x_1)$$

Geometrical interpretation :

The area enclosed by $y = f(x)$, x -axis, ordinates $x = a$ and $x = b$, equals the area contained by the length of the interval $(b - a)$ and the ordinate $f(x_1)$ (fig. 3).

Proof. There arises two cases—

(i) $b = a$ and (ii) $b > a$.



Case (i). If $b = a$ then the theorem is seen to be true but trivial as both sides equal to zero.

Case (ii). Since $f(x)$ is continuous, it must have a minimum and a maximum. Let $m = g.l.b.$ of $f(x)$ and $M = l.u.b.$ of $f(x)$ in $a \leq x \leq b$ (fig. 3). Therefore, for all

$$x = x_i, \quad m \leq f(x_i) \leq M$$

or, multiplying by Δx_i , $\Delta x_i m \leq \Delta x_i f(x_i) \leq \Delta x_i M$

$$\text{Now summing over } i = 1 \text{ to } n, \quad \sum_{i=1}^n \Delta x_i m \leq \sum_{i=1}^n \Delta x_i f(x_i) \leq \sum_{i=1}^n \Delta x_i M$$

Taking limit as $n \rightarrow \infty$,

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \Delta x_i m \leq \lim_{n \rightarrow \infty} \sum_{i=1}^n \Delta x_i f(x_i) \leq \lim_{n \rightarrow \infty} \sum_{i=1}^n \Delta x_i M$$

Since $\sum_{i=1}^n \Delta x_i = b - a$ and m & M being constants, can be taken out

of the summation. Therefore,

$$m(b-a) \leq \int_a^b f(x) dx \leq M(b-a), \text{ using def. of integral}$$

$$\text{or} \quad m \leq \frac{1}{b-a} \int_a^b f(x) dx \leq M \quad (1)$$

$$\text{Let } \lambda = \frac{1}{b-a} \int_a^b f(x) dx. \text{ Therefore } m \leq \lambda \leq M.$$

Now by the Intermediate Value Theorem of differential calculus which states that for a continuous function $f(x)$ in $a \leq x \leq b$ and a number λ such that $m \leq \lambda \leq M$ there is at least one point x_1 such that $a \leq x_1 \leq b$ at which $f(x_1) = \lambda$ (fig. 3 shows 3 such points),

$$f(x_1) = \frac{1}{b-a} \int_a^b f(x) dx \quad \text{from (1)}$$

$$\text{or} \quad \int_a^b f(x) dx = (b-a)f(x_1)$$

which establishes the theorem.

Useful notes.

(i) Theorem does not hold true for discontinuous function.

(ii) It is applicable to the problems of physics and engineering as will be seen in centre of mass, centroid etc. and in economics to calculate average cost and average revenue etc.

(iii) It calculates the average mean value of a function $f(x)$ in a closed interval $a \leq x \leq b$ as

$$f(x_0) = \frac{\int_a^b f(x) dx}{b-a} \text{ is the average ordinate of the rectangle (fig. 3).}$$

(iv) It is used in determining "root mean square" in statistics as

$$\left\{ \frac{1}{b-a} \int_a^b [f(x)]^2 dx \right\}^{1/2}$$

(v) If $f(x)$ is continuous in the interval of integration, then there is a number θ , $0 < \theta < 1$, such that

$$\int_a^b f(x) dx = (b-a) f\{a + \theta(b-a)\}$$

(vi) Extension : If two functions $f(x)$ and $g(x)$ are defined in $a \leq x \leq b$ of which $f(x)$ is continuous in $a \leq x \leq b$ and $g(x)$ does not change sign but is integrable in the interval, then there exists at least one value $x = x_1$ in $a \leq x_1 \leq b$ such that

$$\int_a^b f(x) g(x) dx = f(x_1) \int_a^b g(x) dx.$$

This reduces to the above theorem when $g(x) = 1$.

Hints : Since $f(x)$ is continuous in $a \leq x \leq b$, let $M = l.u.b.$ of $f(x)$ and $m = g.l.b.$ of $f(x)$. So,

$$m \leq f(x) \leq M.$$

or

$$mg(x) \leq f(x) g(x) = Mg(x)$$

or

$$\int_a^b mg(x) dx \leq \int_a^b f(x) g(x) dx \leq \int_a^b Mg(x) dx$$

or

$$m \leq \frac{\int_a^b f(x) g(x) dx}{\int_a^b g(x) dx} \leq M.$$

By Intermediate Value Theorem of Differential Calculus, as before,

$$f(x_1) = \frac{\int_a^b f(x) g(x) dx}{\int_a^b g(x) dx} \text{ etc.}$$

8.6. FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

If $f(x)$ is continuous in the closed interval $[a, b]$ (i) then $f(x)$ has an indefinite integral in $[a, b]$ and (ii) if $F(x)$ is a indefinite integral of $f(x)$ in $[a, b]$, then

$$\int_a^b f(x) dx = F(b) - F(a).$$

The first part of the theorem gives a relation between differentiation and integration while the second part gives an easy method of evaluation of definite integral against the definitions in sec. 8.2 and 8.3.

Proof. (i) Let $F(x) = \int_a^x f(t) dt$, for $a \leq x \leq b$. If we can show that

$F'(x) = f(x)$ then it will be proved by definition 1.1 that $F(x)$ is an indefinite integral of $f(x)$ in $[a, b]$. Now from the definition of derivative,

$$\begin{aligned} F'(x) &= \lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\int_a^{x+h} f(t) dt - \int_a^x f(t) dt \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\int_a^{x+h} f(t) dt + \int_x^a f(t) dt \right], \text{ by Prop. 6 sec. 8.7} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\int_x^{x+h} f(t) dt \right], \text{ by Prop. 7 sec. 8.7} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \cdot h f(x + \theta h), \text{ by note (v) sec. 8.5} \\ &= \lim_{h \rightarrow 0} f(x + \theta h) = f(x) \end{aligned}$$

$\therefore f(x)$ is continuous, so $F'(x)$ exists and $F'(x) = f(x)$.

$\therefore$ the derivative of the definite integral of a continuous function w.r.t. the upper limit, x of integration equals the value of the integrand at this

upper limit, x . Hence it establishes a relation between the derivative and the integral.

$$(ii) \quad \text{Let } G(x) = \int_a^x f(t) dt \quad (1)$$

$$\therefore G'(x) = f(x) = F'(x) \text{ by (i)}$$

$\therefore$ the derivatives of $F(x)$ and $G(x)$ are the same, therefore by Th. 1 sec. 1.1 $G(x) = F(x) + C$ (2)

$$\text{From (1), } G(a) = \int_a^a f(t) dt = 0 \text{ by prop. 1 sec. 8.7}$$

$$\text{Also from (2), } G(a) = F(a) + C$$

$$\therefore 0 = F(a) + C \text{ or } C = -F(a) \quad (3)$$

$$\text{Again from (1) \& (2), } G(b) = \int_a^b f(t) dt = F(b) + C$$

$$\therefore \int_a^b f(t) dt = F(b) - F(a) \text{ by (3)} \quad (4)$$

which establishes the second part of the theorem and gives an easy method for evaluation of definite integral without recourse into the summation procedures. This theorem only requires the knowledge of the indefinite integral of $f(t)$ which can be known from the methods of chapter 1 through 7.

N.B. (a) Normal practice of writing (4) is

$$\int_a^b f(t) dt = \left[F(t) \right]_a^b = F(t) \Big|_a^b = F(b) - F(a)$$

i.e. in the integral of $\int f(t) dt = F(t)$ use the upper limit first then the lower limit and subtract the second result from the first.

(b) Also note that the integral $\int_a^x f(t) dt$ is an indefinite integral as by (ii) of the theorem

$$\int_a^x f(t) dt = G(t) \Big|_a^x = G(x) - G(a)$$

$$= G(x) + C \text{ where } C = -G(a) = \text{const.}$$

and $f(t) = G'(t)$

and the integral is a function of the upper limit x .

8.7. PROPERTIES OF DEFINITE INTEGRAL

The properties of definite integral can be established in two ways—
(i) using summation procedures & Riemann integral, (ii) using Fundamental Theorem of Integral Calculus. The second procedure will be used to establish each of the properties one after another. For first method consult Ferrar, Smith etc.

Prop. 1. $\int_a^a f(x) dx = 0.$

Let $f(x) = G'(x) \therefore \int_a^a f(x) dx = G(x) \Big|_a^a = G(a) - G(a) = 0.$

$\therefore$ area enclosed by $y = f(x)$, x -axis and two coincident ordinates is zero. i.e. the area of a line segment (vertical here) is zero.

Prop. 2. $\int_a^b f(x) dx = \int_a^b f(t) dt.$

i.e. the variable of integration is immaterial in definite integral and it is a dummy variable.

Let $f(x) = G'(x) \therefore \int_a^b f(x) dx = G(b) - G(a)$

Also $f(t) = G'(t) \therefore \int_a^b f(t) dt = G(b) - G(a)$

$\therefore \int_a^b f(x) dx = \int_a^b f(t) dt$ i.e. Area enclosed by identical boundaries in different location is same.

Prop. 3. $\int_a^b k f(x) dx = k \int_a^b f(x) dx$, k is a constant. It follows from indefinite integral (sec. 1.7). k -multiple ordinate multiplies the original area when one side remains unchanged.

Prop. 4. $\int_a^b [f(x) \pm g(x)] dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$

It also follows from sec. 1.7.

Increase or decrease of one side-only increases or decreases the area by the increased or decreased amount of area.

Prop. 5. $\int_a^b dx = b - a$

$$\therefore \int_a^b dx = x \Big|_a^b = b - a \text{ i.e. area contained by unit length and given}$$

breadth equals the breadth only.

$$\text{Prop. 6. } \int_a^b f(x) dx = - \int_b^a f(x) dx$$

$$\text{Let } f(x) = G'(x) . \therefore \int_a^b f(x) dx = G(x) \Big|_a^b = G(b) - G(a)$$

$$\text{Also } - \int_a^b f(x) dx = - [G(x)]_b^a = - [G(a) - G(b)] = G(b) - G(a)$$

$$\therefore \int_a^b f(x) dx = - \int_b^a f(x) dx.$$

(Fig. 2) Area $ABCD$ with increasing abscissa (i.e. anticlock-wise description) equals the negative of the area $BACDB$ with decreasing abscissa (i.e. clockwise description). Or area above the x -axis reflects the area below the x -axis.

Prop. 7. Additive property.

$$\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx, a < c < b.$$

This is the *additive* property of definite integral.

$$\begin{aligned} \text{Let } f(x) = G'(x) . \therefore \int_a^b f(x) dx + \int_b^c f(x) dx \\ = G(x) \Big|_a^b + G(x) \Big|_b^c = G(b) - G(a) + G(c) - G(b) \\ = G(c) - G(a) = \int_a^c f(x) dx. \end{aligned}$$

i.e. sum of two separate areas bounded by a curve, the x -axis and the ordinates at $x = a$, b & b , c , equals the area bounded by the same curve, the x -axis and ordinates at $x = a$ & c only.

Prop. 8. Substitution in definite integral.

The relation $\int_a^b f(g(x)) g'(x) dx = \int_\alpha^\beta f(u) du$ holds true if $u = g(x)$ and (i) $g(x)$ is single-valued and continuous in $[a, b]$ and has $g'(x)$

continuous in $[a, b]$; (ii) as x varies from ' a ' to ' b ', so u must be within $[\alpha = g(a), \beta = g(b)]$

Under these conditions all substitutions in chapters 2 and 4 are valid with *additional* work of changing the limit from old to a new but requires *no* revival of the old variable as in chapters 2 and 4.

Prop. 9. Integration by parts

Procedure remains same as in ch. 5 with exceptional use of limits in proper position as follows :

$$\int_a^b u(x)v(x) dx = \left[u(x) \int v(x) dx \right]_a^b - \int_a^b \left[\frac{d}{dx} u(x) \int v(x) dx \right] dx.$$

$$\text{Prop. 10. } \int_0^a f(x) dx = \int_0^a f(a-x) dx.$$

Let us change the variable in the L.H. integral. Let us substitute $x = a - u$ or $dx = -du$ and the limits become $u = a$ and $u = 0$ corresponding to $x = 0$ and $x = a$. Therefore,

$$\begin{aligned} \int_0^a f(x) dx &= - \int_a^0 f(a-u) du = \int_0^a f(a-u) du, \quad \text{by Prop. 6} \\ &= \int_0^a f(a-x) dx, \text{ by Prop. 2.} \end{aligned}$$

Geometrically, the origin is shifted to $(a, 0)$ and figure rotates through an angle $3\pi/4$ keeping size and area of the original figure unchanged. This can be verified with simple example like $\int_0^2 x dx = \int_0^2 (2-x) dx$ where $y=x$ makes an angle $\pi/4$ but $y=2-x$ makes angle $3\pi/4$ with x -axis and origin is shifted to $(2, 0)$.

$$\begin{aligned} \text{Prop. 11. } \int_{-a}^a f(x) dx &= 2 \int_0^a f(x) dx, \text{ when } f(x) \text{ is even;} \\ &= 0, \text{ when } f(x) \text{ is odd.} \end{aligned}$$

Analytically, for even function $f(-x) = f(x)$ and *geometrically* it is symmetrical about y -axis e.g. x^2 , $\cos x$ etc. Similarly, an odd function is one when $f(-x) = -f(x)$ and geometrically symmetrical about the origin e.g. x , $\sin x$, $x \cos x$ etc. For even functions, figure folds about y -axis and coincides in all respect. Similarly, for odd function when one part of the figure rotates through 180° about the origin, two parts coincide completely.

To prove the property, let us use the additive property 7 as

$$\int_{-a}^a f(x) dx = \int_{-a}^0 f(x) dx + \int_0^a f(x) dx.$$

For the first integral, let $x = -u$ so that $dx = -du$ and the limits are $u = a$ & 0 corresponding to $x = -a$ & 0 .

$$\therefore \int_{-a}^0 f(x) dx = - \int_a^0 f(-u) du = \int_0^a f(-x) dx \text{ by props. 6 \& 2.}$$

$$= \begin{cases} \int_0^a f(x) dx, & \text{if } f(x) \text{ is even} \\ - \int_0^a f(x) dx, & \text{if } f(x) \text{ is odd.} \end{cases}$$

$$\therefore \int_{-a}^a f(x) dx = \begin{cases} 2 \int_0^a f(x) dx, & \text{if } f(x) \text{ is even} \\ 0 & \text{if } f(x) \text{ is odd.} \end{cases}$$

For positively or evenly oriented symmetry, the total area equals twice the half area while for negatively or oddly oriented symmetry produces zero area. So, *care* should be taken to get total area for the second case (ch. 11.).

Prop. 12. $\int_0^{na} f(x) dx = n \int_0^a f(x) dx$ if $f(a+x) = f(x)$.

Using additive prop. 7, the L.H. integral can be written as the sum of n integrals as,

$$\begin{aligned} \int_0^{na} f(x) dx &= \int_0^a f(x) dx + \int_a^{2a} f(x) dx + \int_{2a}^{3a} f(x) dx + \dots \\ &\quad + \int_{(n-1)a}^{na} f(x) dx \end{aligned}$$

Using substitution $x = a + u$ and the given condition each one of the integrals on the right from the second onward can be shown equal to the

first and so, $\int_0^{na} f(x) dx = n \int_0^a f(x) dx$.

Geometrically (fig. 4), the condition $f(a+x) = f(x)$ shows that the curve is periodic with period a . So, the total area can be divided into n equal areas by erecting ordinates at points $(2a, 0)$, .. $((n-1)a, 0)$ and each area

equals to $\int_0^a f(x) dx$.

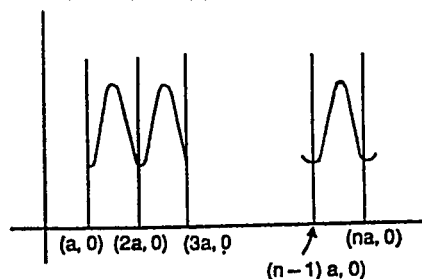


Fig. 4

$$\therefore \int_0^{na} f(x) dx = n \int_0^a f(x) dx$$

if $f(x)$ is periodic with period $= a$ i.e. $f(x+a) = f(x)$.

$$\text{Prop. 13. } \int_0^{2a} f(x) dx = \begin{cases} 2 \int_0^a f(x) dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$$

Again using additive property 7,

$$\int_0^{2a} f(x) dx = \int_0^a f(x) dx + \int_a^{2a} f(x) dx.$$

On the substitution $x = 2a - u$ the second integral on the right reduces to $\int_0^a f(2a-x) dx$. Therefore

$$\begin{aligned} \int_0^{2a} f(x) dx &= \int_0^a f(x) dx + \int_0^a f(2a-x) dx \\ &= \begin{cases} 2 \int_0^a f(x) dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x). \end{cases} \end{aligned}$$

Geometrically, the given condition $f(2a-x) = f(x)$ implies that $f(x)$ is positively oriented symmetric at the line $x=a$ and coincides if folded about $x=a$ and so the area is twice the half area. On the other hand, for the condition $f(2a-x) = -f(x)$ implies $f(x)$ is negatively oriented symmetric about the point $(a, 0)$ and so the area is zero i.e. positive area equals the negative area.

$$\text{Prop. 14. } \left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx$$

In words—the modulus or absolute value of an integral is less than or equal to the integral of the modulus or absolute value of the integrand.

Since integral $\int_a^b f(x) dx = Lt_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x_i$ is a sum and absolute value of a sum is less than or equal to the sum of the absolute values. So

$$\left| \int_a^b f(x) dx \right| \leq Lt_{n \rightarrow \infty} \sum_{i=1}^n |f(x_i^*)| \Delta x_i = \int_a^b |f(x)| dx.$$

Prop. 15. If $f(x)$ and $g(x)$ are continuous in $a \leq x \leq b$ and also $f(x) \leq g(x)$ in the same interval, then $\int_a^b f(x) dx \leq \int_a^b g(x) dx$. This is known as the *general comparison property*.

By hypothesis, $g(x) - f(x) \geq 0$ for $a \leq x \leq b$.

Hence $[g(x) - f(x)]$ is non-negative and continuous in $[a, b]$.

$$\therefore \int_a^b [g(x) - f(x)] dx \geq 0$$

or $\int_a^b g(x) dx - \int_a^b f(x) dx \geq 0$ by linearity property.

$$\therefore \int_a^b f(x) dx \leq \int_a^b g(x) dx.$$

i.e. larger the function, larger the definite integral.

8.8. DIFFERENTIATION UNDER INTEGRAL SIGN

If the definite integral contains a variable parameter, then the integral becomes a function of the parameter. Also a definite integral is a function of its upper and lower limits. Sometimes these integrals cannot be integrated analytically, but it becomes necessary to find the derivative of the integral w.r.t. the parameter. So, a differentiation under the integral sign may arise in the following cases :

- (i) w.r.t. upper or lower limit,
- (ii) w.r.t. parameter in the integrand only,
- (iii) w.r.t. parameter in the limits only,
- (iv) w.r.t. parameter both in the integrand and the limit.

This procedure often helps in evaluating the definite integral (cf. Ex. 16).

Case (i) Since the definite integral is a function of the upper limit 'b' and the lower limit 'a', so,

$$\int_a^b f(x) dx = F(b) - F(a) \text{ when } F'(x) = f(x) \quad (1)$$

$\therefore$ differentiating partially w.r.t. 'a' and 'b' respectively, we have,

$$\frac{\partial}{\partial a} \int_a^b f(x) dx = -\frac{\partial}{\partial a} F(a) = -F'(a) = -f(a) \text{ by (1)}$$

and $\frac{\partial}{\partial b} \int_a^b f(x) dx = \frac{\partial}{\partial b} F(b) = F'(b) = f(b) \text{ by (1)}$

Here continuity of $F(a)$ and $F(b)$ are assumed.

Case (ii) Let $F(\alpha) = \int_a^b f(x, \alpha) dx$. Let $F(\alpha)$ and $\frac{\partial}{\partial \alpha} f(x, \alpha)$ are

continuous in a rectangular region $c \leq \alpha \leq d$ and $a \leq x \leq b$ in the $x\alpha$ -plane and 'a' and 'b' are fixed constants.

$$\text{Then } F'(\alpha) = \int_a^b \frac{\partial}{\partial \alpha} f(x, \alpha) dx.$$

$$\text{Proof: Now } F(\alpha + \Delta\alpha) = \int_a^b f(x, \alpha + \Delta\alpha) dx$$

$\therefore$ by def. of derivative,

$$\begin{aligned} F'(\alpha) &= \lim_{\Delta\alpha \rightarrow 0} \frac{F(\alpha + \Delta\alpha) - F(\alpha)}{\Delta\alpha} \\ &= \lim_{\Delta\alpha \rightarrow 0} \frac{1}{\Delta\alpha} \left[\int_a^b f(x, \alpha + \Delta\alpha) dx - \int_a^b f(x, \alpha) dx \right] \\ &= \lim_{\Delta\alpha \rightarrow 0} \frac{1}{\Delta\alpha} \int_a^b [f(x, \alpha + \Delta\alpha) - f(x, \alpha)] dx \\ &= \lim_{\Delta\alpha \rightarrow 0} \frac{1}{\Delta\alpha} \int_a^b \Delta\alpha \cdot \frac{\partial f(x, \eta)}{\partial \alpha} dx, \alpha < \eta < \alpha + \Delta\alpha \quad \text{by} \end{aligned}$$

mean value theorem for partial derivative, $[f(x+h, y+k) - f(x, y)] = [hf_x(\xi, \eta) + kf_y(\xi, \eta)]$ where $x < \xi < x+h, y < \eta < y+k$.

$\therefore F'(\alpha) = \int_a^b \frac{\partial}{\partial \alpha} f(x, \alpha) dx$, using limit $\Delta\alpha \rightarrow 0$ with which η coincides with α . This result also follows from case (iv) below as $\frac{db}{d\alpha} = \frac{da}{d\alpha} = 0$ there.

Case (iii) Let $F(\alpha) = \int_{a(\alpha)}^{b(\alpha)} f(x) dx$. Let $a(\alpha)$ and $b(\alpha)$ are continuous functions of α . Then

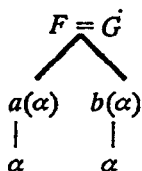
$$F'(\alpha) = f(b(\alpha)) \frac{db}{d\alpha} - f(a(\alpha)) \frac{da}{d\alpha}.$$

Proof: Let $F(\alpha) = G(a(\alpha), b(\alpha)) = \int_{a(\alpha)}^{b(\alpha)} f(x) dx$.

Using chain rule of differentiation of composite functions, we have,

$$F'(\alpha) = \frac{\partial G}{\partial a} \frac{da}{d\alpha} + \frac{\partial G}{\partial b} \frac{db}{d\alpha} \quad (1)$$

$\therefore$ functional tree relation is



Let $Q'(x) = f(x)$. Therefore,

$$\begin{aligned} G(a(\alpha), b(\alpha)) &= \int_{a(\alpha)}^{b(\alpha)} Q'(x) dx \\ &= [Q(x)]_{a(\alpha)}^{b(\alpha)} \\ &= Q(b(\alpha)) - Q(a(\alpha)) \end{aligned}$$

$$\therefore \frac{\partial G}{\partial a} = - \frac{\partial Q(a(\alpha))}{\partial a} = -Q'(a(\alpha)) = -f(a(\alpha))$$

and
$$\frac{\partial G}{\partial b} = \frac{\partial Q(b(\alpha))}{\partial b} = Q'(b(\alpha)) = f(b(\alpha))$$

$$\therefore F'(\alpha) = f(b(\alpha)) \frac{db}{d\alpha} - f(a(\alpha)) \frac{da}{d\alpha} \quad \text{by (1)}$$

This result also follows from case (iv) below as $\frac{\partial}{\partial \alpha} f(x) = 0$ there.

Case (iv) Let $F(\alpha) = \int_{a(\alpha)}^{b(\alpha)} f(x, \alpha) dx$. Let $a(\alpha), b(\alpha)$ and $\frac{\partial}{\partial \alpha} f(x, \alpha)$ are all continuous in rectangular region $c \leq \alpha \leq d, a \leq x \leq b$ in the $x\alpha$ -plane. Then

$$F'(\alpha) = \int_{a(\alpha)}^{b(\alpha)} \frac{\partial}{\partial \alpha} f(x, \alpha) dx + f(b(\alpha), \alpha) \frac{db}{d\alpha} - f(a(\alpha), \alpha) \frac{da}{d\alpha}.$$

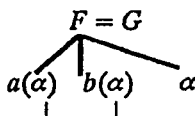
This is known as Liebnitz formula. This also provides a condition for the interchange of differentiation with integration. From this formula, the above three cases follow.

Proof: Let $F(\alpha) = G(\alpha, a(\alpha), b(\alpha)) = \int_{a(\alpha)}^{b(\alpha)} f(x, \alpha) dx$

Now from the chain rule of differentiation of composite functions, we get,

$$F'(\alpha) = \frac{\partial G}{\partial \alpha} + \frac{\partial G}{\partial a} \frac{da}{d\alpha} + \frac{\partial G}{\partial b} \frac{db}{d\alpha} \quad (1)$$

$\therefore$ Functional tree relation is



From case (ii),

$$\frac{\partial G}{\partial \alpha} = \int_{a(\alpha)}^{b(\alpha)} \frac{\partial}{\partial \alpha} f(x, \alpha) dx$$

and from case (iii) it follows that

$$\frac{\partial G}{\partial a} = -f(a(\alpha), \alpha) \text{ \& } \frac{\partial G}{\partial b} = f(b(\alpha), \alpha)$$

$\therefore$ (1) becomes,

$$F'(\alpha) = \int_{a(\alpha)}^{b(\alpha)} \frac{\partial}{\partial \alpha} f(x, \alpha) dx + f(b(\alpha), \alpha) \frac{db}{d\alpha} - f(a(\alpha), \alpha) \frac{da}{d\alpha}.$$

This may be extended to several variables in which case partial derivative is to be used in place of ordinary derivatives.

8.9. INTEGRATION UNDER THE INTEGRAL SIGN

If $F(\alpha) = \int_a^b f(x, \alpha) dx$, $c \leq \alpha \leq d$ and $f(x, \alpha)$ is continuous in x

and α in a region defined by $a \leq x \leq b$, $c \leq \alpha \leq d$ then for constants 'a' and 'b'

$$\int_c^d F(\alpha) d\alpha = \int_c^d \left\{ \int_a^b f(x, \alpha) dx \right\} d\alpha = \int_a^b \left\{ \int_c^d f(x, \alpha) d\alpha \right\} dx.$$

This is known as *integration under the integral sign or interchange of order of integration*.

Proof: Let $\psi(\beta) = \int_a^b \left\{ \int_c^\beta f(x, \alpha) d\alpha \right\} dx$ (1)

Differentiating under the integral sign w.r.t. β ,

$$\begin{aligned} \psi'(\beta) &= \int_a^b \frac{\partial}{\partial \beta} \left\{ \int_c^\beta f(x, \alpha) d\alpha \right\} dx \\ &= \int_a^b f(x, \beta) dx = F(\beta) \end{aligned}$$

Now integrating w.r.t. β , between c to β

$$\psi(\beta) = \int_c^\beta F(\beta) d\beta + K \quad (2)$$

From (1), $\psi(c) = 0$, so $K = 0$ in (2). Therefore from (1), and (2)

$$\int_a^b \left\{ \int_c^\beta f(x, \alpha) d\alpha \right\} dx = \psi(\beta) = \int_c^\beta \left\{ \int_a^b f(x, \beta) dx \right\} d\beta$$

Using $\beta = \alpha$ and finally changing limit from α to d we get the desired result.

N.B. The use of this result will be found in ch. 10 and ch 16.

8·10. ESTIMATION OF INTEGRALS

Sometimes it is necessary to estimate an integral when actual evaluation is *troublesome* or even *impossible* by the known methods in closed form. This is done by using prop. 14, 15 in sec. 8·7, Mean value theorem both normal and extended form in sec. 8·5 and using the relation

$$m(b-a) \leq \int_a^b f(x) dx \leq M(b-a) \quad (1)$$

where $m = g.l.b.$ of $f(x)$ and $M = l.u.b.$ of $f(x)$ on the interval $[a, b]$. From

(1) the lower bound of $\int_a^b f(x) dx$ is $m(b-a)$ which is less than or equal

to the integral $\int_a^b f(x) dx$ and the upper bound of this integral is

$M(b-a)$ which is greater than or equal to the integral.

8·11. SUMMATION OF SERIES

It is seen that definite integral is the sum of a series. So, inversely some finite series of n terms in the limit when $n \rightarrow \infty$ can be expressed as a definite integral and thus the evaluation of the definite integral gives the sum of the series.

$$\text{From sec. 8·2·1, } \int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{r=0}^{n-1} f(a+rh)$$

where $nh = b-a$, can be written as

$$\begin{aligned} \int_0^1 f(x) dx &= \lim_{n \rightarrow \infty} h \sum_{r=0}^{n-1} f(rh) \quad \text{when } a=0, b=1 \text{ and } nh=1 \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=0}^{n-1} f\left(\frac{r}{n}\right) \end{aligned} \quad (1)$$

To use this formula, the following procedures may be adopted :

(i) Normally, the series should be of n terms. But addition or omission of one, two or even more terms will not alter the limiting procedure, the formula (1) can be written as

$$\int_0^1 f(x) dx = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=k}^{n+l} f\left(\frac{r}{n}\right)$$

where k and l are independent of n .

(ii) Always take out a factor, $\frac{1}{n}$ and arrange the remaining terms as function of $\left(\frac{r}{n}\right)$.

(iii) Put the series in summation notation.

(iv) Compare $\left(\frac{r}{n}\right)$ with x , $\left(\frac{1}{n}\right)$ with dx and $\left(\lim_{n \rightarrow \infty} \Sigma\right)$ with $\int$ sign.

(v) Put the series equal to $\int_0^1 f(x) dx$ and perform the integration.

(vi) If, in summation notation, upper range of r is $(an+l)$ instead of $(n+l)$ then upper limit of integration will be ' a ' instead of 1 i.e. the symbol

$$\int_0^1 \text{ will be } \int_0^a \text{ (cf Ex. 15). Because}$$

$$\left(\text{Lower limit of integral}\right) = (\text{Lowest value of the sum as } n \rightarrow \infty)$$

$$= \lim_{n \rightarrow \infty} \frac{k}{n}, \left[\because \frac{r}{n} = \frac{k}{n} \text{ in the first term of the series} \right]$$

$$= 0$$

and

$$\left(\text{Upper limit of integral}\right) = (\text{Highest value of the sum as } n \rightarrow \infty)$$

$$= \lim_{n \rightarrow \infty} \frac{an+l}{n}, \left[\because \frac{r}{n} = \frac{an+l}{n} \text{ in the last term of the series} \right]$$

$$= \lim_{n \rightarrow \infty} \left(a + \frac{l}{n}\right)$$

$$= a$$

N.B. Normally, lower limit is always 0 (zero) and upper limit is 1 or a as r varies from k to $(n+l)$ or $(an+l)$

8.12. REDUCTION FORMULAE AND WALLI'S FORMULAE

Reduction formulae similar to indefinite integral can be derived for definite integrals too. To have an idea two reduction formulae are shown below :

$$(a) (i) \text{ Reduction formula for } \int_0^{\pi/4} \sin^m x dx.$$

$$\text{Let } I_m = \int_0^{\pi/4} \sin^m x dx = \int_0^{\pi/4} \sin^{m-1} x \sin x dx$$

$$= \left[-\sin^{m-1} x \cos x \right]_0^{\pi/4} + \int_0^{\pi/4} (m-1) \sin^{m-2} x \cos^2 x dx$$

$$= -2^{-m/2} + (m-1) \int_0^{\pi/4} \sin^{m-2} x (1 - \sin^2 x) dx$$

$$= -2^{-m/2} + (m-1) I_{m-2} - (m-1) I_m$$

$$\text{or. } I_m = \frac{-2^{-m/2}}{m} + \frac{(m-1)}{m} I_{m-2}, \quad m \geq 0$$

$$(ii) \text{ Reduction formula for } \int_0^{\pi/4} \tan^m x \sec^n x dx$$

As in 6.9 (B) let us reduce the power of $\sec x$. Let

$$\begin{aligned} I_{m,n} &= \int_0^{\pi/4} \tan^m x \sec^n x dx = \int_0^{\pi/4} (\tan^m x \sec^2 x) \sec^{n-2} x dx \\ &= \left[\frac{\tan^{m+1} x}{m+1} \sec^{n-2} x \right]_0^{\pi/4} \\ &\quad - \int_0^{\pi/4} \frac{(n-2)}{m+1} \sec^{n-3} x \sec x \tan x \tan^{m+1} x dx \\ &= \frac{2^{(n-2)/2}}{m+1} - \frac{n-2}{m+1} \int_0^{\pi/4} \tan^m x (\sec^2 x - 1) \sec^{n-2} x dx \\ &= \frac{2^{(n-2)/2}}{m+1} - \frac{n-2}{m+1} (I_{m,n} - I_{m,n-2}) \\ \therefore I_{m,n} &= \frac{2^{(n-2)/2}}{m+n-1} + \frac{n-2}{m+n-1} I_{m,n-2}. \end{aligned}$$

(b) *Walli's Formulae* : These are important for evaluating definite integrals (i) and (ii) below until the gamma function in ch. 10 is known.

$$\begin{aligned} (i) \int_0^{\pi/2} \sin^n x dx &= \int_0^{\pi/2} \cos^n x dx \\ &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}, \quad \text{if } n \text{ is even +ve integer.} \\ &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{4}{5} \cdot \frac{2}{3}, \quad \text{if } n \text{ is odd +ve integer.} \end{aligned}$$

$$\begin{aligned} \text{Let } I_n &= \int_0^{\pi/2} \sin^n x dx = \int_0^{\pi/2} \sin^n \left(\frac{\pi}{2} - x \right) dx, \text{ by prop. 10} \\ &= \int_0^{\pi/2} \cos^n x dx. \end{aligned}$$

$$\begin{aligned}
 \text{Now, } I_n &= \int_0^{\pi/2} \sin^{n-1} x \sin x \, dx \\
 &= -\sin^{n-1} x \cos x \Big|_0^{\pi/2} + \int_0^{\pi/2} (n-1) \sin^{n-2} x \cos^2 x \, dx, \text{ (I.b.p.)} \\
 &= (n-1) \int_0^{\pi/2} \sin^{n-2} x (1 - \sin^2 x) \, dx \\
 &= (n-1) I_{n-2} - (n-1) I_n
 \end{aligned}$$

or $I_n = \frac{n-1}{n} I_{n-2}$ is a reduction formula.

$$\therefore I_n = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-3} \cdots \frac{4}{5} \cdot \frac{2}{3} I_1, & \text{if } n \text{ is odd + ve integer} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-3} \cdots \frac{3}{4} \cdot \frac{1}{2} I_0 & \text{if } n \text{ is even + ve integer.} \end{cases}$$

$$\text{Now } I_1 = \int_0^{\pi/2} \sin x \, dx = -\cos x \Big|_0^{\pi/2} = 1$$

$$\text{and } I_0 = \int_0^{\pi/2} dx = \frac{\pi}{2}$$

$\therefore$ Walli's formula stands as

$$\begin{aligned}
 I_n &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}, \text{ if } n \text{ is even + ve integer} \\
 &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{4}{5} \cdot \frac{2}{3}, \text{ if } n \text{ is odd + ve integer.}
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad \int_0^{\pi/2} \sin^m x \cos^n x \, dx &= \int_0^{\pi/2} \cos^m x \sin^n x \, dx \\
 &= \frac{1.3.5 \cdots (m-1)}{2.4.6 \cdots (m+n)} \cdot \frac{(n-1)}{2} \cdot \frac{\pi}{2}, \text{ both } m \text{ \& } n \text{ +ve even integer}
 \end{aligned}$$

$$= \frac{2.4.6 \cdots (n-1)}{(m+1)(m+2) \cdots (m+n)}, m \text{ even, } n \text{ odd +ve integer}$$

$$= \frac{2.4.6 \cdots (m-1)}{(n+1)(n+2) \cdots (m+n)}, m \text{ odd, } n \text{ even +ve integer}$$

= case 2nd or 3rd according as $m > n$ or $n > m$, both m & n are odd +ve integer.

From sec. 6.2 (B), we have,

$$I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x dx = \left[\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} \right]_0^{\pi/2} + \frac{m-1}{m+n} I_{m-2,n}; m \geq 2$$

$$= \frac{m-1}{m+n} I_{m-2,n}; m \geq 2, \text{ is the reduction formula.} \quad (\text{A})$$

Also from prop. 10,

$$I_{m,n} = \int_0^{\pi/2} \sin^m \left(\frac{\pi}{2} - x \right) \cos^n \left(\frac{\pi}{2} - x \right) dx$$

$$= \int_0^{\pi/2} \cos^m x \sin^n x dx = \int_0^{\pi/2} \sin^n x \cos^m x dx = I_{n,m} \quad (\text{B})$$

Now the four cases are considered below one after another.

Case 1. Both m and n are even. Let $m = 2p$ and $n = 2q$. So, from (A)

$$I_{2p,2q} = \frac{2p-1}{2p+2q} I_{2p-2,2q} = \frac{(2p-1)(2p-3)}{(2p+2q)(2p+2q-2)} I_{2p-4,2q}$$

$$= \frac{(2p-1)(2p-3) \dots 1}{(2p+2q)(2p+2q-2) \dots (2q+2)} I_{0,2q}$$

and $I_{0,2q} = \int_0^{\pi/2} \cos^{2q} x dx$

$$= \frac{2q-1}{2q} \cdot \frac{2q-3}{2q-2} \dots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \text{ by (b) (i)}$$

$$\therefore I_{2p,2q} = \frac{[1.3.5 \dots (2p-1)] [1.3.5 \dots (2q-1)]}{2.4.6 \dots (2p+2q)} \cdot \frac{\pi}{2}$$

$$\text{i.e. } I_{m,n} = \frac{[1.3.5 \dots (m-1)] [1.3.5 \dots (n-1)]}{2.4.6 \dots (m+n)} \cdot \frac{\pi}{2}$$

Case 2. m even and n odd. Let $m = 2p$, $n = 2q-1$.

$\therefore$ from (A),

$$I_{2p,2q-1} = \frac{2p-1}{2p+2q-1} I_{2p-2,2q-1}$$

$$= \frac{(2p-1)(2p-3) \dots 1}{(2p+2q-1)(2p+2q-3) \dots (2q+1)} I_{0,2q-1}$$

and $I_{0,2q-1} = \int_0^{\pi/2} \cos^{2q-1} x dx = \frac{2q-2}{2q-1} \cdot \frac{2q-4}{2q-3} \dots \frac{2}{3} \text{ by (b) (i)}$

$$\therefore I_{2p,2q-1} = \frac{[1.3.5 \dots (2p-1)] [2.4.6 \dots (2q-2)]}{1.3.5 \dots (2p-1)(2p+1)(2p+3) \dots (2p+2q-1)}$$

$$= \frac{2.4.6 \dots (2q-2)}{(2p+1)(2p+3) \dots (2p+2q-1)}$$

$$\text{i.e. } I_{m,n} = \frac{2.4.6 \dots (n-1)}{(m+1)(m+3) \dots (m+n)} \text{ if } m \text{ even \& } n \text{ odd}$$

Case 3. m odd and n even. Let $m = 2p-1$ and $n = 2q$.

$\therefore$ from (B), $I_{m,n} = I_{n,m}$

$\therefore$ from case (2) interchange of m and n gives case 3

$$\text{i.e. } I_{m,n} = \frac{2.4.6 \dots (m-1)}{(n+1)(n+3) \dots (m+n)} \text{ if } m \text{ odd and } n \text{ even.}$$

Case 4. Both m and n odd. Let $m = 2p-1$ and $n = 2q-1$. This reduces to case 2 or case 3 according as $m > n$ or $n > m$ due to cancellation of factors corresponding to the bigger one of the exponents m or n .

An abstract.

In all the 4 cases, note that there are 3 sets of factors ending with $(m-1)$, $(n-1)$ & $(m+n)$ in both numerator and denominator. Start with 1 or 2 according as the end factor is odd or even, proceed on increasing each factor by 2 upto the end factor in each case. Automatic cancellation gives the end result irrespective of oddness or evenness of the exponents.

N.B. In chapter 10, gamma and beta function will give a compact single formula for the evaluation of all integrals in (b).

Examples

1. Evaluate $\int_1^4 x^4 dx$ as a process of summation.

Let us divide the interval into n equal parts by the points $1 = 1, 1+h, 1+2h, \dots, 1+nh=4$ which are in A.P. Therefore $nh = 4-1=3$. Now according to sec. 8.2.1, let us erect ordinates at the left end of each subintervals. Hence,

$$\begin{aligned} \int_1^4 x^4 dx &= \lim_{h \rightarrow 0} h \left[1^4 + (1+h)^4 + (1+2h)^4 + \dots + (1+nh)^4 + \dots \right. \\ &\quad \left. + (1+\overline{n-1})^4 \right] \\ &= \lim_{h \rightarrow 0} h \left[n \cdot 1^4 + {}^4C_1 (h + 2h + \dots + \overline{n-1}h) \right. \\ &\quad + {}^4C_2 (h^2 + 2^2 h^2 + \dots + (n-1)^2 h^2) + \\ &\quad + {}^4C_3 (h^3 + 2^3 h^3 + \dots + (n-1)^3 h^3) + \\ &\quad \left. + {}^4C_4 (h^4 + 2^4 h^4 + \dots + (n-1)^4 h^4) \right] \\ &= \lim_{h \rightarrow 0} h \left[n + 4h(1+2+\dots+\overline{n-1}) + \right. \\ &\quad \left. 6h^2(1^2+2^2+\dots+(n-1)^2) + 4h^3(1^3+ \right. \end{aligned}$$

$$\begin{aligned}
& 2^3 + \dots + (n-1)^3 + h^4 (1^4 + 2^4 + \dots + (n-1)^4) \\
&= \lim_{h \rightarrow 0} h \left[n + 4h \cdot \frac{(n-1)n}{2} + 6h^2 \frac{(n-1)n(2n-1)}{6} + \right. \\
& 4h^3 \left\{ \frac{(n-1)n}{2} \right\}^2 + h^4 \cdot \left\{ \frac{(n-1)^5}{5} + \frac{(n-1)^4}{2} + \frac{(n-1)^3}{3} - \frac{(n-1)}{30} \right\} \Bigg] \\
&= \lim_{h \rightarrow 0} \left[nh + 2(nh-h)nh + (nh-h)nh(2nh-h) \right. \\
& \quad + 4 \left\{ \frac{(nh-h)nh}{2} \right\}^2 + \left\{ \frac{(nh-h)^5}{5} + \frac{h(nh-h)^4}{2} \right. \\
& \quad \left. \left. + \frac{h^2(nh-h)^3}{3} - \frac{h^4(nh-h)}{30} \right\} \right] \\
&= \lim_{h \rightarrow 0} \left[3 + 2(3-h)3 + (3-h)3(6-h) + \{(3-h) \cdot 3\}^2 \right. \\
& \quad \left. + \frac{(3-h)^5}{5} + \frac{h(3-h)^4}{2} + \frac{h^2(3-h)^3}{3} - \frac{h^4(3-h)}{30} \right] \\
& \qquad \qquad \qquad \therefore nh = 3. \\
&= \frac{1023}{5}.
\end{aligned}$$

N.B. (i) $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$,

(ii) $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$

(iii) $1^3 + 2^3 + 3^3 + \dots + n^3 = \left\{ \frac{n(n+1)}{2} \right\}^2$

(iv) $1^4 + 2^4 + 3^4 + \dots + n^4 = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$
 $= \frac{1}{30} n(n+1)^2(2n+1)(3n-1)$

Alter. Let us divide the interval into n parts by the points $1=1, r, r^2, \dots, r^k, \dots, r^n = 4$ which forms a G.P. Therefore $r^n = 4$ or $r = 4^{1/n}$. So, as $n \rightarrow \infty, r \rightarrow 1$. Now according to seq. 8.2, let us erect ordinates at the left end of each subintervals of length $\delta_k = r^{k+1} - r^k = r^k(r-1)$. So,

$$\begin{aligned}
\int_1^4 x^4 dx &= \lim_{r \rightarrow 1} [\delta_1 1^4 + \delta_2 r^4 + \delta_3 r^8 + \dots + \delta_n r^{4(n-1)}] \\
&= \lim_{r \rightarrow 1} [r(r-1) + r^2(r-1)r^4 + r^3(r-1)r^8 + \dots \\
& \qquad \qquad \qquad + r^n(r-1)r^{4(n-1)}]
\end{aligned}$$

$$= \lim_{r \rightarrow 1} r(r-1) [1 + r^5 + r^{10} + \dots + r^{5(n-1)}]$$

$$= \lim_{r \rightarrow 1} r(r-1) \frac{(r^5)^n - 1}{r^5 - 1}, \text{ sum of G.P. series.}$$

$$= \lim_{r \rightarrow 1} r(r-1) \frac{(r^5)^5 - 1}{r^5 - 1} = Lt_{r \rightarrow 1} \frac{r(r-1)(4^5 - 1)}{r^5 - 1}$$

$$\therefore r^n = 4.$$

$$= 1023 \lim_{r \rightarrow 1} \frac{r(r-1)}{r^5 - 1} \text{ which is } \frac{0}{0} \text{ form.}$$

$$= (1023) \frac{1}{5} = \frac{1}{5} (1023) \text{ using L' Hospitals rule.}$$

2. Evaluate $\int_0^1 (\sqrt{x})^3 dx$ as a process of summation.

Here lower limit is zero. So G.P. forming divisional point cannot be taken. As the integrand is irrational, so to get rid of the irrationality squared A.P. points like

$0 = a_0, a_1, a_2, \dots, a_r, \dots, a_n = 1$ where $a_r = r^2 h^2$ are taken as subdivisional points.

[Note—irrationality due to exponent of the form $\frac{p}{q}$, take $a_r = (rh)^q$]. Now erecting ordinates at the left end of the n subintervals of length $\delta_r = a_r - a_{r-1}, r = 1, 2, \dots, n$, we have

$$\begin{aligned} \int_0^1 (\sqrt{x})^3 dx &= \lim_{h \rightarrow 0} [(\sqrt{a_1})^3 \delta_1 + (\sqrt{a_1})^3 \delta_2 + (\sqrt{a_2})^3 \delta_3 + \dots \\ &\quad + (\sqrt{a_{n-1}})^3 \delta_n] \\ &= \lim_{h \rightarrow 0} \left[(\sqrt{0})^3 1^2 h^2 + (\sqrt{1^2 h^2})^3 (2^2 h^2 - 1^2 h^2) + (\sqrt{2^2 h^2})^3 \right. \\ &\quad \left. (3^2 h^2 - 2^2 h^2) + \dots + (\sqrt{(n-1)^2 h^2})^3 (n^2 h^2 - (n-1)^2 h^2) \right] \\ &= \lim_{h \rightarrow 0} h^5 [1^3 (2^2 - 1^2) + 2^3 (3^2 - 2^2) + \dots + (n-1)^3 (n^2 - (n-1)^2)] \\ &= \lim_{h \rightarrow 0} h^5 \sum_{r=1}^{n-1} r^3 [(r+1)^2 - r^2] = Lt_{h \rightarrow 0} h^5 \sum_{r=1}^{n-1} r^3 (2r+1) \end{aligned}$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} h^5 \sum_{r=1}^{n-1} (2r^4 + r^3) \\
&= \lim_{h \rightarrow 0} h^5 \left[2(1^4 + 2^4 + 3^4 + \dots + (n-1)^4) + (1^3 + 2^3 + 3^3 + \dots + (n-1)^3) \right] \\
&= \lim_{h \rightarrow 0} h^5 \left[2 \left\{ \frac{(n-1)^5}{5} + \frac{(n-1)^4}{2} + \frac{(n-1)^3}{3} - \frac{n-1}{30} \right\} + \left\{ \frac{(n-1)n}{2} \right\}^2 \right] \\
&= \lim_{h \rightarrow 0} \left[2 \left\{ \frac{(nh-h)^5}{5} + \frac{h(nh-h)^4}{2} + \frac{h^2(nh-h)^3}{3} - \frac{h^4(nh-h)}{30} \right\} + \right. \\
&\quad \left. h \left\{ \frac{(nh-h)nh}{2} \right\}^2 \right] \\
&= \lim_{h \rightarrow 0} \left[2 \left\{ \frac{(1-h)^5}{5} + \frac{h(1-h)^4}{2} + \frac{h^2(1-h)^3}{3} - \frac{h^4(1-h)}{30} \right\} \right. \\
&\quad \left. + h \left\{ \frac{(1-h)}{2} \right\}^2 \right] \\
&= \frac{2}{5} \quad [\because a_n = n^2 h^2 = 1 \therefore nh = 1]
\end{aligned}$$

3. Evaluate $\int_{\pi/6}^{\pi/2} \sin 2x \, dx$ using summation.

Since finite series of sines and cosines can be summed up only when the angles are in A.P. So, in this case A.P. division is preferred. Let us divide the interval into n equal subintervals by the points $\frac{\pi}{6} = \frac{\pi}{6}, \frac{\pi}{6} + h, \frac{\pi}{6} + 2h, \dots, \frac{\pi}{6} + nh = \frac{\pi}{2}$. So that $nh = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$. Now erecting ordinates at the left end of each subintervals, we get,

$$\begin{aligned}
\int_{\pi/6}^{\pi/2} \sin 2x \, dx &= \lim_{h \rightarrow 0} h \left[\sin 2 \cdot \frac{\pi}{6} + \sin \left(2 \cdot \frac{\pi}{6} + 2h \right) + \sin \left(2 \cdot \frac{\pi}{6} + 4h \right) \right. \\
&\quad \left. + \dots + \sin \left(2 \cdot \frac{\pi}{6} + 2(n-1)h \right) \right] \\
&= \lim_{h \rightarrow 0} h \left[\sin \frac{\pi}{3} + \sin \left(\frac{\pi}{3} + 2h \right) + \right. \\
&\quad \left. \sin \left(\frac{\pi}{3} + 4h \right) + \dots + \sin \left(\frac{\pi}{3} + 2(n-1)h \right) \right]
\end{aligned}$$

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{h}{2 \sin h} \left[2 \sin \frac{\pi}{3} \sin h + 2 \sin \left(\frac{\pi}{3} + 2h \right) \sin h \right. \\
 &\quad \left. + 2 \sin \left(\frac{\pi}{3} + 4h \right) \sin h + \dots \right. \\
 &\quad \left. + 2 \sin \left(\frac{\pi}{3} + 2n-1 h \right) \sin h \right]
 \end{aligned}$$

[$\therefore$ difference of angles = $2h$. To use method of difference for both sine & cosine series multiply by $2 \sin (\frac{1}{2}$ the diff. of angles) & balance]

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{h}{2 \sin h} \left[\left\{ \cos \left(\frac{\pi}{3} - h \right) - \cos \left(\frac{\pi}{3} + h \right) \right\} + \right. \\
 &\quad \left\{ \cos \left(\frac{\pi}{3} + 2h - h \right) - \cos \left(\frac{\pi}{3} + 2h + h \right) \right\} + \\
 &\quad \left\{ \cos \left(\frac{\pi}{3} + 4h - h \right) - \cos \left(\frac{\pi}{3} + 4h + h \right) \right\} + \\
 &\quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
 &\quad \left. \left\{ \cos \left(\frac{\pi}{3} + 2n-1 h - h \right) - \cos \left(\frac{\pi}{3} + 2n-1 h + h \right) \right\} \right] \\
 &= \lim_{h \rightarrow 0} \frac{h}{2 \sin h} \left[\cos \left(\frac{\pi}{3} - h \right) - \cos \left(\frac{\pi}{3} + 2nh - h \right) \right] \\
 &= \lim_{h \rightarrow 0} \frac{h}{2 \sin h} \left[\cos \left(\frac{\pi}{3} - h \right) - \cos \left(\frac{\pi}{3} + \frac{2\pi}{3} - h \right) \right]
 \end{aligned}$$

$$\therefore nh = \frac{\pi}{3}$$

$$\begin{aligned}
 &= \frac{1}{2} \left[\cos \frac{\pi}{3} - \cos \pi \right] = \frac{1}{2} \left[\frac{1}{2} + 1 \right], \quad \left[\because \lim_{h \rightarrow 0} \frac{h}{\sin h} = 1 \right] \\
 &= \frac{3}{4}
 \end{aligned}$$

4. Evaluate $\int_{\pi/12}^{\pi/6} \sec^2 2x \, dx$ by the process of summation.

For trigonometric integrand A.P. division is preferred; because relationship exists for multiple & sub-multiple angles which appear by addition only. Again summation of series of square of sec and cosec can not be obtained easily. So, we form an inequality (2) below to get the result. As usual we divide the interval into n equal parts by the points

$\frac{\pi}{12} = \frac{\pi}{12}, \frac{\pi}{12} + h, \frac{\pi}{12} + 2h, \dots, \frac{\pi}{12} + nh = \frac{\pi}{6}$. Therefore,
 $nh = \frac{\pi}{6} - \frac{\pi}{12} = \frac{\pi}{12}$. Erecting ordinates at the left end of each subintervals,
 we get,

$$\int_{\pi/12}^{\pi/6} \sec^2 2x \, dx = \lim_{h \rightarrow 0} h \left[\sec^2 \frac{\pi}{12} + \sec^2 \left(\frac{\pi}{12} + h \right) + \dots \right. \\ \left. + \sec^2 \left(\frac{\pi}{12} + \overline{n-1}h \right) \right] \quad (1)$$

Since $\sec 2x$ decreases from $\frac{\pi}{12}$ to $\frac{\pi}{6}$, so,

$$\sec 2 \left(\frac{\pi}{12} + \overline{r-1}h \right) < \sec 2 \left(\frac{\pi}{12} + rh \right) < \sec 2 \left(\frac{\pi}{12} + \overline{r+1}h \right) \quad (2)$$

$$\text{or, } \sec 2 \left(\frac{\pi}{12} + rh \right) \sec 2 \left(\frac{\pi}{12} + \overline{r-1}h \right) < \sec^2 2 \left(\frac{\pi}{12} + rh \right) \\ < \sec 2 \left(\frac{\pi}{12} + rh \right) \sec 2 \left(\frac{\pi}{12} + \overline{r+1}h \right)$$

Now instead of finding the R.H. sum of (1). Let us first find

$$\lim_{h \rightarrow 0} h \sum_{r=0}^{n-1} \sec 2 \left(\frac{\pi}{12} + rh \right) \sec 2 \left(\frac{\pi}{12} + \overline{r+1}h \right) \\ = \lim_{h \rightarrow 0} h \sum_{r=0}^{n-1} \frac{1}{\cos 2 \left(\frac{\pi}{12} + rh \right) \cos 2 \left(\frac{\pi}{12} + \overline{r+1}h \right)} \\ = \lim_{h \rightarrow 0} \frac{h}{\sin 2h} \sum_{r=0}^{n-1} \frac{\sin \left[2 \left(\frac{\pi}{12} + \overline{r+1}h \right) - 2 \left(\frac{\pi}{12} + rh \right) \right]}{\cos 2 \left(\frac{\pi}{12} + rh \right) \cos 2 \left(\frac{\pi}{12} + \overline{r+1}h \right)} \\ = \lim_{h \rightarrow 0} \frac{h}{\sin 2h} \sum_{r=0}^{n-1} \left[\tan 2 \left(\frac{\pi}{12} + \overline{r+1}h \right) - \tan 2 \left(\frac{\pi}{12} + rh \right) \right] \\ = \lim_{h \rightarrow 0} \frac{h}{\sin 2h} \left[\tan 2 \left(\frac{\pi}{12} + \overline{n-1}h \right) - \tan 2 \cdot \frac{\pi}{12} \right] \\ = \lim_{h \rightarrow 0} \frac{1}{2} \cdot \frac{2h}{\sin 2h} \left[\tan 2 \left(\frac{\pi}{12} + \frac{\pi}{12} - h \right) - \tan \frac{\pi}{6} \right] \because nh = \frac{\pi}{12} \\ = \frac{1}{2} \left(\tan \frac{\pi}{3} - \tan \frac{\pi}{6} \right) = \frac{1}{\sqrt{3}}.$$

$$\text{In the same way, } \lim_{h \rightarrow 0} h \sum_{r=0}^{n-1} \sec^2 2 \left(\frac{\pi}{12} + rh \right) \sec 2 \left(\frac{\pi}{12} + \overline{r-1} h \right) \\ = \frac{1}{\sqrt{3}}$$

$\therefore$ there can be no number between the same value $\frac{1}{\sqrt{3}}$. Hence

$$\lim_{h \rightarrow 0} \sum_{r=0}^{n-1} \sec^2 2 \left(\frac{\pi}{12} + rh \right) = \frac{1}{\sqrt{3}}.$$

and so from (1) $\int_{\pi/12}^{\pi/6} \sec^2 2x \, dx = \frac{1}{\sqrt{3}}.$

5. (i) For the integral $\int_0^{\pi} \sin x \, dx$, find the upper and the lower sums

corresponding to the division of the closed interval $[0, \pi]$ into 3 and 6 equal subintervals. (ii) Find the upper integral sum and the lower integral sum with more finer subintervals. (iii) Is it Riemann integrable?

Let I denotes the interval $[0, \pi]$ (fig. 5). Let $S(I_i)$ and $s(I_i)$ denote the upper and lower sums corresponding to the division of the closed interval I into i equal subintervals. Now,

$$I_3 \text{ of } [0, \pi] = \left[0, \frac{\pi}{3} \right], \left[\frac{\pi}{3}, \frac{2\pi}{3} \right] \text{ and } \left[\frac{2\pi}{3}, \pi \right]$$

From the accompanying table of values we can read off m_i and M_i , the lower and the upper bounds respectively in each subintervals.

i	I_i	$f(x) = \sin x$	m_i	M_i
1	$[0, \pi/3]$	$0 \rightarrow \sqrt{3}/2$	0	$\sqrt{3}/2$
2	$[\frac{\pi}{3}, 2\pi/3]$	$\frac{\sqrt{3}}{2} \rightarrow 1 \rightarrow \frac{\sqrt{3}}{2}$	$\sqrt{3}/2$	1
3	$[2\pi/3, \pi]$	$\frac{\sqrt{3}}{2} \rightarrow 0$	0	$\sqrt{3}/2$

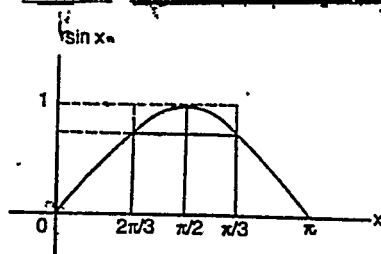


Fig. 5

$$\therefore S(I_3) = (M_1 + M_2 + M_3) \frac{\pi}{3} = \left(\frac{\sqrt{3}}{2} + 1 + \frac{\sqrt{3}}{2} \right) \frac{\pi}{3} \approx 2.86$$

$$\text{and } s(I_3) = (m_1 + m_2 + m_3) \frac{\pi}{3} = \left(0 + \frac{\sqrt{3}}{2} + 0 \right) \frac{\pi}{3} \approx 0.907$$

$$\text{Let } I_6 \text{ of } [0, \pi] = \left[0, \frac{\pi}{6} \right], \left[\frac{\pi}{6}, \frac{\pi}{3} \right], \left[\frac{\pi}{3}, \frac{\pi}{2} \right], \left[\frac{\pi}{2}, \frac{2\pi}{3} \right], \\ \left[\frac{2\pi}{3}, \frac{5\pi}{6} \right], \left[\frac{5\pi}{6}, \pi \right]$$

i	I_i	$f(x) = \sin x$	m_i	M_i
1	$\left[0, \frac{\pi}{6} \right]$	$0 \rightarrow 1/2$	0	$1/2$
2	$\left[\frac{\pi}{6}, \frac{\pi}{3} \right]$	$1/2 \rightarrow \sqrt{3}/2$	$1/2$	$\sqrt{3}/2$
3	$\left[\frac{\pi}{3}, \frac{\pi}{2} \right]$	$\sqrt{3}/2 \rightarrow 1$	$\sqrt{3}/2$	1
4	$\left[\frac{\pi}{2}, 2\pi/3 \right]$	$1 \rightarrow \sqrt{3}/2$	$\sqrt{3}/2$	1
5	$\left[\frac{2\pi}{3}, 5\pi/6 \right]$	$\sqrt{3}/2 \rightarrow 1/2$	$1/2$	$\sqrt{3}/2$
6	$\left[\frac{5\pi}{6}, \pi \right]$	$1/2 \rightarrow 0$	0	$1/2$

$$\therefore S(I_6) = (M_1 + M_2 + M_3 + M_4 + M_5 + M_6) \pi/6 \\ = \left(\frac{1}{2} + \frac{\sqrt{3}}{2} + 1 + 1 + \frac{\sqrt{3}}{2} + \frac{1}{2} \right) \frac{\pi}{6} \approx 2.48$$

$$\text{and } s(I_6) = (m_1 + m_2 + m_3 + m_4 + m_5 + m_6) \frac{\pi}{6} \\ = \left(0 + \frac{1}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{1}{2} + 0 \right) \frac{\pi}{6} \approx 1.43$$

$$\text{(ii) } I_8 \text{ of } [0, \pi] = \left[0, \frac{\pi}{8} \right], \left[\frac{\pi}{8}, \frac{\pi}{4} \right], \left[\frac{\pi}{4}, \frac{3\pi}{8} \right], \left[\frac{3\pi}{8}, \frac{\pi}{2} \right], \left[\frac{\pi}{2}, \frac{5\pi}{8} \right], \\ \left[\frac{5\pi}{8}, \frac{3\pi}{4} \right], \left[\frac{3\pi}{4}, \frac{7\pi}{8} \right], \left[\frac{7\pi}{8}, \pi \right]$$

i	I_i	$f(x) = \sin x$	m_i	M_i
1	$[0, \pi/8]$	$0 \rightarrow .38$	0	.38
2	$[\pi/8, \pi/4]$	$.38 \rightarrow .71$	.38	.71
3	$[\pi/4, 3\pi/8]$	$.71 \rightarrow .92$	.71	.92
4	$[3\pi/8, \pi/2]$	$.92 \rightarrow 1.00$	.92	1.00
5	$[\pi/2, 5\pi/8]$	$1.00 \rightarrow .92$	.92	1.00
6	$[5\pi/8, 3\pi/4]$	$.92 \rightarrow .71$	.71	.92
7	$[3\pi/4, 7\pi/8]$	$.71 \rightarrow .38$	.38	.71
8	$[7\pi/8, \pi]$	$.38 \rightarrow 0$	0	.38

$$\begin{aligned}\therefore S(I_8) &= (M_1 + M_2 + M_3 + M_4 + M_5 + M_6 + M_7 + M_8) \frac{\pi}{8} \\ &\approx (.38 + .71 + .92 + 1.00 + 1.00 + .92 + .71 + .38) (.39) \\ &\approx 2.35\end{aligned}$$

$$\begin{aligned}\text{and } s(I_8) &= (m_1 + m_2 + m_3 + m_4 + m_5 + m_6 + m_7 + m_8) \frac{\pi}{8} \\ &\approx (0 + .38 + .71 + .92 + .92 + .71 + .38 + 0) (.39) \\ &\approx 1.58\end{aligned}$$

$$\begin{aligned}I_{12} \text{ of } [0, \pi] &= \left[0, \frac{\pi}{12}\right], \left[\frac{\pi}{12}, \frac{\pi}{6}\right], \left[\frac{\pi}{6}, \frac{\pi}{4}\right], \left[\frac{\pi}{4}, \frac{\pi}{3}\right], \left[\frac{\pi}{3}, \frac{5\pi}{12}\right], \left[\frac{5\pi}{12}, \frac{\pi}{2}\right], \\ &\quad \left[\frac{\pi}{2}, \frac{7\pi}{12}\right], \left[\frac{7\pi}{12}, \frac{2\pi}{3}\right], \left[\frac{2\pi}{3}, \frac{3\pi}{4}\right], \left[\frac{3\pi}{4}, \frac{5\pi}{6}\right], \left[\frac{5\pi}{6}, \frac{11\pi}{12}\right], \\ &\quad \left[\frac{11\pi}{12}, \pi\right]\end{aligned}$$

i	I_i	$f(x) = \sin x$	m_i	M_i
1	$[0, \pi/12]$	$0 \rightarrow .26$	0	.26
2	$[\pi/12, \pi/6]$	$.26 \rightarrow .50$	.26	.50
3	$[\pi/6, \pi/4]$	$.50 \rightarrow .71$	.50	.71
4	$[\pi/4, \pi/3]$	$.71 \rightarrow .87$	.71	.87
5	$[\pi/3, 5\pi/12]$	$.87 \rightarrow .97$	.87	.97
6	$[5\pi/12, \pi/2]$	$.97 \rightarrow 1.00$	.97	1.00
7	$[\pi/2, 7\pi/12]$	$1.00 \rightarrow .97$	.97	1.00

8	$[7\pi/12, 2\pi/3]$	$\cdot 97 \rightarrow \cdot 87$	$\cdot 87$	$\cdot 97$
9	$[2\pi/3, 3\pi/4]$	$\cdot 87 \rightarrow \cdot 71$	$\cdot 71$	$\cdot 87$
10	$[3\pi/4, 5\pi/6]$	$\cdot 71 \rightarrow \cdot 50$	$\cdot 50$	$\cdot 71$
11	$[5\pi/6, 11\pi/12]$	$\cdot 50 \rightarrow \cdot 26$	$\cdot 26$	$\cdot 50$
12	$[11\pi/12, \pi]$	$\cdot 26 \rightarrow 0$	0	$\cdot 26$

$$\therefore S(I_{12}) = \left(\sum_{i=1}^{12} M_i \right) \frac{\pi}{12} \approx 2.24$$

$$s(I_{12}) = \left(\sum_{i=1}^{12} m_i \right) \frac{\pi}{12} \approx 1.72$$

Similarly, $S(I_{60}) \approx 2.05$ & $s(I_{60}) \approx 2.05$

(iii) It is observed as the subintervals are becoming finer and finer S is becoming smaller & smaller while s is becoming larger & larger. This will happen for all problems. Now, the set

$$S = \{2.86, 2.48, 2.35, 2.24, 2.05\}$$

and $s = \{.907, 1.43, 1.58, 1.72, 2.05\}$

$$\therefore \text{g.l.b. of } S = I = \int_0^{\pi} \sin x \, dx = 2.05$$

and $\text{l.u.b. of } s = I = \int_0^{\pi} \sin x \, dx = 2.05$

$\therefore$ both the lower and the upper sums approaches 2.05. Hence upper and lower integral sums are equal and equal to 2.05 and the integral is Riemann integrable. But the actual value using Fundamental Theorem of

integral calculus, is $\int_0^{\pi} \sin x \, dx = [-\cos x]_0^{\pi} = 2.$

6. Let f be defined by $f(x) = x - [x]$, $0 \leq x \leq 4$. The symbol $[x]$ indicates the maximum integral value of x .

(i) Sketch the graph of f .

(ii) Find $S(\pi)$ and $s(\pi)$ for $\pi = \{0, 1, 2, 3, 4\}$

(iii) Same as (ii), but for $\pi = \left\{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4\right\}$

(iv) What is the value of $\int_0^4 f(x) \, dx$?

(i) The graph can be easily drawn by separately drawing $y_1 = x$ and $y_2 = [x]$ and subtracting them (fig. 6, 7, 8)

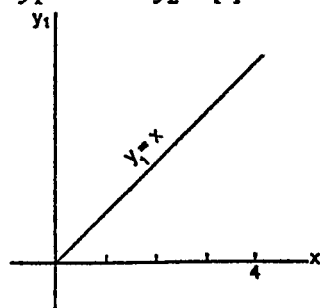


Fig. 6

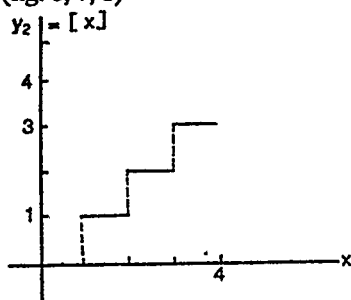


Fig. 7

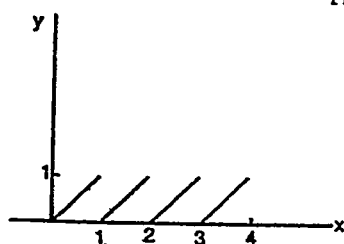


Fig. 8

Actually the graph is of the form $y = x - c$, because $[x] = c = 0, 1, 2, 3$ according as $0 \leq x < 1, 1 \leq x < 2, 2 \leq x < 3, \& 3 \leq x < 4$.

(ii) From fig. 8, m_i and M_i can be easily read off.

$$S(\pi) = 1[1 + 1 + 1 + 1] = 4$$

and

$$s(\pi) = 1[0 + 0 + 0 + 0] = 0$$

(iii)

$$S(\pi) = \frac{1}{2} \left[\frac{1}{2} + 1 + \frac{1}{2} + 1 + \frac{1}{2} + 1 + \frac{1}{2} + 1 \right] = 3$$

$$\therefore f\left(\frac{1}{2}\right) = 1 - 0 = 1, f(1) = 1 - 1 = 0 \text{ etc.}$$

$$s(\pi) = \frac{1}{2} [0 + 0 + 0 + 0 + 0 + 0 + 0 + 0] = 0$$

(iv)

No. of subintervals	16	32	64	128	256	512
S	2.5	2.25	2.12	2.06	2.02	2.01
s	.75	1.75	1.88	1.94	1.98	1.99

$$\therefore S = \{4, 3, 2.5, 2.25, 2.12, 2.06, 2.02, 2.01\}$$

$$s = \{0, 0, .75, 1.75, 1.88, 1.94, 1.98, 1.99\}$$

$$\therefore \text{g.l.b of } S = J = \int_0^4 f(x) dx = 2 \text{ (approx.)}$$

$$\text{and } \text{l.u.b of } s = I = \int_0^4 f(x) dx = 2 \text{ (approx.)}$$

$$\therefore J = I. \text{ Hence it is Riemann integrable and } \int_0^4 f(x) dx = 2.$$

Note that this function is piecewise or sectionally continuous.

$$\text{Q 7. Show that } \int_0^{\pi/2} \frac{x dx}{\sin x + \cos x} = \frac{\pi}{2\sqrt{2}} \log(\sqrt{2} + 1).$$

The integrand is *not* of any form of indefinite integral. So, fundamental theorem is not directly applicable. Let us use properties.

$$\text{Let } I = \int_0^{\pi/2} \frac{x dx}{\sin x + \cos x}.$$

Now, using prop. 10, we have,

$$\begin{aligned} I &= \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right) dx}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} = \int_0^{\pi/2} \frac{\frac{\pi}{2} - x}{\sin x + \cos x} dx \\ &= \frac{\pi}{2} \int_0^{\pi/2} \frac{1}{\sin x + \cos x} dx - I \end{aligned}$$

$$\text{or } 2I = \frac{\pi}{2} \int_0^{\pi/2} \frac{1}{\sin x + \cos x} dx, \text{ integrand is of type 2.12 and prob.}$$

19 exercise 2 (4)

$$\therefore 2I = \frac{\pi}{2} \left[\frac{1}{\sqrt{2}} \log \tan \left(\frac{x}{2} + \frac{\pi}{8} \right) \right]_0^{\pi/2}$$

$$\text{or } I = \frac{\pi}{4} \left[\frac{1}{\sqrt{2}} \left(\log \tan \frac{3\pi}{8} - \log \tan \frac{\pi}{8} \right) \right]$$

$$= \frac{\pi}{4\sqrt{2}} \log \frac{\tan \frac{3\pi}{8}}{\tan \frac{\pi}{8}} = \frac{\pi}{4\sqrt{2}} \log \frac{\sin \frac{3\pi}{8} \cos \frac{\pi}{8}}{\cos \frac{3\pi}{8} \sin \frac{\pi}{8}}$$

$$= \frac{\pi}{4\sqrt{2}} \cdot \frac{1}{2} \log \frac{2 \sin^2 \frac{3\pi}{8} \cdot 2 \cos^2 \frac{\pi}{8}}{2 \cos^2 \frac{3\pi}{8} \cdot 2 \sin^2 \frac{\pi}{8}}$$

$$\begin{aligned}
 &= \frac{\pi}{8\sqrt{2}} \log \frac{\left(1 - \cos \frac{3\pi}{4}\right) \left(1 + \cos \frac{\pi}{4}\right)}{\left(1 + \cos \frac{3\pi}{4}\right) \left(1 - \cos \frac{\pi}{4}\right)} \\
 &= \frac{\pi}{8\sqrt{2}} \log \frac{\left(1 + \frac{1}{\sqrt{2}}\right) \left(1 + \frac{1}{\sqrt{2}}\right)}{\left(1 - \frac{1}{\sqrt{2}}\right) \left(1 - \frac{1}{\sqrt{2}}\right)} = \frac{\pi}{4\sqrt{2}} \log \frac{\sqrt{2} + 1}{\sqrt{2} - 1} \\
 &= \frac{\pi}{2\sqrt{2}} \log (\sqrt{2} + 1).
 \end{aligned}$$

8. Evaluate $\int_0^1 \cot^{-1} (1 - x + x^2) dx$.

This integrand is *not* also of any type of indefinite integral. So, let

$$\begin{aligned}
 I &= \int_0^1 \cot^{-1} [1 - x(1 - x)] dx \\
 &= - \int_0^1 \cot^{-1} [x(1 - x) - 1] dx \\
 &= - \int_0^1 [\cot^{-1} x + \cot^{-1} (1 - x)] dx \\
 &\quad \left[\because \cot^{-1} x + \cot^{-1} y = \frac{xy - 1}{y + x} \right] \\
 &= - \int_0^1 \cot^{-1} x dx - \int_0^1 \cot^{-1} (1 - x) dx \\
 &= - \int_0^1 \cot^{-1} x dx - \int_0^1 \cot^{-1} x dx \text{ by prop. 10} \\
 &= -2 \int_0^1 \cot^{-1} x dx \\
 &= -2 \left\{ \left[x \cot^{-1} x \right]_0^1 - \int_0^1 \frac{x}{-(1 + x^2)} dx \right\}, \text{ I.b.p} \\
 &= -2 \left\{ \frac{\pi}{4} + \frac{1}{2} \left[\log (1 + x^2) \right]_0^1 \right\} \\
 &= -\frac{\pi}{2} - \log 2 \qquad \because \log 1 = 0.
 \end{aligned}$$

N.B. $\cot^{-1} [1 + x(x-1)] = \cot^{-1}(x-1) - \cot^{-1}x$ may also be used.

9. Evaluate $\int_{-2}^2 x^9 (1-x^2)^7 dx$.

Let $f(x) = x^9 (1-x^2)^7$.

Then $f(-x) = -f(x)$

$\therefore f(x)$ is an odd function.

$\therefore$ by prop. 11, $\int_{-1}^2 x^9 (1-x^2)^7 dx = 0$.

10. Use Fundamental Theorem to evaluate $\int_0^4 (x - [x]) dx$ in Ex.6.

The integrand is *not* continuous from 0 to 4. It is piecewise continuous

fig. 8. So, it will be sum of 4 integrals as $\int_0^4 (x - [x]) dx =$
 $\int_0^1 x dx + \int_1^2 (x-1) dx + \int_2^3 (x-2) dx + \int_3^4 (x-3) dx = 2$.

11. The typing speed w , in words per minute, of a certain secretary is $w = 20t - 5t^2 + 50$, where t is the elapsed time in hours. Find (a) the secretary's average speed per minute, (b) the total number of words the secretary typed over the 3 hour interval $0 \leq t \leq 3$.

(a) It is a problem on *mean value theorem*. Here $t = 3$ hours $= 3 \times 60 =$ minutes. Therefore, average speed

$$= \frac{\int_0^{3 \times 60} (20t - 5t^2 + 50) dt}{(3 \times 60 - 0)} = \frac{11,700}{3 \times 60} = 65 \text{ words/minute}$$

$$\begin{aligned} \text{(b) Total number of words typed} &= \int_0^{3 \times 60} (20t - 5t^2 + 50) dt \\ &= 11,700. \end{aligned}$$

12. Evaluate

$$Lf_{n \rightarrow \infty} \left[\frac{1}{\sqrt{n^2-1}} + \frac{1}{\sqrt{n^2-2^2}} + \frac{1}{\sqrt{n^2-3^2}} + \dots + \frac{1}{\sqrt{2n-1}} \right]$$

$$\text{Let } S = Lf_{n \rightarrow \infty} \left[\frac{1}{\sqrt{n^2-1}} + \frac{1}{\sqrt{n^2-2^2}} + \frac{1}{\sqrt{n^2-3^2}} + \dots + \frac{1}{\sqrt{2n-1}} \right]$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \left[\frac{n}{\sqrt{n^2-1}} + \frac{n}{\sqrt{n^2-2^2}} + \frac{n}{\sqrt{n^2-3^2}} + \dots + \frac{n}{\sqrt{n^2-(n-1)^2}} \right]$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \left[\frac{1}{\sqrt{1-\frac{1^2}{n^2}}} + \frac{1}{\sqrt{1-\frac{2^2}{n^2}}} + \frac{1}{\sqrt{1-\frac{1^2}{n^2}}} + \dots + \frac{1}{\sqrt{1-\frac{(n-1)^2}{n^2}}} \right]$$

Removing n from numerators.

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^{n-1} \frac{1}{\sqrt{1-\frac{r^2}{n^2}}} = \int_0^1 \frac{1}{\sqrt{1-x^2}} dx$$

$$\left[\because \frac{r}{n} = \frac{1}{n} \text{ and } \frac{n-1}{n} \text{ in the 1st and last term respectively.} \right]$$

$$\therefore \text{lower limit} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0 \text{ and}$$

$$\text{upper limit} = \lim_{n \rightarrow \infty} \frac{n-1}{n} = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n} \right) = 1$$

$$\therefore S = \left[\sin^{-1} x \right]_0^1 = \frac{\pi}{2}.$$

13. Evaluate

$$\lim_{n \rightarrow \infty} \left[\left(1 + \frac{1}{n^2} \right)^{2/n^2} \left(1 + \frac{2^2}{n^2} \right)^{4/n^2} \left(1 + \frac{3^2}{n^2} \right)^{6/n^2} \dots \left(1 + \frac{n^2}{n^2} \right)^{2n/n^2} \right]$$

This is a product. So, we have to take log to make sum. Let

$$A = \lim_{n \rightarrow \infty} \left[\left(1 + \frac{1}{n^2} \right)^{2/n^2} \left(1 + \frac{2^2}{n^2} \right)^{4/n^2} \left(1 + \frac{3^2}{n^2} \right)^{6/n^2} \dots \left(1 + \frac{n^2}{n^2} \right)^{2n/n^2} \right]$$

$$\begin{aligned}
 \therefore \log A &= \lim_{n \rightarrow \infty} \left[\frac{2}{n^2} \log \left(1 + \frac{1}{n^2} \right) + \frac{4}{n^2} \log \left(1 + \frac{2^2}{n^2} \right) + \right. \\
 &\quad \left. \frac{6}{n^2} \log \left(1 + \frac{3^2}{n^2} \right) + \dots + \frac{2n}{n^2} \log \left(1 + \frac{n^2}{n^2} \right) \right] \\
 &= 2 \lim_{n \rightarrow \infty} \frac{1}{n} \left[\frac{1}{n} \log \left(1 + \frac{1}{n^2} \right) + \frac{2}{n} \log \left(1 + \frac{2^2}{n^2} \right) + \frac{3}{n} \log \left(1 + \frac{3^2}{n^2} \right) \right. \\
 &\quad \left. + \dots + \frac{n}{n} \log \left(1 + \frac{n^2}{n^2} \right) \right] \\
 &= 2 \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \frac{r}{n} \log \left(1 + \frac{r^2}{n^2} \right) \\
 &= 2 \int_0^1 x \log(1+x^2) dx = \int_1^2 \log u du
 \end{aligned}$$

$\left[\because \text{lower limit} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0 \text{ and} \right.$	$\text{where } 1+x^2 = u$						
$\text{upper limit} = \lim_{n \rightarrow \infty} \frac{n}{n} = \lim_{n \rightarrow \infty} 1 = 1 \left. \right]$	$2x dx = du$						
and	<table border="1" style="margin-left: auto; margin-right: auto; border-collapse: collapse;"> <tr> <td style="padding: 5px; text-align: center;">x</td> <td style="padding: 5px; text-align: center;">u</td> </tr> <tr> <td style="padding: 5px; text-align: center;">0</td> <td style="padding: 5px; text-align: center;">1</td> </tr> <tr> <td style="padding: 5px; text-align: center;">1</td> <td style="padding: 5px; text-align: center;">2</td> </tr> </table>	x	u	0	1	1	2
x	u						
0	1						
1	2						

$$\begin{aligned}
 \therefore \log A &= \left[u \log u \right]_1^2 - \int_1^2 u \cdot \frac{1}{u} du \\
 &= 2 \log 2 - \left[u \right]_1^2, \because \log 1 = 0 \\
 &= 2 \log 2 - 1 = \log 4 - \log e = \log 4/e \\
 \therefore A &= \frac{4}{e}, \text{ taking anti-log.}
 \end{aligned}$$

14. Evaluate $\lim_{n \rightarrow \infty} \sum_{r=1}^{3n} \frac{n^2}{(3n+r)^3}$.

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \sum_{r=1}^{3n} \frac{n^2}{(3n+r)^3} &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^{3n} \frac{n^3}{(3n+r)^3} \\
 &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^{3n} \frac{1}{\left(3 + \frac{r}{n} \right)^3} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^{3n} \frac{1}{\left(3 + \frac{r}{n} \right)^3}
 \end{aligned}$$

$$= \int_0^3 \frac{1}{(3+x)^3} dx \quad \because \text{lower limit} = Lt_{n \rightarrow \infty} \frac{1}{n} = 0$$

$$= \frac{1}{24} \quad \text{upper limit} = Lt_{n \rightarrow \infty} \frac{3n}{n} = Lt_{n \rightarrow \infty} 3 = 3$$

[Note that $\frac{r}{n} = \frac{1}{n}$ and $\frac{3n}{n}$ in the 1st and the last term respectively as r varies from 1 to $3n$.]

15. Differentiating w.r.t. the parameter evaluate the integral

$$\int_0^\pi \log(1 + \alpha \cos x) dx.$$

N.B. Normal procedure for evaluating this definite integral is *not* possible. So, differentiation w.r.t. α reduces it to simpler integral. Steps are (1) differential with respect to α , (2) integration of the derivative of α , (3) calculation of const. of integration,

$$\text{Let } f(\alpha) = \int_0^\pi \log(1 + \alpha \cos x) dx \quad (a)$$

$$\begin{aligned} \therefore f'(\alpha) &= \int_0^\pi \frac{\cos x}{1 + \alpha \cos x} dx = \int_0^\pi \frac{\frac{1}{\alpha}(1 + \alpha \cos x) - \frac{1}{\alpha}}{1 + \alpha \cos x} dx \\ &= \frac{1}{\alpha} \int_0^\pi \left[1 - \frac{1}{1 + \alpha \cos x} \right] dx \\ &= \frac{1}{\alpha} \left\{ [x]_0^\pi - \int_0^\pi \frac{1}{1 + \alpha \cos x} dx \right\} \\ &= \frac{1}{\alpha} \left[\pi - \frac{\pi}{\sqrt{1 - \alpha^2}} \right] = \pi \left[\frac{1}{\alpha} - \frac{1}{\alpha \sqrt{1 - \alpha^2}} \right] \end{aligned}$$

Integrating w.r.t. α

$$f(\alpha) = \pi \log(1 + \sqrt{1 - \alpha^2}) + c \quad (b)$$

But from (a), $f(0) = 0$

$$\therefore \text{from (b), } 0 = \pi \log 2 + c \quad \text{or } c = -\pi \log 2$$

$$\therefore \text{the reqd. integral, } f(x) = \pi \log \frac{1 + \sqrt{1 - \alpha^2}}{2}$$

16. Find the first partial derivative of

$$G(x, y) = \int_{xy}^{x+y} \frac{\sin(x+y)t}{t} dt \text{ w.r.t. } y.$$

Differentiating partially w.r.t. y , we get,

$$\begin{aligned} G_y &= \frac{\partial G}{\partial y} = \frac{\partial}{\partial y} \int_{xy}^{x+y} \frac{\sin(x+y)t}{t} dt \\ &= \int_{xy}^{x+y} \frac{\partial}{\partial y} \left(\frac{\sin(x+y)t}{t} \right) dt + \frac{\sin[(x+y)(x+y)]}{x+y} \frac{\partial}{\partial y}(x+y) \\ &\quad - \frac{\sin[(x+y)xy]}{xy} \frac{\partial}{\partial y}(xy) \text{ by Leibnitz rule.} \\ &= \int_{xy}^{x+y} \cos(x+y)t dt + \frac{\sin(x+y)^2}{x+y} - \frac{\sin[(x+y)xy]}{xy} \quad (x) \\ &= \left[\frac{\sin(x+y)t}{x+y} \right]_{t=xy}^{x+y} + \frac{\sin(x+y)^2}{x+y} - \frac{\sin[xy(x+y)]}{y} \\ &= \frac{2\sin(x+y)^2}{x+y} - \left(\frac{1}{y} + \frac{1}{x+y} \right) \sin[xy(x+y)] \end{aligned}$$

17. Estimate the integral $\int_{\pi/4}^{\pi/3} \frac{\sin x}{x} dx$. State its bounds.

This integral *cannot* be integrated by known means. So, it is to be estimated. The integrand $f(x) = \frac{\sin x}{x}$ decreases in the interval $\left[\frac{\pi}{4}, \frac{\pi}{3} \right]$, since its derivative $f'(x) = \frac{x \cos x - \sin x}{x^2} = \frac{(x - \tan x) \cos x}{x^2} < 0$. Hence, the least value of the function, $m = f(\pi/3) = \frac{3\sqrt{3}}{2\pi}$ and its greatest value being $M = f(\pi/4) = \frac{2\sqrt{2}}{\pi}$.

$\therefore$ by mean value theorem, sec. 8.5 eqn. (1)

$$\begin{aligned} \frac{3\sqrt{3}}{2\pi} &\leq \frac{\int_{\pi/4}^{\pi/3} \frac{\sin x}{x} dx}{(\pi/3) - (\pi/4)} \leq \frac{2\sqrt{2}}{\pi} \\ \text{i.e. } 0.22 &\approx \frac{\sqrt{3}}{8} \leq \int_{\pi/4}^{\pi/3} \frac{\sin x}{x} dx \leq \frac{\sqrt{2}}{6} \approx .24 \end{aligned}$$

$\therefore$ the upper and the lower bounds of the given integral are .24 and .22 respectively.

Exercises 8

1. (a) Let $f(x) = [x]$, $0 \leq x \leq 4$.

(i) Sketch the graph of $f(x)$.

(ii) If $\pi = \{0, 1, 2, 3, 4\}$ find $S(\pi)$ and $s(\pi)$.

(iii) If $\pi = \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4\}$ find $S(\pi)$ and $s(\pi)$.

(iv) Find $\int_0^4 f(x) dx$ from Riemann analytic definition.

(b) Same as Ex. 6 only for $f(x) = x + [x]$, $0 \leq x \leq 4$.

(c) Find upper and lower sums of $f(x) = 1 - x^2$ for the subintervals $\left\{ \left[0, \frac{1}{2}\right], \left[\frac{1}{2}, 1\right], \left[1, \frac{3}{2}\right], \left[\frac{3}{2}, 2\right] \right\}$.

(d) Find upper and lower sums of $f(x) = 2x^2$ in the interval $[-1, 1]$ making 2 equal subintervals.

(e) Find $\int_0^1 x dx$ and $\int_0^1 x dx$. Is it integrable?

2. Evaluate using summation procedure of definite integral :

(a) $\int_0^a x^3 dx$, (b) $\int_0^{\pi/2} \sin x dx$, (c) $\int_0^a \cos x dx$,

(d) $\int_0^{\pi/4} \sec^2 x dx$, (e) $\int_{\pi/4}^{\pi/2} \operatorname{cosec}^2 x dx$, (f) $\int_a^b \frac{1}{x^2} dx$,

(g) $\int_a^b \frac{1}{\sqrt{x}} dx$, (i) $\int_0^5 e^{2x} dx$, (i) $\int_1^8 \sqrt[3]{x} dx$.

3. Find the value of the following series :—

$$(i) \lim_{n \rightarrow \infty} \left[\frac{1}{n^4} + \frac{2^3}{n^4} + \frac{3^3}{n^4} + \dots + \frac{n^3}{n^4} \right],$$

$$(ii) \lim_{n \rightarrow \infty} \left[\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n} \right],$$

$$(ii) \lim_{n \rightarrow \infty} \frac{1}{n} \left[\sin 2k \left(\frac{\pi}{2n} \right) + \sin 2k \left(\frac{2\pi}{2n} \right) + \sin 2k \left(\frac{3\pi}{2n} \right) + \dots + \sin 2k \left(\frac{\pi}{2} \right) \right],$$

$$(iv) \lim_{n \rightarrow \infty} \left[\frac{1}{n^2} \sec^2 \left(\frac{1}{n^2} \right) + \frac{2}{n^2} \sec^2 \left(\frac{4}{n^2} \right) + \frac{3}{n^2} \sec^2 \left(\frac{9}{n^2} \right) + \dots + \frac{1}{n} \sec^2 (1) \right],$$

$$(v) \lim_{n \rightarrow \infty} \left[\frac{n}{n^2 + 1^2} + \frac{n}{n^2 + 2^2} + \frac{n}{n^2 + 3^2} + \dots + \frac{n}{n^2 + n^2} \right],$$

$$(vi) \lim_{n \rightarrow \infty} \left[\frac{1^2}{n^3 + 1^3} + \frac{2^2}{n^3 + 2^3} + \frac{3^2}{n^3 + 3^3} + \dots + \frac{n^2}{n^3 + n^3} \right],$$

$$(vii) \lim_{n \rightarrow \infty} \left[\left(1 + \frac{1}{n^2} \right) \left(1 + \frac{2^2}{n^2} \right) \left(1 + \frac{3^2}{n^2} \right) \dots \left(1 + \frac{n^2}{n^2} \right) \right]^{1/n},$$

$$(viii) \lim_{n \rightarrow \infty} \left[\frac{n^{1/2}}{n^{3/2}} + \frac{n^{1/2}}{(n+3)^{3/2}} + \frac{n^{1/2}}{(n+6)^{3/2}} + \dots + \frac{n^{1/2}}{\{n+3(n-1)\}^{3/2}} \right]$$

$$(ix) \lim_{n \rightarrow \infty} \left[\left(1 + \frac{1}{n} \right) \left(1 + \frac{2}{n} \right) \dots \left(1 + \frac{n}{n} \right) \right]^{1/n}$$

$$(x) \lim_{n \rightarrow \infty} \sum_{r=0}^{2n} \frac{1}{\sqrt{4n^2 + r^2}}$$

$$(xi) \lim_{n \rightarrow \infty} \left[\left(2 + \frac{1}{n^2} \right)^{1/n^2} \left(2 + \frac{2^2}{n^2} \right)^{2/n^2} \left(2 + \frac{3^2}{n^2} \right)^{3/n^2} \dots \left(2 + \frac{n^2}{n^2} \right)^{n/n^2} \right]$$

$$(xii) \lim_{n \rightarrow \infty} \left[\left(1 + \frac{1}{n^2} \right)^{2/n^2} \left(1 + \frac{2^2}{n^2} \right)^{4/n^2} \left(1 + \frac{3^2}{n^2} \right)^{6/n^2} \dots \left(1 + \frac{n^2}{n^2} \right)^{2n/n^2} \right]$$

$$(xiii) \lim_{n \rightarrow \infty} \left[\frac{1}{na} + \frac{1}{na+1} + \frac{1}{na+2} + \dots + \frac{1}{nb} \right]$$

$$\left[\text{Hints. write } nb = na + n(b-a), \int_0^{b-a} \frac{dx}{a+x} \right]$$

$$(xiv) \lim_{n \rightarrow \infty} \left[\frac{1}{n+m} + \frac{1}{n+2m} + \frac{1}{n+3m} + \dots + \frac{1}{n+nm} \right]$$

$$\left[\text{Hints. } \int_0^1 \frac{dx}{1+mx} \right]$$

$$(xv) \lim_{n \rightarrow \infty} \frac{1 + 2^{11} + 3^{11} + \dots + n^{11}}{n^{12}}$$

4. Evaluate the following —

$$(i) \int_0^1 \frac{\sqrt{\tan^{-1} x}}{1+x^2} dx, (ii) \int_0^1 \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx, (iii) \int_0^{\pi/2} \frac{dx}{5+3 \cos x},$$

$$(iv) \int_8^{15} \frac{dx}{(x-3)\sqrt{x+1}}, (v) \int_0^{\pi/2} \frac{\cos x}{(1+\sin x)(2+\sin x)} dx,$$

$$(vi) \int_0^{16} \frac{x^{1/4}}{1+x^{1/2}} dx, (vii) \int_{1/3}^1 \frac{(x-x^3)^{1/3}}{x^4} dx, (viii) \int_0^{\pi} \frac{dx}{(2+\cos x)^2}$$

$$(ix) \int_0^{\pi/2} \frac{d\theta}{4+5 \cos^2 \theta}, (x) \int_0^{\pi/6} \cos^4 3\theta \sin^2 6\theta d\theta,$$

$$(xi) \int_0^{\pi/2} x^3 \sin 3x dx, (xii) \int_0^1 \frac{\sqrt{1-x^2}}{1-x^2 \sin^2 \alpha} dx,$$

$$(xiii) \int_0^a \frac{x^4}{(a^2+x^2)^4} dx, (xiv) \int_0^{\pi/2} \frac{dx}{1+\cot x},$$

$$(xv) \int_0^{\pi/2} \frac{\sin x \cos x}{a^2 \sin^2 x + b^2 \cos^2 x} dx, (xvi) \int_0^{\pi/2} \frac{\sin^2 \theta \cos \theta}{1+e^2 \sin^2 \theta} d\theta,$$

$$(xvii) \int_0^{\pi/2} \frac{\sin x \cos x}{\cos^2 x + 3 \cos x + 2} dx, (xviii) \int_0^{\pi} \frac{dx}{a^2 - 2a \cos x + 1},$$

$$(xix) \int_0^1 \frac{dx}{(1+x^2)\sqrt{1-x^2}}, (xx) \int_3^{29} \frac{(x-2)^{2/3}}{(x-2)^{2/3}+3} dx.$$

5. Verify that $\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx$ for the following problems :

$$(i) \int_{-2}^3 6(x+2-x^2) dx, (ii) \int_{-1}^4 (3x-x^2) dx,$$

$$(iii) \int_{-1}^2 (x^2-3x+2) dx. [\text{Hints. for (iii)} x^2-3x+2 = (x-2)(x-1);$$

$$|x^2 - 3x + 2| = \begin{cases} x^2 - 3x + 2 & \text{if } x \leq 1 \text{ or } x \geq 2 \text{ and} \\ -(x^2 - 3x + 2) & \text{if } 1 < x < 2. \end{cases} \text{ So,}$$

$$\int_{-1}^2 |x^2 - 3x + 2| dx = \int_{-1}^1 |x^2 - 3x + 2| dx + \int_1^2 |x^2 - 3x + 2| dx$$

$$= \int_{-1}^1 (x^2 - 3x + 2) dx + \int_1^2 -(x^2 - 3x + 2) dx = 4\frac{5}{6}$$

6. Show that

$$(i) \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} = \frac{\pi^2}{4}; \quad (ii) \int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx = \pi \left(\frac{\pi}{2} - 1 \right);$$

$$(iii) \int_0^1 \frac{\log(1+x) dx}{1+x^2} = \frac{\pi}{8} \log 2 = \int_0^{\pi/4} \log(1 + \tan \theta);$$

$$(iv) \int_0^{\pi/2} \frac{(\sin x)^{3/2}}{(\sin x)^{3/2} + (\cos x)^{3/2}} dx = \frac{\pi}{4};$$

$$(v) \int_{-a}^a \frac{x^3 \sin(1+x^2)}{(1+x^2)(1+x^8)} dx = 0;$$

$$(vi) \int_0^a \frac{adx}{(x + \sqrt{a^2 - x^2})^2} = \frac{1}{\sqrt{2}} \log(1 + \sqrt{2});$$

$$(vii) \int_0^{\pi} \frac{x}{1 + \sin x} dx = \pi;$$

$$(viii) \int_0^{\pi/2} \frac{x \sin 2x}{\sin x + \cos x} dx = \frac{\pi}{2} + \frac{\pi}{2\sqrt{2}} \log(\sqrt{2} - 1);$$

$$(ix) \int_{-1/2}^{1/2} \cos x \log \frac{1+x}{1-x} dx = 0; \quad [\text{Hints. } f(x) = \cos x \log \frac{1+x}{1-x} \text{ or}$$

$$f(-x) = \cos x \log \frac{1-x}{1+x} = -\cos x \log \frac{1+x}{1-x} = -f(x)]$$

$$(x) \int_0^{\pi} \frac{x^3 \cos^4 x \sin^2 x}{\pi^2 - 3\pi x + 3x^2} dx = \frac{\pi^2}{32}; \quad [\text{Hints. Prop. 10 \& simplify}].$$

$$(xi) \int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cos x} dx = 0;$$

$$(xii) \int_0^{\pi/2} \frac{\sin^2 x}{1 + \sin x \cos x} dx = \frac{\pi}{3\sqrt{3}};$$

$$(xiii) \int_0^{\pi} x \cos^4 x \, dx = \frac{3}{16} \pi^2;$$

$$(xiv) \int_0^{\pi/2} \log(\tan x + \cot x) \, dx = \pi \log 2;$$

$$(xv) \int_0^{\pi} x \sin^6 x \cos^4 x \, dx = \frac{3\pi^2}{512}.$$

7. (a) Find the derivatives w.r.t. α of the following integrals without first integrating and check by first integrating and then differentiating :

$$(i) \int_0^{\alpha x} \cos(x + \alpha) \, dx; (ii) \int_0^{\sqrt{\alpha}} x \, dx;$$

$$(iii) \int_0^x \sin^{-1} \frac{x}{\alpha} \, dx; (iv) \int_{\alpha^2}^{\alpha^3} (x^2 + \alpha^2) \, dx.$$

(b) Differentiating w.r.t. parameter evaluate the following integral :

$$\int_0^1 \frac{x^{\alpha} - 1}{\log x} \, dx.$$

(c) Differentiate the following integrals w.r.t. the parameter :

$$(i) F(y) = \int_{\sin y}^y \sqrt{1+x^3} \, dx, (ii) F(a) = \int_0^1 \frac{adx}{\sqrt{1-a^2x^2}},$$

$$(iii) F(x) = \int_0^x e^{-x^2t^2} \, dt, (iv) F(x) = \int_{x^2}^{\sin x} (x^2 - t^2)^n \, dt,$$

$$(v) F(u) = \int_0^4 \log(1 - u^2x^2) \, dx.$$

(d) Evaluate the following :

$$(i) \frac{\partial}{\partial x} \int_0^{xy} \frac{\sin(x+y)t}{t} \, dt, (ii) \frac{\partial^2}{\partial x \partial y} \int_1^x \exp\left(\frac{x+y}{t}\right) \, dt, x, y > 0,$$

(e) The Bessels function of order n , $J_n(x)$ may be defined by

$$J_n(x) = \frac{1}{\pi} \int_0^{\pi} \cos(n\theta - x \sin \theta) \, d\theta.$$

(i) Show that this $J_n(x)$ satisfies the recurrence formula

$$2J'_n(x) = J_{n-1}(x) - J_{n+1}(x).$$

(ii) Show that $x^2 J''_n(x) + x J'_n(x) - (n^2 - x^2) J_n(x) = 0$.

(f) Let $f(\varphi)$ be continuous for $0 \leq \varphi \leq 2\pi$. Prove that the function $u(r, \theta)$ defined by the Poisson integral formula

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 + r^2 - 2r \cos(\theta - \varphi)} f(\varphi) d\varphi$$

satisfies Laplace's eqn. $\nabla^2 u = 0$ in polar coordinates r & θ in the unit disk

$$r < 1. \quad \left[\text{Hints: } \nabla^2 u \equiv \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right. \\ \left. = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right]$$

(g) (i) By successive differentiation of $\int_0^1 x^n dx = \frac{1}{n+1}, n \neq -1,$

$$\text{obtain } \int_0^1 x^n (\log x)^m dx = (-1)^m \frac{m!}{(n+1)^{m+1}}.$$

$$(ii) \text{ From } \int_0^\pi \frac{dx}{\alpha - \cos x} = \frac{\pi}{\sqrt{\alpha^2 - 1}}, \alpha > 1$$

$$\text{find } \int_0^\pi \log \frac{b - \cos x}{a - \cos x} = \pi \log \frac{b + \sqrt{b^2 - 1}}{a + \sqrt{a^2 - 1}}$$

$$(iii) \text{ Prove that } \int_0^{\pi/2} \frac{\log(1 + \cos \alpha \cos x)}{\cos x} dx = \frac{1}{2} \left(\frac{\pi^2}{4} - \alpha^2 \right)$$

$$(iv) \text{ Evaluating } \int_0^a \frac{dx}{a^2 + x^2} \text{ show that } \int_0^a \frac{dx}{(a^2 + x^2)^2} = -\frac{\pi + 2}{8a^3}.$$

$$(v) \text{ Evaluate } \int_0^a \frac{\log(1 + ax)}{1 + x^2} dx \text{ and show that}$$

$$\int_0^1 \frac{\log(1+x)}{1+x^2} dx = \frac{\pi}{8} \log 2.$$

8. (a) Estimate the following integrals:

$$(i) I = \int_1^3 \sqrt{3+x^3} dx, (ii) I = \int_0^1 \sqrt{1+x^4} dx.$$

$$(b) \text{ Which of the two integrals } \int_0^1 \sqrt{x} dx, \int_0^1 x^3 dx \text{ is greater?}$$

(c) Prove the inequalities (i) $1 < \int_0^1 e^{x^2} dx < e$ and

(ii) $\int_0^{\pi/2} e^{-R \sin x} dx < \frac{\pi}{2R} (1 - e^{-R})$, $R > 0$.

[Hints : Use $f(x) = \frac{\sin x}{x}$, deduce $\sin x > \frac{2}{\pi}x$ in $[0, \pi/2]$]

9. (i) If $I_{m,n} = \int_0^1 x^m (\log x)^n dx$, prove that $I_{m,n} = -\frac{n}{m+1} I_{m,n-1}$.

Hence show that $\int_0^1 x^4 (\log x)^3 dx = -\frac{6}{625}$.

(ii) If $I_{m,n} = \int_0^1 x^m (1-x)^{n-1} dx$, then establish the reduction

formula $I_{m,n} = \frac{n-1}{m+1} I_{m,n-2}$.

(iii) Prove that, if n is a positive integer,

$$\int_0^{\pi/2} \cos x \cos nx dx = \frac{\pi}{2^{n+1}}.$$

(iv) If $I_n = \int_0^{\pi/2} \theta \sin^n \theta d\theta$ and $n > 1$, prove that

$$I_n = \frac{n-1}{n} I_{n-2} + \frac{1}{n^2}. \text{ Deduce that } I_5 = \frac{149}{225}.$$

(v) Given that $I_{m,n} = \int_0^{\pi/2} \cos^m x \sin nx dx$, prove that

$$I_{m,n} = \frac{1}{m+n} + \frac{m}{m+n} I_{m-1,n-1}. \text{ Calculate } I_{3,3}.$$

10. (a) (i) A metal plate is heated to $50t$ degrees Fahrenheit at time t . What is the average temperature A of the plate from $t = 0$ to $t = 4$?

(ii) Find a number c such that $\int_a^b f(x) dx = f(c)(b-a)$ where

$a = -3$, $b = 4$ and $f(x) = 8x^3 + 2$.

(iii) Determine the average length of all vertical chords of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ over the interval $a \leq x \leq 2a$.

(iv) Find the average length of all positive ordinates of the circle $x^2 + y^2 = 1$. [Hints : $\frac{1}{2} \int_{-1}^1 \sqrt{1-x^2} dx$]

(v) Find the average value of the square of the electromotive force $(E^2)_m$ where $E = E_0 \sin \frac{2\pi t}{T}$ over the interval from $t = 0$ to $t = T/2$

(b) Evaluate the following :

(i) $\int_0^{\pi/2} \cos^6 \theta d\theta$, (ii) $\int_0^1 x^{3/2} (1-x)^{3/2} dx$,

(iii) $\int_0^{\pi/6} \cos^4 3\theta \sin^2 6\theta d\theta$

11. Prove that

(i) $\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x} dx = \frac{\pi}{4}$ for all real values of n .

(ii) $\int_0^{\pi} \frac{x}{1 + \cos^2 x} dx = \frac{\pi^2}{2\sqrt{2}}$.

(iii) $\int_0^{\pi/2} \frac{\sqrt{\tan x}}{\sqrt{\tan x} + \sqrt{\cot x}} dx = \frac{\pi}{4}$.

12. Prove that

(i) $\left| \int_0^1 \frac{\cos nx}{x+1} dx \right| \leq \log 2$ for all n

(ii) $\left| \int_1^{\sqrt{3}} \frac{e^{-x} \sin x}{x^2 + 1} dx \right| \leq \frac{\pi}{12e}$ [Hints : Use property 14.]

(iii) $\lim_{m \rightarrow \infty} \int_0^m \frac{dx}{x^4 + 4} = \frac{\pi}{8}$.

Answers

1. (a) (ii) $S = 10, s = 6$; (iii) $S = 8, s = 6$; (iv) 6.

(b) (ii) $S = 20, s = 12$; (iii) $S = 16.5, s = 12.5$; (iv) 14.

(c) $S = .25, s = -1.75$. (d) $S = 4, s = 0$; (e) $\frac{1}{2}, \frac{1}{2}$; yes

2. (a) $a^4/4$, (b) 1, (c) $\sin a$, (d) 1, (e) 1, (f) $\frac{1}{a} - \frac{1}{b}$, (g) $2(\sqrt{b} - \sqrt{a})$,

(i) $\frac{1}{2}(e^{10} - 1)$, (j) $45/4$.

3. (i) $\frac{1}{4}$, (ii) $\log 2$, (iii) $(2k)!(2^k k!)^2$, (iv) $\frac{1}{2} \tan 1$, (v) $\frac{\pi}{4}$, (vi) $\frac{1}{3} \log 2$,
(vii) $2e^{1/2}(\pi - 4)$, (viii) $\frac{1}{3}$, (ix) $4/c$, (x) $\log(1 + \sqrt{2})$, (xi) $\sqrt{27/4} e^{-1/2}$, (xii) $4/c$,
(xiii) $\log \frac{b}{a}$, (xiv) $\frac{1}{m} \log(1 + m)$, (xv) $\frac{1}{12}$.

4. (i) $\frac{\pi^{3/2}}{12}$, (ii) $\frac{\pi^2}{8}$, (iii) $\frac{1}{2} \tan^{-1} \frac{1}{2}$, (iv) $\frac{1}{4} \log \frac{5}{3}$, (v) $\log 3/4$,

(vi) $\frac{8}{3} + 4 \tan^{-1} 2$, (vii) 6, (viii) $\frac{2\pi}{3\sqrt{3}}$, (ix) $\frac{\pi}{12}$, (x) $\frac{5\pi}{192}$, (xi) $\frac{2}{27} - \frac{\pi^2}{12}$,
(xii) $\frac{\pi}{4} \sec^2 \frac{\alpha}{2}$, (xiii) $\frac{1}{16a^3} \left(\frac{\pi}{4} - \frac{1}{3} \right)$, (xiv) $\frac{\pi}{4}$, (xv) $\frac{1}{a^2 - b^2} \log \frac{b}{a}$ if $a^2 \neq b^2$ & $\frac{1}{2a^2}$
if $a^2 = b^2$, (xvi) $\frac{e - \tan^{-1} e}{e^3}$, (xvii) $\log 9/8$, (xviii) $\frac{\pi}{1 - a^2}$ if $a < 1$ & $\frac{\pi}{a^2 - 1}$
if $a > 1$, (xix) $\frac{\pi}{2\sqrt{2}}$, (xx) $\left(8 + \frac{3\pi\sqrt{3}}{2} \right)$.

7. (b) $\log(\alpha + 1)$, $\alpha > -1$.

7. (c) (i) $F'(y) = -\sqrt{1 + \sin^3 y} \cos y + \sqrt{1 + e^{3y}} e^y$, (ii) $(1 - a^2)^{-1/2}$ if integra-
tion is carried out after differentiation, (iii) $e^{-x^4 - 2x} \int_0^x t^2 e^{-x^2 t^2} dt$;

(iv) $(x^2 - \sin^2 x)^n \cos x - (x^2 - x^4)^n 2x + \int_{x^2}^{\sin x} 2xt (x^2 - t^2)^{n-1} dt$,

(v) $F'(u) = \int_0^4 \frac{-2ux^2}{1 - u^2 x^2} dx$.

7. (d) (i) $\left[(2x+y)/x(x+y) \right] \sin xy (x+y)$, (ii) $\frac{y}{x(x+y)} e^{(x+y)/x} + \frac{e^{x+y}}{x+y}$.

8. (a) (i) $4 \leq I \leq 10.95$, (ii) $1 < I < 1.41$

9. (v) $\frac{5}{12}$

10. (a) (i) $100^\circ F$, (ii) $\sqrt[3]{25/4}$, (iii) $b [2\sqrt{3} - \log(2 + \sqrt{3})]$, (iv) $\frac{\pi}{4}$,
(v) $E\delta^2/2$.

10. (b) (i) $5\pi/32$; (ii) $3\pi/128$, (iii) $5\pi/192$.

IMPROPER (OR INFINITE) INTEGRAL

9.1. PRELIMINARY REMARKS

The definite integral $\int_a^b f(x) dx$ conveys two information—(i) the

function $f(x)$ is continuous in $[a, b]$ i.e. $f(x)$ is bounded in $[a, b]$ means there is a positive constant M such that $|f(x)| < M$ for all x in the interval [bounded intervals are (a, b) , $[a, b]$, $[a, b)$, $(a, b]$ and unbounded intervals are (a, ∞) , $(-\infty, a)$, $[a, \infty)$, $(-\infty, a]$, $(-\infty, \infty)$] and (ii) the value of the integral is finite at the end points 'a' and 'b'. If any one of (or both) the information is (or are) not fulfilled then there arises the need for a new explanation of the definite integral. Specially these possibilities arise in probability and statistics, in the study of electric potential and in many physical applications.

DEFINITION :

If either any one of (or both) the limits of the integral is (or are) infinite or " $f(x)$ is infinitely discontinuous or indeterminate at 'a' or 'b' or both or at any one or more points between 'a' and 'b' " then the integral is called an improper (or infinite) integral.

KINDS :

By definition the improper integral can be classified into three—
(1) *First kind* in which integrand $f(x)$ is continuous but either one of (or both) the limits is (or are) infinite; (2) *Second kind* in which limits are finite but the integrand is infinitely discontinuous in the interval; (3) *Third kind* or mixed kind, a combination of First and Second kinds.

From this discussion the definite integral $\int_a^b f(x) dx$ should better

be named as *proper* (or *finite*) in contrast to *improper* (or *infinite*). Both the names improper and infinite for this type of integrals are in vogue among the authors. Some prefers the name improper and some prefers the name infinite. Those who are in favour of infinite they find analogy with infinite

series, $\sum_{n=0}^{\infty} a_n$ as tabulated below :

$\int_0^{\infty} f(x) dx$, integral	$\sum_{n=0}^{\infty} a_n$, infinite series
1. Variable is x 2. Continuous interval from 0 to ∞ 3. $f(x)$ 4. $\int_0^{\infty} (\dots) dx$ 5. Partial integral, $\int_0^b f(x) dx$ 6. May or may not have finite value	1. Variable is n 2. Discrete values of n are 0, 1, 2, ... ∞ 3. a_n 4. $\sum_{n=0}^{\infty}$ 5. Partial sum, $\sum_{n=0}^k a_n$ 6. May or may not have finite value.

This argument is valid for the first kind of integral whereas the second kind cannot fit within this structure. So, combined name of these two kinds cannot be accepted as infinite integral. It is logical to call them *improper*. Without going into any controversy, it is better to call these types as *Generalized* integrals and all other integrals as *Not Generalized*. As these names are obsolete, so improper will be used onward.

DEFINITIONS :

Improper integral having finite value is called *convergent* or *exists* ; whereas one having no finite value is called *divergent* or *does not exist*. Nothing of these sorts, they are *oscillatory*. It is to be noted that definite integrals are always convergent as they possess definite finite values.

9.2. IMPROPER INTEGRALS OF THE FIRST KIND

This kind can be further classified into three categories — (i) Upper limit infinite, (ii) Lower limit infinite, (iii) Both limits infinite. The ways of evaluation of each of them is described below :

(i) This category is of the form $\int_a^{\infty} f(x) dx$. This is evaluated from

the knowledge of definite integral and limiting value, As $f(x)$ is continuous in $\{a, \infty\}$ there exists a finite number $b > a$ such that the definite integral

$\int_a^b f(x) dx$ can be set up. As $b \rightarrow \infty$ this definite integral becomes the given improper integral i.e.

$\int_a^\infty f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx$. If the limit exists i.e. has a finite number A then the improper integral $\int_a^\infty f(x) dx$ is convergent and has the value A ; otherwise divergent.

(ii) This category is of the form $\int_{-\infty}^b f(x) dx$. With the same reasoning as in (i), there exists a finite number $a < b$ such that the definite integral $\int_a^b f(x) dx$ can be set up and as $a \rightarrow (-\infty)$ this integral $\rightarrow$ the required form i.e., $\int_{-\infty}^b f(x) dx = \lim_{a \rightarrow -\infty} \int_a^b f(x) dx$. The improper integral $\int_{-\infty}^b f(x) dx$ is convergent if the limit exists i.e., finite otherwise divergent.

(iii) This category is of the form $\int_{-\infty}^{+\infty} f(x) dx$. This is a combination of the above two procedures. Because for some constant c between $-\infty$ to $+\infty$, this integral can be expressed as sum of (i) and (ii) i.e., $\int_{-\infty}^{+\infty} f(x) dx = \int_{-\infty}^c f(x) dx + \int_c^{+\infty} f(x) dx$, $-\infty < c < \infty$ and c is arbitrary.

Use (i) and (ii) for the two R.H. integrals. The improper integral $\int_{-\infty}^{+\infty} f(x) dx$ is convergent if both the R.H. integrals are convergent, otherwise divergent if any one (or both) is (or are) divergent. For principal value see sec. 9.5.

9.3. IMPROPER INTEGRALS OF THE SECOND KIND

By definition, here the limits $[a, b]$ are finite but the integrand is infinite or unbounded or indeterminate some where in $[a, b]$. The point at which the integrand is infinite or unbounded is called a singular point (S.P.). In other words at the s.p. the integrand "blows up". This s.p. may either be the end point (or points) of the interval or some other point (or finite number of points) within the interval. According to the position of s.p., this kind of improper integral is further classified into four categories—(i) s.p. only at the left end 'a', (ii) s.p. only at right end 'b', (iii) s.p. only at 'c' where $a < c < b$ and (iv) s.p. at both 'a' and 'b'. They are described below :

(i) *S.p. at 'a'*. If $x = a$ be a only s.p. of $f(x)$ then there exists a small number $\varepsilon > 0$. such that $f(x)$ is continuous in $[a + \varepsilon, b]$. Hence,

$$\int_a^b f(x) dx = \lim_{\varepsilon \rightarrow 0} \int_{a+\varepsilon}^b f(x) dx.$$

It is to be noted that if $\int_a^b f(x) dx = F(x) + c$ then

$$\lim_{\varepsilon \rightarrow 0} \int_{a+\varepsilon}^b f(x) dx = \lim_{\varepsilon \rightarrow 0} [F(b) - F(a + \varepsilon)].$$

So the convergence or divergence of the improper integral $\int_a^b f(x) dx$ depends on the existence

or non-existence of the limit $\lim_{\varepsilon \rightarrow 0} F(a + \varepsilon)$ respectively.

(ii) *S.p. at 'b'*. If $x = b$ be a only s.p. of $f(x)$ then there exists a small $\varepsilon > 0$ such that $f(x)$ is continuous in $[a, b - \varepsilon]$. Hence,

$$\int_a^b f(x) dx = \lim_{\varepsilon \rightarrow 0} \int_a^{b-\varepsilon} f(x) dx.$$

As in (i), convergence or divergence of the improper integral

$\int_a^b f(x) dx$ here depends on the existence or non-existence of

$$\lim_{\varepsilon \rightarrow 0} F(b - \varepsilon).$$

(iii) *S.p. only at c*, where $a < c < b$. If $x = c$ s.t. $a < c < b$ be only a s.p. of $f(x)$, then the integral can be decomposed into sum of two integrals

of type (i) and (ii) above i.e., $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$.

$$= \lim_{\varepsilon \rightarrow 0} \int_a^{c-\varepsilon} f(x) dx + \lim_{\varepsilon' \rightarrow 0} \int_{c+\varepsilon'}^b f(x) dx$$

Thus the improper integral $\int_a^b f(x) dx$ with s.p. at $x = c$,

$a < c < b$, converges if each one of decomposed integral in the R.H.S. is convergent ; otherwise divergent. In case of divergence, the Cauchy's principal value may exist (Ref. sec. 9.5 below).

N.B. This can be extended to any number of s.p.s in $[a, b]$.

(iv) S.p. at both 'a' & 'b'. If 'a' and 'b' are the only s.p. of $f(x)$, then there exists a number 'c' s.t. $a < c < b$.

$$\begin{aligned}\text{So, } \int_a^b f(x) dx &= \int_a^c f(x) dx + \int_c^b f(x) dx \\ &= \lim_{\varepsilon \rightarrow 0} \int_{a+\varepsilon}^c f(x) dx + \lim_{\varepsilon' \rightarrow 0} \int_c^{b-\varepsilon'} f(x) dx.\end{aligned}$$

The integral is convergent if each of the R.H. integral is convergent, otherwise divergent.

N.B. (iii) and (iv) may occur in combination and in that case both procedures are to be extended and the problem will be complicated.

N.B. Henceforth any *definite integral* in form should be carefully observed for the existence of any singular point before integration.

9.4. IMPROPER INTEGRAL OF THE THIRD KIND.

Integral can be improper in more than one way. It may be a combination of first and second kinds of secs. 9.2 and 9.3 respectively. That is, an integral may have both (i) infinite or unbounded limit or limits and

(ii) infinite discontinuous or unbounded integrand. For example $\int_0^\infty \frac{1}{x^3} dx$

has upper limit infinite and integrand with infinite discontinuity or unboundedness near $x = 0$. Let $\int_a^\infty f(x) dx$ be an improper integral of the third

kind when $f(x)$ has a s.p. at $x = c$ where $a < c < d$ and $c < d < \infty$. Then the integral can be decomposed as

$$\int_a^\infty f(x) dx = \int_a^d f(x) dx + \int_d^\infty f(x) dx \quad (1)$$

Therefore, the first integral is of second kind as s.p. $x = c$ lies in $[a, d]$ and the second integral is of first kind as the upper limit is infinite.

Now, the integral $\int_a^\infty f(x) dx$ is *convergent* if both the R.H. integrals are

convergent. It is *divergent* if any one of the R.H. integrals is divergent. So, to decompose as in (1) any integral of the third kind with more than one s.p., choose some $x = d$'s arbitrarily which are not s.p.'s in the given unbounded interval and the integral is *convergent* if all the decomposed integrals are convergent and *divergent* if any one of them is divergent.

9.5. PRINCIPAL VALUE OF DIVERGENT INTEGRAL

Normally divergent integral has no fixed value. But in some conditional cases, divergent integral may have value in a continuous physical process. Here Cauchy's method of principal value of divergent integral is considered. This method uses $\epsilon = \epsilon'$ of sec. 9.3 (iii), (iv) and writes

$$\text{V.P. } \int_a^b f(x) dx = \lim_{\epsilon \rightarrow 0} \left[\int_a^{c-\epsilon} f(x) dx + \int_{c+\epsilon}^b f(x) dx \right]$$

where V.P. stands for French *valeur principale*. An integral of this type is called *singular integral* to distinguish it from proper and convergent improper integrals which are called *regular integrals*. Similarly, the

$$\text{V.P. } \int_{-\infty}^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_{-b}^b f(x) dx.$$

For example, $\int_{-1}^2 \frac{1}{x} dx$ is divergent by 9.3 (iii) as $x = 0$ is a s.p. in the given interval. But

$$\begin{aligned} \text{V.P. } \int_{-1}^2 \frac{1}{x} dx &= \lim_{\epsilon \rightarrow 0} \left[\int_{-1}^{0-\epsilon} \frac{1}{x} dx + \int_{0+\epsilon}^2 \frac{1}{x} dx \right] \\ &= \lim_{\epsilon \rightarrow 0} \left\{ [\log x]_{-1}^{-\epsilon} + [\log x]_{\epsilon}^2 \right\} \\ &= \lim_{\epsilon \rightarrow 0} \{ \log \epsilon - \log 1 + \log 2 - \log \epsilon \} \because \log x \equiv \log |x| \\ &= \lim_{\epsilon \rightarrow 0} \log 2 = \log 2 = 0.693 \end{aligned}$$

$\therefore$ the integral is singular.

Also the

$$\begin{aligned} \text{V.P. } \int_{-\infty}^{\infty} \sin x dx &= \lim_{b \rightarrow \infty} \int_{-b}^b \sin x dx = \lim_{b \rightarrow \infty} (-\cos x) \Big|_{-b}^b \\ &= \lim_{b \rightarrow \infty} [-\cos b + \cos b] = 0 \end{aligned}$$

i.e. principal value exists whereas $\int_{-\infty}^{\infty} \sin x dx$ is divergent. Therefore, the integral is singular.

N.B. Types 9.2 (iii) and 9.3 (iv) should be tested for singular i.e. those are divergent, consider the V.P. for them.

9.6. OPERATIONS WITH IMPROPER INTEGRALS

Improper integrals obey all the operations of definite integrals except the linearity. The operations are discussed below :

(i) **Constant factor of integrand:** As in the case of definite or indefinite integrals, a constant factor, K may be taken outside the improper integral i.e.

$$\int_a^b Kf(x) dx = K \int_a^b f(x) dx;$$

because in limiting process K may remain inside or outside the limit sign without affecting the limiting process.

(ii) **Linearity :** This property may not always true for improper integrals as in the case of definite integrals. That is, improper integral of a linear combination like

$$\int_a^b [c_1 f_1(x) + c_2 f_2(x) + \dots + c_n f_n(x)] dx \text{ may not always be}$$

equal to the linear combinations of the integrals i.e.

$$c_1 \int_a^b f_1(x) dx + c_2 \int_a^b f_2(x) dx + \dots + c_n \int_a^b f_n(x) dx$$

due to the divergence of any (some) of the individual integrals.

For example,

$$\int_2^\infty \left(\frac{1}{x-1} - \frac{1}{x} \right) dx = \lim_{b \rightarrow \infty} \int_2^b \left(\frac{1}{x-1} - \frac{1}{x} \right) dx = \lim_{b \rightarrow \infty} \left[\log \frac{x-1}{x} \right]_2^b$$

$$= \lim_{b \rightarrow \infty} \left[\log \left(1 - \frac{1}{x} \right) \right]_2^b = \lim_{b \rightarrow \infty} \left[\log \left(1 - \frac{1}{b} \right) + \log 2 \right] = \log 2$$

= convergent ;

$$\text{whereas } \int_2^\infty \frac{dx}{x-1} = \lim_{b \rightarrow \infty} \int_2^b \frac{dx}{x-1} = \lim_{b \rightarrow \infty} \left[\log (x-1) \right]_2^b$$

$$= \lim_{b \rightarrow \infty} \left[\log (b-1) - \log 1 \right] = \infty = \text{divergent.}$$

$$\text{and } \int_2^\infty \left(-\frac{1}{x} \right) dx = \lim_{b \rightarrow \infty} \int_2^b \left(-\frac{1}{x} \right) dx = \lim_{b \rightarrow \infty} \left[-\log x \right]_2^b$$

$$= \lim_{b \rightarrow \infty} \left[\log \frac{1}{x} \right]_2^b = -\infty = \text{divergent}.$$

$$\therefore \int_2^{\infty} \left(\frac{1}{x-1} - \frac{1}{x} \right) dx \neq \int_2^{\infty} \frac{1}{x-1} dx + \int_2^{\infty} \left(-\frac{1}{x} \right) dx.$$

(iii) **Integration by parts:** This is same as definite integral i.e.

$$\int_a^b u(x) v(x) dx = \left[u(x) \int v(x) dx \right]_a^b - \int_a^b \left[\frac{d}{dx} u(x) \int v(x) dx \right] dx$$

But the existence depends on the existence of two of the three integrals.

(iv) **Substitution:** Appropriate substitution sometimes transfers an improper integral of the first kind into an ordinary definite integral.

For example—

$$\int_0^{\infty} \frac{x^2}{(1+x^2)^2} dx = \int_0^{\pi/2} \sin^2 \theta d\theta \quad \text{when } x = \tan \theta.$$

Also a second kind integral can be transformed to a first kind by simple substitution like $y = \frac{1}{b-x}$ when $x = b$ is a s.p.

(v) **Additive property:** Additive property holds for a convergent improper integral $\int_a^{\infty} f(x) dx$ if there is some $b > a$ i.e.

$$\int_a^{\infty} f(x) dx = \int_a^b f(x) dx + \int_b^{\infty} f(x) dx$$

and the latter improper integral $\int_b^{\infty} f(x) dx$ must be convergent because of the convergence of the former.

9.7. TEST FOR CONVERGENCE OF IMPROPER INTEGRAL.

Test of convergence conveys dual meaning — either convergence or divergence of an improper integral. Sometimes it happens that the improper integral like $\int_0^{\infty} e^{-x^2} dx$ cannot be evaluated from the

definitions (sec. 9.2 and sec. 9.3). But its convergence is required. Also these tests are useful when the converging character is necessary only instead of the value of the integral. So a limited number of tests are stated below without proofs.

9.7.1. CONVERGENCE TESTS : FIRST KINDS

Test number TF_i denotes the i th number test of first kind in this book. These numbers are not standard symbols.

The improper integrals $\int_{-\infty}^b f(x) dx$ and $\int_{-\infty}^{\infty} f(x) dx$ are both related to $\int_a^{\infty} f(x) dx$. Because if x is replaced by $(-y)$ then

$$\int_{-\infty}^b f(x) dx = \int_{-b}^{\infty} f(-y) dy \text{ and } \int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^0 f(x) dx + \int_0^{\infty} f(x) dx$$

$$= \int_0^{\infty} f(-y) dy + \int_0^{\infty} f(x) dx. \text{ So, if a test of convergence is}$$

applicable to $\int_a^{\infty} f(x) dx$, it is equally applicable to $\int_{-\infty}^b f(x) dx$ and $\int_{-\infty}^{\infty} f(x) dx$.

That is why, all the tests discussed below are stated for $\int_a^{\infty} f(x) dx$ only.

TF₁. Comparison test : If $\int_a^{\infty} f(x) dx$ and $\int_b^{\infty} g(x) dx$ are such that the integrals are positive and (i) $f(x) \leq g(x)$ for all x beyond a certain point $x = c$, and also $\int_b^{\infty} g(x) dx$ is convergent, then $\int_a^{\infty} f(x) dx$ is convergent, but if (ii) $g(x) \leq f(x)$ and $\int_b^{\infty} g(x) dx$ is divergent, then $\int_a^{\infty} f(x) dx$ is divergent.

The p th integral $\int_a^{\infty} \frac{1}{x^p} dx$, p is constant, $p > 0$, may be considered as comparison integral. This integral is convergent for $p > 1$ and divergent for $p \leq 1$. Because

$$\int_a^{\infty} \frac{1}{x^p} dx = \lim_{b \rightarrow \infty} \frac{1}{1-p} \left[x^{1-p} \right]_{x=a}^b$$

$$= \begin{cases} \lim_{b \rightarrow \infty} \frac{-1}{p-1} \left[\frac{1}{x^{p-1}} \right]_{x=a}^b, & p > 1 \\ \lim_{b \rightarrow \infty} \frac{1}{1-p} \left[x^{1-p} \right]_{x=a}^b, & p < 1 \\ \infty, & p = 1 \end{cases}$$

$$= \begin{cases} \frac{1}{p-1} \cdot \frac{1}{a^{p-1}}, & p > 1 \\ \infty, & p < 1 \\ \infty, & p = 1 \end{cases}$$

Hence the ascertainment.

TF₂. Limit form of comparison test : If $\int_a^\infty f(x) dx$ and $\int_b^\infty g(x) dx$ are such that the integrands are positive and $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = L$, then (i) $\int_a^\infty f(x) dx$ is convergent when $L = 0$ and $\int_b^\infty g(x) dx$ is convergent, and (ii) $\int_a^\infty f(x) dx$ is divergent when $L = \infty$ and $\int_b^\infty g(x) dx$ is divergent; (iii) Both integrals are either convergent or divergent if L exists but $\neq 0$.

TF₃. Cauchy's convergence criterion : If $f(x) \geq 0$ and $\int_a^\infty f(x) dx$ is convergent and $g(x)$ is bounded i.e. $|g(x)| \leq M$ for some +ve const. M , then $\int_a^\infty f(x)g(x) dx$ is convergent.

This criterion is only meant for convergence and remained silent about its divergence.

TF₄. Tests for convergence of integral when integrand is both +ve and -ve.

If the integrand crosses the x -axis once or more then it is both +ve and -ve. In that case the above tests are not applicable. The following definition as well as tests are useful.

Definition : (1) If $\int_a^\infty |f(x)| dx$ is convergent then $\int_a^\infty f(x) dx$ is *absolutely convergent*. That is, if $f(x) \geq 0$ as well as $f(x) \leq 0$ in $[a, \infty)$ then absolute value $|f(x)|$ is +ve for which the tests TF_1 to TF_3 are applicable and if $\int_a^\infty |f(x)| dx$ is found convergent then $\int_a^\infty f(x) dx$ is absolutely convergent. In other words, absolutely convergent integrals are convergent but the converse may not be true.

Definition : (2) If $\int_a^\infty f(x) dx$ is convergent but $\int_a^\infty |f(x)| dx$ is divergent then $\int_a^\infty f(x) dx$ is *conditionally convergent*.

TF₅. Dirichlet's test : If $f(x)$, $g(x)$ and $g'(x)$ are all continuous in $[a, \infty)$ and $f(x)$, $g(x)$ satisfy the following three conditions then $\int_a^\infty f(x) g(x) dx$ is convergent.

(i) $\lim_{x \rightarrow \infty} g(x) = 0$; (ii) $\int_a^\infty |g'(x)| dx$ is convergent and

(iii) $F(r) = \int_a^r f(x) dx$ is bounded i.e. $|F(r)| \leq M$ for some +ve constant M .

TF₆. Uniform convergence : The question of uniform convergence arises when the improper integral contains a parameter, x like

$$g(x) = \int_a^\infty f(t, x) dt \quad (1)$$

Definition : The integral (1) converges uniformly w.r.t. the parameter x in the interval $c \leq x \leq d$ if corresponding to any $\varepsilon > 0$ there exists a number T independent of x , such that

$$\left| \int_{t_0}^\infty f(t, x) dt \right| < \varepsilon$$

for every value of x in $[c, d]$ whenever $t_0 \geq T$.

Sometimes the use of this definition directly for testing uniform convergence becomes difficult. So, let us state below M -test, the simplest and most useful test for uniform convergence.

Weierstrass M-test : If there exists a positive continuous function $M(t)$ such that $|f(x, t)| \leq M(t)$, $t \geq a$, $c \leq x \leq d$ then the improper integral $\int_a^\infty f(x, t) dt$ converges uniformly and absolutely for energy x in the interval

$[c, d]$ if $\int_a^\infty M(t) dt$ converges.

9.7.2. CONVERGENCE TESTS : SECOND KINDS

The convergence tests for different s.p. are identical. Hence only tests for s.p. at $x = b$ the upper limit are stated without proof. Test number TS_i denotes i th test of the second kind.

TS_1 . Comparison test : Let $x = b$ be a s.p. for both $f(x)$ and $g(x)$ in interval $[a, b]$ and. (i) $0 \leq f(x) \leq g(x)$ everywhere except at $x = b$ then $\int_a^b f(x) dx$ is convergent if $\int_a^b g(x) dx$ is convergent; (ii) $f(x) \geq g(x) \geq 0$

everywhere except at $x = b$ then $\int_a^b f(x) dx$ is divergent if $\int_a^b g(x) dx$ is divergent.

A well known comparison integral is $\int_a^b \frac{dx}{(b-x)^p}$ with s.p.

at $x = b$, the upper limit. It is convergent if $p < 1$ and divergent if $p \geq 1$. This statement can be easily verified.

TS_2 . Limit form of comparison test : (i) If $f(x)$ and $g(x)$ are positive and $\lim_{x \rightarrow b} \frac{f(x)}{g(x)} = L$ where L is neither zero nor infinite, then the two integrals

$\int_a^b f(x) dx$ and $\int_a^b g(x) dx$ either both converge or both diverge at the s.p.

$x = b$; (ii) If $L = 0$ in (i) and $\int_a^b g(x) dx$ converges, then $\int_a^b f(x) dx$ converges;

(iii) If $L = \infty$ in (i) and $\int_a^b f(x) dx$ diverges then $\int_a^b g(x) dx$ diverges.

TS₃ . Cauchy's convergence criterion : Let $f(x) \geq 0$ and let it has a s.p. at $x = b$ (or $x = a$). Let $g(x)$ be bounded i.e. $|g(x)| \leq M$ for some +ve constant M . Now if $\int_a^b f(x) dx$ is convergent then $\int_a^b f(x) g(x) dx$ is convergent. As in TF₃ it remains silent about the divergence.

TS₄ . Integrand is both +ve and -ve. As in TF₁ the integrand is both +ve and -ve in $[a, b]$. Let $x = b$ be a s.p. of $f(x)$ also. Now if $\int_a^b |f(x)| dx$ is convergent then $\int_a^b f(x) dx$ is *absolutely convergent* while if $\int_a^b f(x) dx$ is convergent but $\int_a^b |f(x)| dx$ is divergent then $\int_a^b f(x) dx$ is *conditionally convergent*.

TS₅ . Uniform convergence : Weierstrass M-test : As in TF₆ the integrand with parameter falls under this test. Let $F(x) = \int_a^b f(t, x) dt$, x is a parameter; (1)

Let $f(t, x)$ has a s.p. at $t = b$ and continuous in $a \leq t \leq b$, $x_0 \leq x \leq x_1$. Also for some +ve function $M(t)$,

$$|f(t, x)| \leq M(t) \text{ when } a \leq t \leq b, x_0 \leq x \leq x_1.$$

If $\int_a^b M(t) dt$ is convergent then $F(x)$ i.e. $\int_a^b f(t, x) dt$ is *uniformly convergent*.

9.7.3. CONVERGENCE TEST : THIRD KINDS

As third kind improper integral is a linear combination of first and second kind's, so there arise no new test. Decomposing the integral into 1st and 2nd kinds and using the TF_i and TS_i for each part conclusion should be drawn from definition of 3rd kind.

9.8 . DIFFERENTIATION UNDER INTEGRAL SIGN

Differentiation of an improper integral w.r.t. a parameter is sometimes necessary. The integral $F(x) = \int_a^\infty f(t, x) dt$ is differentiable if

$\frac{\partial f}{\partial x}$ is continuous in t and x for $a \leq t < \infty$ and $c \leq x \leq d$ and the improper integral $\int_a^\infty f(t, x) dt$ is convergent and $\int_a^\infty \frac{\partial}{\partial x} f(t, x) dt$ is uniformly convergent for $c \leq x \leq d$, then

$$F'(x) = \int_a^\infty \frac{\partial}{\partial x} f(t, x) dt. \text{ Similarly for other improper integrals.}$$

Application to evaluate improper integral is shown in examples 14, 15 and prob. 13 in exercises.

Examples

1. Evaluate $\int_1^\infty \frac{x+2}{x(x+1)} dx$.

The integrand is continuous in $[1, \infty)$ and the upper limit of the integral is infinite. So, it is an improper integral of the first kind. Therefore, by def. 9.2 (i)

$$\begin{aligned} \int_1^\infty \frac{x+2}{x(x+1)} dx &= \lim_{b \rightarrow \infty} \int_1^b \frac{x+2}{x(x+1)} dx = \lim_{b \rightarrow \infty} \int_1^b \left(\frac{2}{x} - \frac{1}{x+1} \right) dx \\ &= \lim_{b \rightarrow \infty} \left[\log \frac{x^2}{x+1} \right]_1^b = \lim_{b \rightarrow \infty} \left[\log \frac{b^2}{b+1} + \log 2 \right] \\ &= \lim_{b \rightarrow \infty} \left[\log \frac{1}{\frac{1}{b} + \frac{1}{b^2}} + \log 2 \right] = \log 0 + \log 2 \\ &= -\infty + \log 2 = -\infty = \text{infinite} \end{aligned}$$

$\therefore$ the given integral is divergent as it has no finite value.

2. Evaluate $\int_0^\infty \frac{\tan^{-1} x}{1+x^2} dx$.

This is also of type 9.2 (i). Let $\tan^{-1} x = \theta$. So, $\frac{1}{1+x^2} dx = d\theta$ and

x	0	∞
θ	0	$\pi/2$

Therefore,

$$\int_0^\infty \frac{\tan^{-1} x}{1+x^2} dx = \int_0^{\pi/2} \theta d\theta = \frac{\pi^2}{8} = \text{finite}$$

∴ the given improper integral is convergent.

3. Evaluate $\int_{-\infty}^0 \sinh x \, dx$.

The integrand is continuous in $(-\infty, 0]$, but the lower limit is $(-\infty)$. So, it is an improper integral of the first kind. Therefore, by def. 9.2 (ii)

$$\begin{aligned}\int_{-\infty}^0 \sinh x \, dx &= \lim_{a \rightarrow -\infty} \int_a^0 \sinh x \, dx = \lim_{a \rightarrow -\infty} [\cosh x]_a^0 \\ &= \lim_{a \rightarrow -\infty} [1 - \cosh a] = 1 - \cosh(-\infty) = 1 - \cosh \infty = 1 - \infty \\ &= -\infty = \text{infinite}\end{aligned}$$

∴ the given integral is divergent.

4. Evaluate $\int_{-\infty}^{+\infty} \frac{x}{x^4 + 1} \, dx$.

The integrand is continuous in $(-\infty, \infty)$ but both the upper and the lower limits are infinite. So, it is an improper integral of the first kind and of type 9.2 (iii).

Therefore, by def.

$$I = \int_{-\infty}^{+\infty} \frac{x}{x^4 + 1} \, dx = \lim_{a \rightarrow -\infty} \int_a^c \frac{x}{x^4 + 1} \, dx + \lim_{b \rightarrow \infty} \int_c^b \frac{x}{x^4 + 1} \, dx$$

$$= \lim_{a^2 \rightarrow \infty} \frac{1}{2} \int_{a^2}^{c^2} \frac{du}{u^2 + 1} + \lim_{b^2 \rightarrow \infty} \frac{1}{2} \int_{c^2}^{b^2} \frac{du}{u^2 + 1} \quad \text{when } x^2 = u$$

and as $a \rightarrow -\infty, a^2 \rightarrow \infty$ and as $b \rightarrow \infty, b^2 \rightarrow \infty$.

$$\therefore I = \frac{1}{2} \lim_{a^2 \rightarrow \infty} [\tan^{-1} u]_{a^2}^{c^2} + \frac{1}{2} \lim_{b^2 \rightarrow \infty} [\tan^{-1} u]_{c^2}^{b^2}$$

$$= \frac{1}{2} \left[\lim_{a^2 \rightarrow \infty} (\tan^{-1} c^2 - \tan^{-1} a^2) + \lim_{b^2 \rightarrow \infty} (\tan^{-1} b^2 - \tan^{-1} c^2) \right]$$

$$= \frac{1}{2} \left[\tan^{-1} c^2 - \frac{\pi}{2} + \frac{\pi}{2} - \tan^{-1} c^2 \right] = 0 = \text{finite}$$

∴ the improper integral is convergent.

5. Evaluate $\int_0^2 \frac{dx}{2-x}$.

The limits of the integral are finite but the integrand is infinitely discontinuous at $x = 2$. So, it is an improper integral of the second kind of type 9.3 (i). Therefore by def. (s.p. at $x = 2$)

$$\begin{aligned}\int_0^2 \frac{dx}{2-x} &= \lim_{\delta \rightarrow 0} \int_0^{2-\delta} \frac{dx}{2-x} = \lim_{\delta \rightarrow 0} \left[-\log(2-x) \right]_0^{2-\delta} \\ &= \lim_{\delta \rightarrow 0} \log \frac{2}{\delta} = \log \infty = \infty = \text{infinite}\end{aligned}$$

$\therefore$ the improper integral is divergent.

6. Evaluate $\int_{-1}^1 \frac{dx}{x^3}$.

The integrand is infinitely discontinuous at $x = 0$ which is a point interior to the interval $[-1, 1]$. So, the integral is of the type 9.3 (iii).

Therefore, by def.

$$\begin{aligned}\int_{-1}^1 \frac{1}{x^3} dx &= \int_{-1}^0 \frac{1}{x^3} dx + \int_0^1 \frac{1}{x^3} dx \\ &= \lim_{\delta_1 \rightarrow 0} \int_{-1}^{0-\delta_1} \frac{dx}{x^3} + \lim_{\delta_2 \rightarrow 0} \int_{0+\delta_2}^1 \frac{1}{x^3} dx \\ &= \lim_{\delta_1 \rightarrow 0} \left[-\frac{1}{2x^2} \right]_{-1}^{-\delta_1} + \lim_{\delta_2 \rightarrow 0} \left[-\frac{1}{2x^2} \right]_{\delta_2}^1 \\ &= -\frac{1}{2} \left[\lim_{\delta_1 \rightarrow 0} \left(\frac{1}{\delta_1^2} - 1 \right) + \lim_{\delta_2 \rightarrow 0} \left(1 - \frac{1}{\delta_2^2} \right) \right] = -\frac{1}{2} (\infty - \infty) \\ &= \text{an indeterminate form}\end{aligned}$$

But if $\delta_1 = \delta_2 = \delta$, then

$$\begin{aligned}\text{V.P. } \int_{-1}^1 \frac{1}{x^3} dx &= \lim_{\delta \rightarrow 0} \left[\int_{-1}^{-\delta} \frac{dx}{x^3} + \int_{\delta}^1 \frac{dx}{x^3} \right] \\ &= \lim_{\delta \rightarrow 0} \left[-\frac{1}{2} \left(\frac{1}{\delta^2} - 1 + 1 - \frac{1}{\delta^2} \right) \right] = 0\end{aligned}$$

$\therefore$ principal value, V.P. $\int_{-1}^1 \frac{dx}{x^3} = 0$. So, the V.P. of integral exists, and hence the integral is singular.

7. Prove that $\int_0^{\pi/2} \log \sin x \, dx = \int_0^{\pi/2} \log \cos x \, dx = -\frac{\pi}{2} \log 2$.

The integrands are infinitely discontinuous and they are improper integrals of the second kind. Yet the problem can be handled using properties of definite integrals of chapter 8. Some authors include this problem under definite integrals.

Let $I = \int_0^{\pi/2} \log \sin x \, dx$. Then by prop. 10

$$I = \int_0^{\pi/2} \log \sin \left(\frac{\pi}{2} - x \right) dx = \int_0^{\pi/2} \log \cos x \, dx$$

$$\therefore \int_0^{\pi/2} \log \sin x \, dx = \int_0^{\pi/2} \log \cos x \, dx.$$

Now adding the two values of I , we get,

$$\begin{aligned} 2I &= \int_0^{\pi/2} \log \sin x \, dx + \int_0^{\pi/2} \log \cos x \, dx \\ &= \int_0^{\pi/2} \log \sin x \cos x \, dx = \int_0^{\pi/2} \log \frac{1}{2} \cdot \sin 2x \, dx \\ &= \log \frac{1}{2} \int_0^{\pi/2} \sin 2x \, dx + \int_0^{\pi/2} \log \sin 2x \, dx \\ &= -\log 2 \left[x \right]_0^{\pi/2} + I_1, \text{ where } I_1 = \int_0^{\pi/2} \log \sin 2x \, dx \\ &= -\frac{\pi}{2} \log 2 + I_1. \end{aligned}$$

Now $I_1 = \int_0^{\pi/2} \log \sin 2x \, dx$

$$\therefore I_1 = \frac{1}{2} \int_0^{\pi} \log \sin u \, du$$

$$= \frac{1}{2} \int_0^{2, \pi/2} \log \sin u \, du$$

$$= \frac{1}{2} \cdot 2 \int_0^{\pi/2} \log \sin u \, du \quad \text{by prop. 13.}$$

$$= \int_0^{\pi/2} \log \sin x \, dx = I, \quad \text{by prop. 2.}$$

$$\therefore 2I = -\frac{\pi}{2} \log 2 + I$$

$$\text{or } I = -\frac{\pi}{2} \log 2.$$

8. Prove that $\int_0^{\infty} \log \left(x + \frac{1}{x} \right) \frac{dx}{1+x^2} = \pi \log 2.$

Let $x = \tan \theta \therefore dx = \sec^2 \theta \, d\theta$ and

x	0	∞
θ	0	$\pi/2$

$$\therefore L.H.S. = \int_0^{\infty} \log \left(x + \frac{1}{x} \right) \frac{dx}{1+x^2} = \int_0^{\pi/2} \log (\tan \theta + \cot \theta) \, d\theta$$

$$= \int_0^{\pi/2} \log \frac{1}{\sin \theta \cos \theta} \, d\theta = -2 \int_0^{\pi/2} \log \sin \theta \, d\theta$$

$$= \pi \log 2 = R.H.S. \quad \because \int_0^{\pi/2} \log \sin \theta \, d\theta = \frac{\pi}{2} \log \frac{1}{2}$$

9. Prove that $\int_0^1 \frac{\log(1-x)}{x} \, dx = \int_0^1 \frac{\log x}{1-x} \, dx = -\frac{\pi^2}{6}.$

Let $1-x = y$ in first integral. $\therefore dx = -dy$

and

x	0	1
y	1	0

 Therefore,

$$\int_0^1 \frac{\log(1-x)}{x} \, dx = - \int_1^0 \frac{\log y}{1-y} \, dy = \int_0^1 \frac{\log x}{1-x} \, dx.$$

[or prop. 10 may be used]

$$\begin{aligned}
 \text{Again, } \int_0^1 \frac{\log(1-x)}{x} dx &= - \int_0^1 \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \dots \infty \right) \frac{1}{x} dx \\
 &= - \int_0^1 \left(1 + \frac{x}{2} + \frac{x^2}{3} + \dots \infty \right) dx = - \left[\frac{x}{1^2} + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \dots \infty \right]_0^1 \\
 &= - \left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \infty \right] = - \frac{\pi^2}{6}. \\
 \therefore \int_0^1 \frac{\log(1-x)}{x} dx &= \int_0^1 \frac{\log x}{1-x} dx = - \frac{\pi^2}{6}.
 \end{aligned}$$

N.B. $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ can be obtained from Fourier Series. Derivation is not shown here.

10. Show that $\int_0^{\infty} \frac{x}{(1+x)^3} dx = \frac{1}{2} \int_0^{\infty} \frac{dx}{(1+x)^2} = \frac{1}{2}.$

Using integration by parts for the first integral,

$$\begin{aligned}
 \int_0^{\infty} \frac{x}{(1+x)^3} dx &= \lim_{b \rightarrow \infty} \left[-\frac{x}{2(1+x)^2} \right]_0^b + \frac{1}{2} \int_0^{\infty} \frac{dx}{(1+x)^2} \\
 &= \lim_{b \rightarrow \infty} \left[-\frac{b}{2(1+b)^2} \right] + \frac{1}{2} \int_0^{\infty} \frac{dx}{(1+x)^2} \\
 &= -\frac{1}{2} \lim_{b \rightarrow \infty} \left[\frac{1}{\frac{1}{b} + b + 2} \right] + \frac{1}{2} \int_0^{\infty} \frac{dx}{(1+x)^2} \\
 &= 0 + \frac{1}{2} \int_0^{\infty} \frac{dx}{(1+x)^2} = \frac{1}{2} \int_0^{\infty} \frac{dx}{(1+x)^2} \\
 \int_0^{\infty} \frac{x}{(1+x)^3} dx &= \frac{1}{2} \int_0^{\infty} \frac{dx}{(1+x)^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{Again } \int_0^{\infty} \frac{dx}{(1+x)^2} &= \lim_{b \rightarrow \infty} \int_0^b \frac{dx}{(1+x)^2} = \lim_{b \rightarrow \infty} \left[-\frac{1}{1+x} \right]_0^b \\
 &= \lim_{b \rightarrow \infty} \left[-\frac{1}{1+b} + 1 \right] = 1
 \end{aligned}$$

$$\therefore \int_0^{\infty} \frac{x}{(1+x)^3} dx = \frac{1}{2} \int_0^{\infty} \frac{dx}{(1+x)^2} = \frac{1}{2}.$$

$$11. \text{ Show that } \int_0^{\infty} \frac{\log x}{1+x^2} dx = 0.$$

The integral is a mixed type. Because the upper limit is infinite and at the lower limit $\log x = -\infty$. So, dividing the interval $[0, \infty]$ into two parts like $[0, 1]$ and $[1, \infty]$, we can write the integral as

$$\int_0^{\infty} \frac{\log x}{1+x^2} dx = \int_0^1 \frac{\log x}{1+x^2} dx + \int_1^{\infty} \frac{\log x}{1+x^2} dx.$$

$$\begin{aligned} \text{Now, } \int_1^{\infty} \frac{\log x}{1+x^2} dx &= - \int_0^1 \frac{\log y}{1+y^2} dy \quad \text{when } x = \frac{1}{y} \\ &= - \int_0^1 \frac{\log x}{1+x^2} dx. \end{aligned}$$

$$\therefore \int_0^{\infty} \frac{\log x}{1+x^2} dx = \int_0^1 \frac{\log x}{1+x^2} dx - \int_0^1 \frac{\log x}{1+x^2} dx = 0.$$

N.B. The choice of $x = 1$ as a breaking point is **ARBITRARY**. Any other +ve value may be used.

$$12. \text{ Test the integral } \int_0^{\infty} \frac{dx}{1+2x^2+3x^4} \text{ for convergence.}$$

Test TF₁ of 9.7.1 may be used. Here

$$f(x) = \frac{1}{1+2x^2+3x^4} < \frac{1}{x^4} = g(x) \text{ for all } x. \text{ Also}$$

$$\begin{aligned} I &= \int_1^{\infty} g(x) dx = \int_1^{\infty} \frac{1}{x^4} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x^4} dx = \lim_{b \rightarrow \infty} \left[-\frac{1}{3x^3} \right]_1^b \\ &= \lim_{b \rightarrow \infty} \left[-\frac{1}{3b^3} + \frac{1}{3} \right] = \frac{1}{3} = \text{finite} \\ \therefore I &\text{ is convergent.} \end{aligned}$$

$$\text{Hence } \int_0^{\infty} f(x) dx = \int_0^{\infty} \frac{1}{1+2x^2+3x^4} dx \text{ is convergent.}$$

Alter. Test TF₂ of 9.7.1 may be used. Let $f(x) = \frac{1}{1+2x^2+3x^4}$ and

$$g(x) = \frac{1}{x^4}, \text{ both are +ve in } [0, \infty) \text{ and}$$

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{1}{1+2x^2+3x^4} \times \frac{x^4}{1} = \lim_{x \rightarrow \infty} \frac{1}{\frac{1}{x^4} + \frac{2}{x^2} + 3} = \frac{1}{3} \neq 0$$

Also $\int_1^{\infty} g(x) dx = \int_1^{\infty} \frac{1}{x^4} dx$ is convergent. Therefore by TF₂ (iii) of

9.7.1, the given integral is convergent.

N.B. $g(x)$ may be obtained from $f(x)$ as the highest approximation of $f(x)$ for large x .

13. Test the integral $\int_0^1 \left(\frac{1}{x} - x\right)^{1/2} \cdot \frac{\log(1+x^{1/3})}{\sin x} dx$ for

convergence.

This is an improper integral of the second kind with s.p. at $x=0$. To use TS₂ of 9.7.2, $g(x)$ is obtained by approximating the component parts of

$f(x)$ i.e. $g(x) = \frac{1}{\sqrt{x}} \cdot \frac{x^{1/3}}{x} = \frac{1}{x^{7/6}} \because$ smallest powers of x in the expansions of

$\left(\frac{1}{x} - x\right)^{1/2}$, $\log(1+x^{1/3})$ and $\sin x$ at $x=0$ are $\frac{1}{\sqrt{x}}$, $x^{1/3}$ and x

respectively. Now, $\int_0^1 \frac{1}{x^{7/6}} dx = \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^1 \frac{1}{x^{7/6}} dx$

$$= \lim_{\varepsilon \rightarrow 0} \left[-6 \frac{1}{x^{1/6}} \right]_{\varepsilon}^1 \text{ is divergent and}$$

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow 0} \frac{\left(\frac{1}{x} - x\right)^{1/2} \log(1+x^{1/3})}{\sin x} \times x^{7/6} \\ &= \lim_{x \rightarrow 0} \frac{\sqrt{1-x^2}}{\sqrt{x}} \left(\frac{\log(1+x^{1/3})}{x^{1/3}} \right) \left(\frac{x}{\sin x} \right) \cdot x^{1/2} = 1 \neq 0 \end{aligned}$$

$\therefore$ by TS₂ of 9.7.2, the integral is divergent.

N.B. Smallest x 's are chosen for $g(x)$, because the series concerned are infinite.

14. Prove that $\int_0^{\infty} \frac{\sin x}{x} dx$ is convergent.

The integrand of the given 1st kind integral in the interval is sometimes +ve and sometimes -ve. So to use TF₄ the integral is decomposed into two integrals. Let

$$I = \int_0^{\infty} \frac{\sin x}{x} dx = \int_0^{\pi/2} \frac{\sin x}{x} dx + \int_{\pi/2}^{\infty} \frac{\sin x}{x} dx \quad (1)$$

The 1st one in (1) is a definite integral as $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$. So it is convergent.

Let us integrate by parts the 2nd one in (1).

$$\begin{aligned} \int_{\pi/2}^{\infty} \frac{\sin x}{x} dx &= \lim_{b \rightarrow \infty} \int_{\pi/2}^b \frac{\sin x}{x} dx \\ &= \lim_{b \rightarrow \infty} \left[-\frac{\cos x}{x} \right]_{\pi/2}^b - \int_{\pi/2}^b \frac{\cos x}{x^2} dx \\ &= - \int_{\pi/2}^{\infty} \frac{\cos x}{x^2} dx, \left[\because \lim_{b \rightarrow \infty} \cos b \text{ is finite, so, } \right. \\ &\quad \left. \lim_{b \rightarrow \infty} \frac{\cos b}{b} = 0 \right] \end{aligned}$$

Also for TF₄, $\int_{\pi/2}^{\infty} \frac{\cos x}{x^2} dx$ is absolutely convergent. Because

$$\begin{aligned} \left| \frac{\cos x}{x^2} \right| &\leq \frac{1}{x^2} \text{ and } \int_{\pi/2}^{\infty} \frac{dx}{x^2} = -\frac{1}{x} \Big|_{\pi/2}^{\infty} = \frac{2}{\pi} = \text{finite} \\ &\left[\because \frac{1}{x^2} \text{ is always +ve} \right] \end{aligned}$$

$\therefore \int_0^{\infty} \frac{\sin x}{x} dx$ is (convergent + convergent) = convergent.

N.B. Similarly, $\int_{\pi/2}^{\infty} \frac{\cos x}{x} dx$ is convergent.

15. Prove that $\int_{\pi/2}^{\infty} \frac{|\sin x|}{x} dx$ is divergent.

In the given interval the integrand is +ve.

$$\text{Let } f(x) = \frac{|\sin x|}{x} \geq \frac{\sin^2 x}{x} = \frac{1 - \cos^2 x}{2x} = g(x) \quad (\text{say})$$

$$\therefore f(x) \geq g(x).$$

$$\begin{aligned} \text{Again } \int_{\pi/2}^{\infty} g'(x) dx &= \lim_{b \rightarrow \infty} \int_{\pi/2}^b \frac{1 - \cos 2x}{2x} dx \\ &= \lim_{b \rightarrow \infty} \left[\frac{1}{2} \log x \right]_{\pi/2}^b - \frac{1}{2} \int_{\pi/2}^b \frac{\cos 2x}{x} dx \\ &= \lim_{b \rightarrow \infty} \left[\frac{1}{2} \log b - \frac{1}{2} \log \frac{\pi}{2} \right] - \frac{1}{2} \int_{\pi/2}^{\infty} \frac{\cos 2x}{x} dx \quad (1) \end{aligned}$$

$$\begin{aligned} \text{Also } \int_{\pi/2}^{\infty} \frac{\cos 2x}{x} dx &= \lim_{b \rightarrow \infty} \left[\frac{\sin 2x}{2x} \right]_{\pi/2}^b + \int_{\pi/2}^b \frac{\sin 2x}{x^2} dx \\ &= 0 - 0 + \int_{\pi/2}^{\infty} \frac{\sin 2x}{x^2} dx. \end{aligned}$$

The right hand integral $\int_{\pi/2}^{\infty} \frac{\sin 2x}{x^2} dx$ is absolutely convergent

because, $\left| \frac{\sin 2x}{x^2} \right| \leq \frac{1}{x^2}$ and $\int_{\pi/2}^{\infty} \frac{1}{x^2} dx$ is convergent. Therefore from (1)

$$\begin{aligned} \int_{\pi/2}^{\infty} g(x) dx &= \infty - \frac{1}{2} \log \frac{\pi}{2} + (\text{convergent to } \pi/2) \\ &= \text{Infinite} + \text{convergent} \end{aligned}$$

$$\therefore \int_{\pi/2}^{\infty} \frac{|\sin x|}{x} dx \text{ is divergent.}$$

16. Show that $\int_0^{\infty} \frac{\sin x}{x}$ is conditionally convergent.

From def. TF₄ $\int_0^{\infty} \frac{\sin x}{x}$ is convergent and

$$\int_0^{\infty} \left| \frac{\sin x}{x} \right| dx = \int_0^{\infty} \frac{|\sin x|}{x} dx = \int_0^{\pi/2} \frac{|\sin x|}{x} dx + \int_{\pi/2}^{\infty} \frac{|\sin x|}{x} dx$$

is to be shown divergent. Ref. above two examples 14 and 15.

17. Prove that $\int_0^{\infty} e^{-\alpha x} \frac{\sin \beta x}{x} dx$, $\alpha \geq 0$ is uniformly convergent.

Here $\lim_{x \rightarrow 0} \frac{\sin \beta x}{x} = \beta$. So given integral is of first kind and it contains two parameters α, β . Therefore, TF_6 is to be used.

$$\text{Let } F(\alpha, \beta) = \int_0^{\infty} e^{-\alpha x} \frac{\sin \beta x}{x} dx \quad \text{and}$$

$$f(\alpha, \beta, x) = e^{-\alpha x} \frac{\sin \beta x}{x}$$

$$\therefore |f(\alpha, \beta, x)| = \left| e^{-\alpha x} \frac{\sin \beta x}{x} \right| \leq \beta e^{-\alpha x}, \quad \alpha > 0, \beta \neq 0$$

$$\begin{aligned} \text{Now } \int_0^{\infty} \beta e^{-\alpha x} dx &= \lim_{b \rightarrow \infty} \frac{\beta}{\alpha} [e^{-\alpha x}]_{x=0}^b = -\frac{\beta}{\alpha} (0 - 1) \\ &= \frac{\beta}{\alpha} = \text{finite for } \alpha > 0 \end{aligned}$$

$\therefore$ it is convergent for $\alpha > 0$.

$\therefore$ by Weierstrass M-test $F(\alpha, \beta)$ is uniformly convergent for $\alpha > 0$.

Again when $\alpha \rightarrow 0+$, $F(\alpha, \beta)$ is to be tested for convergence. Now

$$|f(\alpha, \beta, x)| = \left| e^{-\alpha x} \frac{\sin \beta x}{x} \right| \leq \frac{|\sin \beta x|}{x}, \because e^{-\alpha x} \leq 1 \text{ for } \alpha \rightarrow 0+$$

By Ex. 14 above $\int_0^{\infty} \frac{\sin \beta x}{x} dx$ is convergent. So Weierstrass M-test

fails to give any decision.

Now integrating by parts, $r \geq 0$

$$\begin{aligned} &\int_r^{\infty} \frac{e^{-\alpha x} \sin \beta x}{x} dx \\ &= \lim_{b \rightarrow \infty} \left[-\frac{e^{-\alpha x} \cos \beta x}{\beta x} \right]_{x=r}^b - \int_r^{\infty} \frac{(1 + \alpha x) e^{-\alpha x} \sin \beta x}{\beta x^2} dx \\ &= \frac{e^{-\alpha r} \cos \beta r}{\beta r} - \int_r^{\infty} \frac{(1 + \alpha x) e^{-\alpha x} \sin \beta x}{\beta x^2} dx \end{aligned}$$

$\therefore$ taking absolute value of both sides

$$\left| \int_r^{\infty} e^{-\alpha x} \frac{\sin \beta x}{x} dx \right| \leq \frac{1}{\beta r} + \int_r^{\infty} \frac{1}{\beta x^2} dx = \frac{2}{\beta r}.$$

The R.H.S. of the inequality does not depend on α and $\lim_{r \rightarrow \infty} \frac{2}{\beta r} = 0$. Therefore, from the def. of uniform convergence, the given integral is uniformly convergent for $\alpha \geq 0$.

18. Evaluate $\int_0^{\infty} \frac{e^{-\alpha x} \sin \beta x}{x} dx$, $\alpha \geq 0$ and deduce that

$$\int_0^{\infty} \frac{\sin \beta x}{x} dx = \begin{cases} \frac{\pi}{2}, & \beta > 0 \\ 0, & \beta = 0 \\ -\frac{\pi}{2}, & \beta < 0 \end{cases}$$

Hence find the value of $\int_0^{\infty} \frac{\sin x}{x} dx$.

These integrals are very useful. The integral $\int_0^{\infty} \frac{\sin x}{x} dx$ is known as *sine integral* function of infinity and is denoted by $S_i(\infty)$. None of these integrals can be integrated by normal procedures. Let

$$F(\alpha, \beta) = \int_0^{\infty} \frac{e^{-\alpha x} \sin \beta x}{x} dx, \quad (1)$$

$\therefore$ (1) is uniformly convergent by Ex. 17 above.

Let us use differentiation under the integral sign.

If the integrand is differentiated partially w.r.t. α or β then x of the denominator cancels, and the integral can be easily integrated. So,

$$\begin{aligned} \frac{\partial F}{\partial \beta} &= \int_0^{\infty} e^{-\alpha x} \cos \beta x dx \\ &= \frac{\alpha}{\alpha^2 + \beta^2}, \text{ if } \alpha \geq 0, \text{ which is uniformly convergent.} \end{aligned}$$

Integrating w.r.t. β , we get,

$$F(\alpha, \beta) = \tan^{-1} \frac{\beta}{\alpha} + c \quad (2)$$

where c is a constant of integration. From (1), we have,

$$F(\alpha, 0) = 0 \quad (3)$$

$\therefore$ from (2), using $\beta = 0$, $c = 0$. So

$$F(\alpha, \beta) = \tan^{-1} \frac{\beta}{\alpha}, \quad \alpha \geq 0 \quad (4)$$

$$\text{i.e. } \int_0^{\infty} e^{-\alpha x} \frac{\sin \beta x}{x} dx = \tan^{-1} \frac{\beta}{\alpha}, \quad \alpha \geq 0.$$

Second part :

$\therefore F(\alpha, \beta)$ is assumed to be continuous function of α for $\alpha \geq 0$. We have from (1)

$$F(0, \beta) = \int_0^{\infty} \frac{\sin \beta x}{x} dx.$$

$$\text{Also from (4), } \lim_{\alpha \rightarrow 0} F(\alpha, \beta) = \lim_{\alpha \rightarrow 0} \tan^{-1} \frac{\beta}{\alpha} = \begin{cases} \pi/2, & \beta > 0 \\ 0, & \beta = 0 \\ -\pi/2, & \beta < 0 \end{cases}$$

And because of continuity, $\lim_{\alpha \rightarrow 0} F(\alpha, \beta) = F(0, \beta)$

$$\therefore \int_0^{\infty} \frac{\sin \beta x}{x} dx = \begin{cases} \pi/2, & \beta > 0 \\ 0, & \beta = 0 \\ -\pi/2, & \beta < 0 \end{cases}$$

Third part :

When $\beta = 1$ in second part, we get

$$\int_0^{\infty} \frac{\sin x}{x} dx = \pi/2$$

19. Show that for $y > 0$, $\int_0^{\infty} e^{-xy} \frac{\sin x}{x} dx = \frac{\pi}{2} - \tan^{-1} y$.

$$\text{Let } F(y) = \int_0^{\infty} e^{-xy} \frac{\sin x}{x} dx \quad (1)$$

$$\begin{aligned} \therefore F'(y) &= - \int_0^{\infty} e^{-xy} \sin x dx = \lim_{b \rightarrow \infty} \left[- \frac{e^{-xy} (-y \sin x - \cos x)}{y^2 + 1} \right]_{x=0}^b \\ &= - \frac{1}{y^2 + 1} \end{aligned}$$

$$\therefore \text{integrating, } F(y) = - \tan^{-1} y + c, \quad y > 0 \quad (2)$$

$$\text{From (1), } F(0) = \int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2} \text{ by Ex. 13}$$

$$\therefore F(y) = \frac{\pi}{2} - \tan^{-1} y \text{ i.e. } \int_0^{\infty} e^{-xy} \frac{\sin x}{x} dx = \frac{\pi}{2} - \tan^{-1} y, y > 0.$$

Exercises 9

1. Classify the following improper integrals :

$$\begin{aligned} \text{(i)} & \int_0^{\infty} \frac{dx}{x^2 - 1}; \text{ (ii)} \int_0^1 \frac{\log x}{1+x} dx; \text{ (iii)} \int_0^{\pi} \frac{dx}{1 + \cos x}; \\ \text{(iv)} & \int_{-1}^1 \frac{dx}{\sqrt{x(x+1)}}; \text{ (v)} \int_0^{\infty} \frac{dx}{1 + \tan x}; \text{ (vi)} \int_{-\infty}^{\infty} \frac{x^2 dx}{x^4 + x^2 + 1} \end{aligned}$$

2 (a) Evaluate the following integrals :

$$\text{(i)} \int_0^{\infty} e^{-x} dx, \text{ (ii)} \int_1^{\infty} \frac{\tan^{-1} x}{1+x^2} dx, \text{ (iii)} \int_0^{\infty} \frac{x^3}{16+x^4} dx.$$

$$\text{(b) Prove that : (i)} \int_0^1 \log \left(\frac{1}{1-x} \right) dx = \int_1^{\infty} \frac{\log x}{x^2} dx,$$

$$\text{(ii)} \int_0^1 \left(\log \frac{1}{u} \right)^3 du = \int_0^{\infty} u^3 e^{-u} du.$$

3. Find the value of :

$$\text{(i)} \int_{-1}^2 x^{-1/3} dx, \text{ (ii)} \int_0^3 \frac{x dx}{(x^2 - 1)^{2/3}}, \text{ (iii)} \int_0^2 \frac{dx}{\sqrt{2-x}}.$$

4. Show that

$$\text{(i)} \int_0^{\infty} \frac{x dx}{(1+x)(1+x^2)} = \frac{\pi}{4};$$

$$\text{(ii)} \int_0^{\infty} \frac{\log(1+x^2)}{1+x^2} dx = \pi \log 2, [\text{Hints : } x = \tan \theta],$$

$$\text{(iii)} \int_0^{\infty} \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)(x^2+c^2)} = \frac{\pi}{2(a+b)(b+c)(c+a)}$$

$$\text{(iv)} \int_{-\infty}^{+\infty} \frac{dx}{(x^2 \pm ax + a^2)(x^2 \pm bx + b^2)} = \frac{2\pi}{\sqrt{3}} \cdot \frac{a+b}{ab(a^2 + ab + b^2)}$$

$$\text{(v)} \int_{-\infty}^0 \frac{dx}{1+x^2} = \int_0^{\infty} \frac{dx}{1+x^2} = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \frac{\pi}{2}.$$

5. Evaluate the following (i) $\int_0^{\pi} \frac{\sin x}{\cos^2 x} dx$;

(ii) $\int_0^{\pi} \log(1 + \cos x) dx$; (iii) $\int_0^{\pi} \frac{dx}{1 + \cos x}$; (iv) $\int_0^1 \frac{\log x}{1+x} dx$;

(v) $\int_0^1 \frac{\sin^{-1} x}{x} dx$; (vi) $\int_0^{\pi} \log(1 - \cos x) dx$.

6. If $(a + b) > 0$, then show that $\int_0^{\infty} \frac{dx}{(x+a)\sqrt{x-b}} = \frac{\pi}{\sqrt{a+b}}$,

[Hints : put $x - b = u^2$]

7. Prove that

(i) $\int_0^1 \frac{dx}{(1+x)(2+x)\sqrt{x(1-x)}} = \pi \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{6}} \right)$. [Hints: $x = \sin^2 \theta$]

(ii) $\int_1^{\infty} \frac{\sqrt{x}}{(1+x)^2} dx = \frac{1}{2} + \frac{\pi}{4}$. [Hints: put $x = t^2$,

$2 \int_1^{\infty} \frac{t^2 dt}{(1+t^2)^2} = - \int_1^{\infty} t \frac{d}{dt} \left(\frac{1}{1+t^2} \right) dt$, integrate by parts.]

8. Evaluate the following :

(i) $\int_1^{\infty} \frac{dx}{(1+x)\sqrt{x}}$; (ii) $\int_0^{\infty} \frac{dx}{(1+x^2)^n}$, n is a +ve integer.

(iii) $\int_0^{\infty} \frac{dx}{(1+x^2)^{n+1/2}}$, n is a +ve integer;

[Hints : (i) put $x = t^2$, for (ii) & (iii) put $x = \cot \theta$] ;

(iv) $\int_0^{\infty} \frac{8a^3}{x^2 + 4a^2} dx$; (v) $\int_0^{2a} \frac{x^{3/2}}{\sqrt{2a-x}} dx$;

(vi) $\int_0^{\infty} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} , k^2 < 1$, [Hints : put $x = \sin \theta$] ;

(vii) $\int_0^5 \frac{dx}{\sqrt{25-x^2}}$.

9. Prove the following :

(i) $\int_0^{\infty} e^{-x} x^n dx = n!$, n is a +ve integer ;

$$(ii) \int_0^{\infty} e^{-ax} \cos bx \, dx = \frac{a}{a^2 + b^2}, \quad a > 0;$$

$$(iii) \int_0^{\infty} e^{-ax} \sin bx \, dx = \frac{b}{a^2 + b^2}, \quad a > 0;$$

$$(iv) \int_0^{\infty} \frac{x^2}{(x^2 + a^2)(x^2 + b^2)} \, dx = \frac{\pi}{2(a+b)}, \quad (a, b > 0);$$

$$(v) \int_{-\infty}^{\infty} \frac{dx}{x^2 + 2x + 2} = \pi;$$

$$(vi) \int_0^{\infty} \frac{dx}{a^2 e^x + b^2 e^{-x}} = \frac{1}{ab} \tan^{-1} \frac{b}{a}.$$

10. Find the value of

$$(i) \int_{-a}^0 \frac{1}{x+a} \, dx; (ii) \int_0^1 \frac{\sin^{-1} x}{\sqrt{1-x^2}} \, dx; (iii) \int_0^{\pi/2} \frac{\sin \theta}{\sqrt{1-\cos \theta}} \, d\theta;$$

$$(iv) \int_0^1 \frac{\sin(1/x)}{x^2} \, dx; (v) \int_1^{\infty} \frac{1}{x \log x} \, dx; (vi) \int_{-\infty}^{-1} \frac{x}{(1+x^2)^2} \, dx;$$

$$(vii) \int_{-\infty}^{+\infty} \frac{2}{e^x + e^{-x}} \, dx.$$

11. Prove that $\int_0^{\infty} \frac{dx}{(1+x)\sqrt{x}}$ is a convergent integral.

[Hints: Break the interval $[0, \infty)$ into $[0, 1]$ and $[1, \infty)$.]

12. Test the following integrals for convergence:

$$(i) \int_1^{\infty} \frac{dx}{x^{1/3}(1+x)^{1/2}} \quad [\text{Hints: } g(x) = \frac{1}{x^{5/6}}];$$

$$(ii) \int_{1/2}^1 \frac{dx}{x^{1/2}(1-x)^{1/3}} \quad [\text{Hints: } g(x) = \frac{1}{(1-x)^{1/3}}];$$

$$(iii) \int_a^{\infty} \frac{x^4}{(x^2+a^2)^{5/2}} \, dx, \quad [\text{Hints: } g(x) = \frac{1}{x}];$$

$$(iv) \int_0^{\pi/2} \frac{x^2}{(\sin x)^3} \, dx, \quad [\text{Hints: } g(x) = \frac{1}{x}];$$

(v) $\int_0^2 \frac{\log(1 + \sqrt[5]{x^3})}{e^{\sin x} - 1} dx$, [Hints : $g(x) = \frac{x^{3/5}}{x} = \frac{1}{x^{2/5}}$ from lowest power of the series, $\lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow 0} \frac{\log(1 + x^{3/5})}{x^{3/5}} \cdot \frac{x}{e^{\sin x} - 1} = 1$];

(vi) $\int_0^1 \frac{dx}{x - \sin x}$,

[Hints : $g(x) = \frac{6}{x^3}$, $\lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow 0} \frac{1}{x - \sin x} \cdot \frac{x^3}{6} = 1$ using L'Hospital's rule]

(vii) $\int_0^{\pi/4} \frac{x^3 dx}{\sin(x^2)(\tan x)^3}$, [Hints : $g(x) = \frac{1}{x^2}$],

(viii) $\int_0^1 \left(\frac{1}{x} - x\right)^{1/2} \frac{\log(1 + x^{1/3})}{\sin x} dx$, [Hints : $g(x) = \frac{1}{x^{7/6}}$];

(ix) $\int_1^\infty \frac{1 - 4 \sin 2x}{x^3 + \sqrt[3]{x}} dx$, [Hints : integrand changes sign. So, test

$$\int_1^\infty \frac{|1 - 4 \sin 2x|}{x^3 + \sqrt[3]{x}} dx, \quad g(x) = \frac{5}{x^3}]$$

(x) $\int_0^\infty e^{-x^2} dx$, [Hints : $e^{-x^2} < \frac{1}{x^2}$. So, $g(x) = \frac{1}{x^2}$].

13. (i) From $\int_0^\infty e^{-\alpha x} dx = \frac{1}{\alpha}$ obtain by differentiation

$$\int_0^\infty x^n e^{-\alpha x} dx = \frac{n!}{\alpha^{n+1}}, \quad \alpha \neq 0.$$

(ii) Let $F(x) = \int_0^\infty e^{-t^2} \cos xt \, dt$. Show that $F'(x) = -\frac{1}{2} x F(x)$. Then

find $F(x)$.

(iii) Prove that $\int_0^\infty \frac{1 - e^{-xt^2}}{t^2} dt = \sqrt{\pi x}$, $x \geq 0$.

[Hints : differentiate under the integral sign.]

14. Prove that

$$(i) \int_0^{\infty} \frac{\sin^3 x}{x} dx = \frac{\pi}{4}, \quad (ii) \int_0^{\infty} \frac{\sin^5 x}{x} dx = \frac{3}{16} \pi.$$

[Hints. Express $\sin^n x$ in multiple angles for $n = 2, 3, 4 \dots$ etc. and use

the result $\int_0^{\infty} \frac{\sin \beta x}{x} dx = \frac{\pi}{4}, \beta > 0$ from w.o. example 14.]

15. Prove that (i) $\int_0^{\infty} \frac{\sin^2 mx}{x^2} dx = \frac{\pi}{2} m$ or $-\frac{\pi}{2} m$ according as $m > 0$ or

$m < 0$.

$$(ii) \int_0^{\infty} \left(\frac{\sin x}{x} \right)^2 dx = \frac{\pi}{2}.$$

$$(iii) \int_0^{\infty} \left(\frac{\sin x}{x} \right)^3 dx = \frac{3\pi}{8}.$$

[Hints. For $\int_0^{\infty} \frac{\sin^n x}{x^n} dx, n > 1$ integer, integrate by parts taking $\frac{1}{x^n}$ as

second function. The integrals are improper integrals of the third kind. So,

write each of them as $\int_0^{\infty} \frac{\sin^n x}{x^n} dx = \int_0^1 \frac{\sin^n x}{x^n} dx + \int_1^{\infty} \frac{\sin^n x}{x^n} dx$

$= \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^1 \frac{\sin^n x}{x^n} dx + \lim_{b \rightarrow \infty} \int_b^1 \frac{\sin^n x}{x^n} dx$. Use L' Hospital's rules to

evaluate $\lim_{\varepsilon \rightarrow 0} \frac{\sin^n x}{x^{n-1}}$ which is always zero and $\lim_{b \rightarrow \infty} \frac{\sin^n b}{b^{n-1}} = 0$ as $\sin b$ is

always finite. Also, use $\int_0^{\infty} \frac{\sin \beta x}{x} dx = \frac{\pi}{2}$ if $\beta > 0$ from w.o. example 14]

16. (i) Find $\int_0^{\infty} \frac{dx}{(x^2 + \lambda)^{n+1}}, n > 0$ integer and $\lambda > 0$

[Hints. Differentiate $\int_0^{\infty} \frac{dx}{n^2 + \lambda}$ n times]

(ii) Find $I(\lambda) = \int_0^{\infty} e^{-x^2 - \lambda^2/x^2} dx$

[Hints: Find $\frac{dI}{d\lambda}$ and integrate $\frac{dI}{d\lambda}$. Note that $I(0) = \sqrt{\pi}/2$]

Answers

1. (i) Third Kind; (ii) Second Kind; (iii) Second Kind; (iv) Second Kind; (v) First Kind; (vi) First Kind.

2. (a) (i) 1, conv., (ii) $3\pi^2/32$, conv., (iii) No finite value, divergent.

3. (i) $\frac{3}{2}(\sqrt[3]{4} - 1)$, conv., (ii) $\frac{9}{2}$, conv., (iii) $2\sqrt{2}$, conv.

5. (i) No finite value, Div.; (ii) $\pi \log \frac{1}{2}$, conv.; (iii) No fixed value, divergent;
(iv) $-\frac{\pi^2}{12}$, conv.; (v) $\frac{\pi}{2} \log 2$, conv.; (vi) $\pi \log \frac{1}{2}$, conv.

8. All Conv. (i) $\frac{\pi}{2}$ (ii) $\frac{1 \cdot 3 \cdot 5 \dots (2n-3)}{2 \cdot 4 \cdot 6 \dots (2n-2)} \cdot \frac{\pi}{2}$; (iii) $\frac{2 \cdot 4 \cdot 6 \dots (2n-2)}{3 \cdot 5 \cdot 7 \dots (2n-1)}$,
(iv) $2\pi a^2$; (v) $3\pi a^2$; (vi) $\frac{\pi}{2} \left\{ 1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 k^4 + \dots \right\}$; (vii) $\frac{\pi}{2}$.

10. (i) No definite value, div.; (ii) $\frac{\pi^2}{8}$, conv.; (iii) 2, conv.; (iv) No definite value, div.;
(v) No definite value, div.; (vi) $-\frac{1}{4}$, conv. (vii) π , conv.

12. (i) div., (ii) conv., (iii) div., (iv) div., (v) conv., (vi) div., (vii) div., (viii) div.,
(ix) absolutely conv., (x) conv.

13. (ii) $F(x) = \frac{\sqrt{\pi}}{2} e^{-\frac{1}{4}x^2}$.

16. (i) $\frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2 \cdot 4 \cdot 6 \dots (2n)} \cdot \frac{\pi}{2^n \sqrt{\lambda}}$

(ii) $I = \frac{\sqrt{\pi}}{2} e^{-2\lambda}$

GAMMA AND BETA FUNCTIONS

10.1. INTRODUCTION

Gamma and beta are two of the many special functions (higher transcendental functions or functions of mathematical physics). Euler in 1729 and 1730 answered the questions raised by Starling, Goldberg and Deniel Burnoulli about the value of $\left(4\frac{1}{2}\right)!$ when $4! = 24$ and the like. The first answer was in the form of infinite products and the second was in integral form. These answers are the now a day's gamma and beta functions with the notations introduced by Legendre. The integral forms of those answers are known as the first and the second integral of Euler. The first integral is the beta function and the second integral is the gamma function. During 19th century gamma function finds its way upto complex variables. It is normally studied as a part of the theory of complex variables and is very important in the mathematical analysis of various branches. Their definitions, properties and relationships will be discussed in this chapter.

10.2. GAMMA FUNCTION

The following are the definitions of gamma functions :

- (i) The convergent improper integral $\int_0^{\infty} e^{-x} x^{n-1} dx, n > 0$,

known as the second Eulerian integral, is called the gamma function of n and is expressed as

$$\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx, n > 0.$$

- (ii) Weierstrass defined it as an infinite product by $\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{n}\right) e^{-z/n} \right]$ for non-zero and non-negative complex number z and γ is an Euler's constant.

- (iii) Euler also gave two other definitions under the same conditions as Weierstrass. They are

$$\Gamma(z) = \frac{1}{z} \prod_{n=1}^{\infty} \left[\left(1 + \frac{1}{n}\right)^z \left(1 + \frac{z}{n}\right)^{-1} \right]$$

and $\Gamma z = \lim_{n \rightarrow \infty} P_n(z) = \lim_{n \rightarrow \infty} \frac{n! n^z}{z(z+1) \dots (z+n)}$ respectively.

Of these four definitions, first one will be used here for further discussions.

10.2.1. DIFFERENT FORMS OF Γn

The integral definition (i) of 10.2 can be expressed in different forms by change of variables.

(i) Let us substitute $x = ay$ in $\Gamma n = \int_0^\infty x^{n-1} e^{-x} dx$. So,
 $dx = a dy$ and

x	0	∞
y	0	∞

, then

$$\Gamma n = a^n \int_0^\infty y^{n-1} e^{-ay} dy = a^n \int_0^\infty x^{n-1} e^{-ax} dx$$

or, $\frac{\Gamma n}{a^n} = \int_0^\infty x^{n-1} e^{-ax} dx.$

(ii) Substituting $x = y^2$, we get $dx = 2y dy$ and

x	0	∞
y	0	∞

$$\therefore \Gamma n = 2 \int_0^\infty y^{2n-1} e^{-y^2} dy = 2 \int_0^\infty x^{2n-1} e^{-x^2} dx.$$

(iii) Lastly on the substitution $x = -(m+1) \log y$,

$$dx = -(m+1) \frac{1}{y} dy \quad \text{and} \quad \tableborder{1}{x}{y}{0}{1}{\infty}{\infty} \quad \text{So,}$$

$$\Gamma n = (m+1)^n \int_0^1 y^n \left(\log \frac{1}{y} \right)^{n-1} dy = (m+1)^n \int_0^1 x^n \left(\log \frac{1}{x} \right)^{n-1} dx.$$

10.2.2. PROPERTIES OF GAMMA FUNCTION

The following properties of gamma function, Γn will be proved and discussed from the integral definition in subsequent sections :

- (i) $\Gamma 1 = 1$;
- (ii) $\Gamma(n+1) = n \Gamma n$;
- (iii) $\Gamma(n+1) = n!$;

$$(iv) \Gamma(n) \Gamma(1-n) = \frac{\pi}{\sin n\pi}, \quad 0 < n < 1;$$

$$(v) 2^{2n-1} \Gamma(n) \Gamma(n + \frac{1}{2}) = \pi \Gamma(2n);$$

$$(vi) \Gamma(1/2) = \sqrt{\pi}.$$

In general Γn cannot be calculated by integration. Only for few specific cases (say $n = 1$) this integral can be evaluated. In all other cases the above properties and the following table are used to determine Γn .

The Value of the Gamma-Function Γn (for $1 \leq n \leq 2$)

n	$\Gamma(n)$	n	$\Gamma(n)$	n	$\Gamma(n)$	n	$\Gamma(n)$
1.00	1.0000	1.25	0.9064	1.50	0.8862	1.75	0.9191
1.01	0.9943	1.26	9044	1.51	8866	1.76	9214
1.02	9888	1.27	9025	1.52	8870	1.77	9238
1.03	9835	1.28	9007	1.53	8876	1.78	9262
1.04	9784	1.29	8990	1.54	8882	1.79	9288
1.05	9735	1.30	8975	1.55	8889	1.80	9314
1.06	9687	1.31	8960	1.56	8896	1.81	9341
1.07	9642	1.32	8946	1.57	8905	1.82	9368
1.08	9597	1.33	8934	1.58	8914	1.83	9397
1.09	9555	1.34	8922	1.59	8924	1.84	9426
1.10	9514	1.35	8912	1.60	8935	1.85	9456
1.11	9474	1.36	8902	1.61	8947	1.86	9487
1.12	9436	1.37	8893	1.62	8959	1.87	9518
1.13	9399	1.38	8885	1.63	8972	1.88	9551
1.14	9364	1.39	8879	1.64	8986	1.89	9584
1.15	9330	1.40	8873	1.65	9001	1.90	9618
1.16	9298	1.41	8868	1.66	9017	1.91	9652
1.17	9267	1.42	8864	1.67	9033	1.92	9688
1.18	9237	1.43	8860	1.68	9050	1.93	9724
1.19	9209	1.44	8858	1.69	9068	1.94	9761
1.20	9182	1.45	8857	1.70	9086	1.95	9799
1.21	9156	1.46	8856	1.71	9106	1.96	9837
1.22	9131	1.47	8856	1.72	9126	1.97	9877
1.23	9108	1.48	8857	1.73	9147	1.98	9917
1.24	9085	1.49	8859	1.74	9168	1.99	9958
						2.00	1.0000

10.2.3. CONVERGENCE OF GAMMA FUNCTION

From 10.2 (i) the gamma function is

$$\Gamma n = \int_0^{\infty} x^{n-1} e^{-x} dx.$$

when $n \geq 1$, the integrand is continuous and the integral is then the improper integral of the first kind and this can be expressed as the sum of two integrals as

$$\Gamma n = \int_0^1 x^{n-1} e^{-x} dx + \int_1^\infty x^{n-1} e^{-x} dx \quad (1)$$

Of the two integrals on the right of (1), first one is a definite integral and is convergent. Let us test the second one for convergence at ∞ . Let

$$f(x) = x^{n-1} e^{-x} \text{ and } g(x) = \frac{1}{x^2}. \text{ So, } \int_1^\infty g(x) dx = \int_1^\infty \frac{1}{x^2} dx =$$

and it is convergent. Also,

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow \infty} \frac{x^{n-1} e^{-x}}{\frac{1}{x^2}} = \lim_{n \rightarrow \infty} x^{n+1} e^{-x} \\ &= \lim_{x \rightarrow \infty} \frac{x^{n+1}}{e^x} = \frac{(n+1)!}{\infty} = 0, \text{ (L'Hospital's rule)} \end{aligned}$$

$$\therefore \int_1^\infty f(x) dx \text{ is convergent by TF}_2 \text{ of 9.7.1.}$$

$\therefore \Gamma n$ is convergent when $n \geq 1$.

Again the integrand of Γn is discontinuous for $n < 1$ at $x = 0$ and the integral becomes a mixed kind of improper integral. In that case Γn can also be written as (1). Of these the second integral is an improper integral of the first kind and is convergent as shown just before. The first integral becomes an improper integral of the second kind. Let us test it for convergence.

$$\text{Let } f(x) = x^{n-1} e^{-x} = \frac{e^{-x}}{x^{1-n}} \text{ and } g(x) = \frac{1}{x^{1-n}}.$$

$$\text{So, } \int_0^1 g(x) dx = \int_0^1 \frac{1}{x^{1-n}} dx \text{ is convergent when}$$

$$0 < 1 - n < 1 \text{ i.e. } n > 0 \text{ and } \lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow 0} \frac{e^{-x}}{x^{1-n}} \times \frac{x^{1-n}}{1} = 1 =$$

exists. Therefore $\int_0^1 x^{n-1} e^{-x} dx$, $n > 0$ is convergent by

TS₃ of 9.7.2. Hence Γn is convergent for $n > 0$.

10.2.4. PROOF OF $\Gamma 1 = 1$

In definition (i) sec. 10.2, let $n = 1$. So,

$$\begin{aligned}\Gamma 1 &= \int_0^{\infty} e^{-x} dx = \lim_{b \rightarrow \infty} \int_0^b e^{-x} dx = \lim_{b \rightarrow \infty} [-e^{-x}]_0^b = \lim_{b \rightarrow \infty} (1 - e^{-b}) \\ &= 1, \quad \because e^{-\infty} = 0.\end{aligned}$$

10.2.5. PROOF OF $\Gamma(n+1) = n \Gamma n$.

Using $\Gamma(n+1)$ for n in definition (i) sec. 10.2, we have,

$$\begin{aligned}\Gamma(n+1) &= \int_0^{\infty} x^n e^{-x} dx = \lim_{b \rightarrow \infty} \int_0^b x^n e^{-x} dx \\ &= \lim_{b \rightarrow \infty} \left\{ \left[-x^n e^{-x} \right]_0^b + \int_0^b n x^{n-1} e^{-x} dx \right\}, \text{ I. b. p.} \\ &= - \lim_{b \rightarrow \infty} b^n e^{-b} + n \int_0^{\infty} x^{n-1} e^{-x} dx,\end{aligned}$$

[the first limit is in determinate form.]

$$\begin{aligned}&= - \lim_{b \rightarrow \infty} \frac{b^n}{e^b} + n \Gamma n, \text{ by def. (i) sec. 10.2} \\ &= - \lim_{b \rightarrow \infty} \frac{n!}{e^b} + n \Gamma n, \text{ using, L'Hospital's rule } n \text{ times} \\ &= n \Gamma n\end{aligned} \tag{1}$$

Cor. (i). The p th application of the formula (1) can be written as

$$\Gamma(n+p) = (n+p-1)(n+p-2) \dots (n+1)n \Gamma n \tag{2}$$

Cor. (ii). Gamma function for -ve arguments, n : From relation (1),

$$\Gamma n = \frac{\Gamma(n+1)}{n} \tag{3}$$

Using $n = 0, -1, -2, \dots$ etc. in (3), we have

$$\Gamma 0 = \frac{\Gamma 1}{0} = \infty; \Gamma(-1) = \frac{\Gamma 0}{-1} = -\infty; \Gamma(-2) = \frac{\Gamma(-1)}{-2} = \infty \text{ etc.}$$

It is seen that for odd and even -ve integer, Γn is $-\infty$ and $+\infty$ respectively. Therefore, Γn is undefined for zero or -ve integers.

The truth of these statements can be embodied into a single formula. Using $(-n)$ for n in (3), we have

$$\begin{aligned}\Gamma(-n) &= \frac{\Gamma(-n+1)}{-n} = \frac{\Gamma(-n+2)}{n(n-1)} = - \frac{\Gamma(-n+3)}{n(n-1)(n-2)} \\ &\dots = (-1)^n \frac{\Gamma 0}{n!} = (-1)^n \cdot \infty\end{aligned}$$

i.e. $\Gamma(-n) = (-1)^n \cdot \infty, n = 1, 2, 3, \dots$ (4)

But the definition of Γn can be extended to negative fractions. Because from (3) it follows that $\Gamma(n+1)$ is meaningful for

$-1 < n < 0$ i.e. $\Gamma n = \frac{\Gamma(n+1)}{n}, -1 < n < 0$, is defined and meaningful. In general (2) is defined and meaningful for $-p < n < -(p-1)$ i.e.

$$\Gamma n = \frac{\Gamma(n+p)}{n(n+1)(n+2) \dots (n+p-2)(n+p-1)}, -p < n < -(p-1), \quad (5)$$

when p is any positive integer.

Let $n+p=q$. So, $n=q-p$ and the above formula becomes,

$$\Gamma n = \Gamma(q-p) = \frac{(-1)^p \Gamma q}{(1-q)(2-q) \dots (p-q)}, -p < n < (p-1) \quad (6)$$

which governs the sign of Γn when n is a negative fraction and $p > 0$ integer.

For example, let $p=1, n=-\frac{1}{2}$. So $q=n+p=1-\frac{1}{2}=\frac{1}{2}$. Therefore from (6)

$$\Gamma(-1/2) = \frac{(-1)^1 \Gamma(1/2)}{1 - (1/2)} = -2 \Gamma(1/2) = -2\sqrt{\pi} \quad [\text{Ref. sec. 10.2.9.}]$$

Cor. (iii) From relation (1),

$$\begin{aligned} \Gamma(n+1) &= n \Gamma((n-1)+1) = n(n-1) \Gamma(n-1) = n(n-1)(n-2) \Gamma(n-2) \\ &= n(n-1)(n-2) \dots 3 \cdot 2 \cdot 1 \cdot \Gamma 1, \text{ if } n \text{ is a +ve integer} \\ &= n(n-1)(n-2) \dots 3 \cdot 2 \cdot 1, \text{ by sec. 10.2.3} \\ &= n!, \text{ if } n \text{ is a +ve integer.} \end{aligned} \quad (7)$$

This is true when $p=1$ in (2). Cor. (i) now permits to evaluate the factorial of +ve, -ve integer or fraction using (7). Probs. 28 for examples.

10.2.6. CONTINUITY OF Γn

$$\begin{aligned} \text{Let } \Gamma n &= \int_0^\infty x^{n-1} e^{-x} dx \\ &= \int_0^1 x^{n-1} e^{-x} dx + \int_1^\infty x^{n-1} e^{-x} dx \end{aligned} \quad (1)$$

To test the continuity of Γn , let $a \leq n \leq b$ such that $0 < a < b$. Now if $x \geq 1$ and $a \leq n \leq b$ for a and b finite then

$0 < x^{n-1} e^{-x} \leq x^{b-1} e^{-x}$. So, $\int_1^\infty x^{b-1} e^{-x} dx$ is convergent and

hence $\int_1^{\infty} x^{n-1} e^{-x} dx$ is uniformly convergent by sec. 10.2.3.

Therefore, its integrand must be continuous. Again if $n > 1$ then the first integral of (1) is a definite integral and is also a continuous function of n . If the first integral is an improper one in that case also

$\int_0^1 x^{a-1} e^{-x} dx$ is convergent, because $0 < x^{n-1} e^{-x} \leq x^{a-1} e^{-x}$

when $0 < x \leq 1$ and $0 < a \leq n \leq b$. Therefore, for $n > 0$, Γn is a continuous function being a sum of two continuous functions.

10.2.7. SUCCESSIVE DERIVATIVES OF Γn

The function is continuous and its associated improper integral is uniformly convergent. So, it can be differentiated under the integral sign w.r.t. the parameter, n . Since $\frac{d}{dx}(a^x) = a^x \log a$, therefore,

$$\Gamma'(n) = \int_0^{\infty} \frac{\partial}{\partial n} (x^{n-1} e^{-x}) dx = \int_0^{\infty} x^{n-1} e^{-x} (\log x) dx$$

$$\Gamma''(n) = \int_0^{\infty} x^{n-1} e^{-x} (\log x)^2 dx$$

$$\Gamma'''(n) = \int_0^{\infty} x^{n-1} e^{-x} (\log x)^3 dx$$

$$\vdots \quad \quad \quad \vdots$$

$$\Gamma^{(n)}(n) = \int_0^{\infty} x^{n-1} e^{-x} (\log x)^n dx.$$

It is to be noted that the improper integrals associated with all the successive derivatives are uniformly convergent.

10.2.8. GRAPH OF Γn .

From sec. 10.2.7 it is clear that $\Gamma''(n) > 0$. So, the curve $y = \Gamma n$ for all $n > 0$ is concave upwards. Again

$$\begin{aligned} \Gamma'(n) &= \int_0^1 x^{n-1} e^{-x} (\log x) dx + \int_1^{\infty} x^{n-1} e^{-x} (\log x) dx \\ &= \int_1^{\infty} x^{n-1} e^{-x} (\log x) dx - \int_0^1 x^{n-1} e^{-x} \log \left(\frac{1}{x} \right) dx \end{aligned}$$

$$= p(n) - q(n), \text{ (say).}$$

Here the function $p(n)$ is a monotonic increasing function of n whereas $q(n)$ is a monotonic decreasing one. Hence $\Gamma'(n)$ increases from $(-\infty)$ to ∞ as n increases from 0 to ∞ and becomes zero only once. It changes sign from +ve to -ve when n increases from smaller to bigger values. So, Γn has a minimum value. As $\Gamma 1 = \Gamma 2 = 1$, the minimum value lies between 1 and 2.

When $n < 0$ then Γn has more than one -ve maximum value and they lie in $(0, -1)$, $(-2, -3)$, $(-4, -5)$ etc. intervals. In similar cases, its +ve minimum values lie in the intervals $(-1, -2)$, $(-3, -4)$, $(-5, -6)$ etc. Under these discussions the graph of Γn looks like the following (fig. 9) :

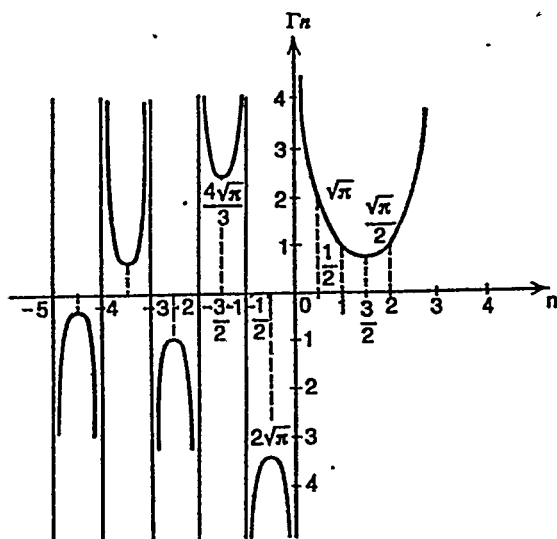


Fig. 9

10.2.9. EVALUATION OF $\Gamma(1/2)$

In 10.2.1(ii) let us use $n = \frac{1}{2}$. Therefore, $\Gamma(1/2) = 2 \int_0^{\infty} e^{-x^2} dx$.

But this integral cannot be evaluated by normal procedures of substitution etc. A multiple integral is constructed and transferred into polar co-ordinates for evaluation as follows :

$$\text{Let } I = \int_0^{\infty} e^{-x^2} dx = \int_0^{\infty} e^{-y^2} dy$$

$$\therefore I^2 = \int_0^\infty e^{-x^2} dx \cdot \int_0^\infty e^{-y^2} dy = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy \quad (1)$$

Let $x = r \cos \theta$, $y = r \sin \theta$. Therefore, from ch. 15, $dx dy = r dr d\theta$. As x and y varies from 0 to ∞ then r varies from 0 to ∞ and θ varies from 0 to $\frac{\pi}{2}$. So, (1) becomes,

$$\begin{aligned} I^2 &= \int_0^{\pi/2} \int_0^\infty e^{-r^2} r dr d\theta = \int_0^{\pi/2} \left[-\frac{1}{2} e^{-r^2} \right]_0^\infty d\theta \\ &= \frac{1}{2} \int_0^{\pi/2} d\theta = \frac{\pi}{4} \end{aligned}$$

$$\therefore I = \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

$$\therefore \Gamma(1/2) = 2 \cdot \frac{\sqrt{\pi}}{2} = \sqrt{\pi} \quad (2)$$

Cor. From 10.2.2 (ii), $\Gamma n = \frac{\Gamma(n+1)}{n}$

$$\therefore \Gamma(-1/2) = \frac{\Gamma(1/2)}{-1/2} = -2\sqrt{\pi} \quad \text{by (2)}$$

$$\text{Also } \Gamma(-3/2) = \frac{\Gamma(-3/2+1)}{-3/2} = -\frac{2}{3} \Gamma(-1/2) = \frac{4}{3}\sqrt{\pi}$$

In this way $\Gamma(-5/2)$, $\Gamma(-7/2)$, etc. can be determined.

10.2.10. THE INTEGRAL FORM OF $\log \Gamma n$

The integral form of $\log \Gamma n$ is useful for further study of Γ -function. For this purpose $\log x$ is first expressed as an integral as follows :

It is known that $\int_0^\infty e^{-b\alpha} d\alpha = \frac{1}{b}$

Integrating w.r.t. 'b' from 1 to x, we get,

$$\int_1^x \left(\int_0^\infty e^{-b\alpha} d\alpha \right) db = \int_1^x \frac{1}{b} db = \log x$$

$$\therefore \log x = \int_0^\infty \left(\int_1^x e^{-b\alpha} db \right) d\alpha ; \text{ [by interchanging the order of integration, (for condition ref. to ch. 16)]}$$

$$\begin{aligned}
 &= \int_0^{\infty} \left[-\frac{e^{-b\alpha}}{\alpha} \right]_{b=1}^x d\alpha \\
 &= \int_0^{\infty} \frac{e^{-\alpha} - e^{-\alpha x}}{\alpha} d\alpha \quad (1)
 \end{aligned}$$

This is the *integral* form of $\log x$ and the integral on the right is known as the *Frullani integral*. [The general Frullani integral is

$$\int_0^{\infty} [f(ax) - f(bx)] \frac{dx}{x} = f(0+) \log \frac{b}{a}. \quad \text{If } f(ax) = e^{-ax} \text{ and } f(bx) = e^{-bx} \text{ with } a=1, b=x, x=\alpha \text{ are used then (1) can be found.}]$$

$$\text{Again } \Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x} dx \quad (2)$$

Diff. (2) w.r.t. n , we have,

$$\begin{aligned}
 \Gamma'(n) &= \int_0^{\infty} x^{n-1} e^{-x} \log x dx \\
 &= \int_0^{\infty} x^{n-1} e^{-x} \left(\int_0^{\infty} \frac{e^{-\alpha} - e^{-\alpha x}}{\alpha} d\alpha \right) dx, \text{ by (1)} \\
 &= \int_0^{\infty} \frac{e^{-\alpha}}{\alpha} \left(\int_0^{\infty} x^{n-1} e^{-x} dx \right) d\alpha \\
 &\quad - \int_0^{\infty} \left(\int_0^{\infty} x^{n-1} e^{-(\alpha+1)x} dx \right) \frac{d\alpha}{\alpha}, \\
 &\quad \text{[interchanging the order of integration]}
 \end{aligned}$$

$$= \int_0^{\infty} \frac{e^{-\alpha}}{\alpha} \Gamma(n) d\alpha - \int_0^{\infty} \frac{\Gamma(n)}{\alpha (1+\alpha)^n} d\alpha, \text{ by 10.2.1 (i)}$$

$$= \Gamma(n) \int_0^{\infty} \left(e^{-\alpha} - \frac{1}{(1+\alpha)^n} \right) \frac{d\alpha}{\alpha} \quad (3)$$

$$\text{or, } \frac{\Gamma'(n)}{\Gamma(n)} = \int_0^{\infty} \frac{e^{-\alpha}}{\alpha} d\alpha - \int_0^{\infty} \frac{1}{\alpha (1+\alpha)^n} d\alpha$$

Substituting $1+\alpha = e^t$ in the 2nd integral on the right, we get,

$$\begin{aligned}
 \int_0^{\infty} \frac{1}{(1+\alpha)^n} \frac{d\alpha}{\alpha} &= \int_0^{\infty} \frac{1}{e^{nt}} \cdot \frac{e^t}{e^t - 1} dt \\
 &= \int_0^{\infty} \frac{e^{-nt}}{1 - e^{-t}} dt = \int_0^{\infty} \frac{\alpha e^{-n\alpha}}{1 - e^{-\alpha}} \cdot \frac{d\alpha}{\alpha} \\
 \therefore \frac{\Gamma(n)}{\Gamma(n)} &= \int_0^{\infty} \frac{e^{-\alpha}}{\alpha} d\alpha - \int_0^{\infty} \frac{\alpha e^{-n\alpha}}{1 - e^{-\alpha}} \cdot \frac{d\alpha}{\alpha} \\
 &= \int_0^{\infty} \left(e^{-\alpha} - \frac{\alpha e^{-n\alpha}}{1 - e^{-\alpha}} \right) \frac{d\alpha}{\alpha}
 \end{aligned}$$

Integrating w.r.t. 'n',

$$\log \Gamma n = \int_0^{\infty} \left(n e^{-\alpha} + \frac{e^{-n\alpha}}{1 - e^{-\alpha}} \right) \frac{d\alpha}{\alpha} + C$$

$$\text{For } n = 1, \quad C = - \int_0^{\infty} \left(e^{-\alpha} + \frac{e^{-\alpha}}{1 - e^{-\alpha}} \right) \frac{d\alpha}{\alpha}$$

$$\therefore \log \Gamma n = \int_0^{\infty} \left[(n-1) e^{-\alpha} - \frac{e^{-\alpha} - e^{-n\alpha}}{1 - e^{-\alpha}} \right] \frac{d\alpha}{\alpha} \quad (4)$$

This is the integral form of $\log \Gamma n$.

Cor. (i) $\log \Gamma n$ is the sum of two integrals as $\log \Gamma n = P(n) + \mu(n)$

where

$$P(n) = \int_0^{\infty} \left[(n-1) e^{-\alpha} - \frac{e^{-\alpha}}{1 - e^{-\alpha}} + \left(\frac{1}{\alpha} + \frac{1}{2} \right) e^{-n\alpha} \right] \frac{d\alpha}{\alpha} \quad (5)$$

and

$$\mu(n) = \int_0^{\infty} \left[\frac{1}{1 - e^{-\alpha}} - \frac{1}{\alpha} - \frac{1}{2} \right] \frac{e^{-n\alpha}}{\alpha} d\alpha \quad (6)$$

$$\text{Cor. (ii). From (5), } P(1) = \int_0^{\infty} \left[\left(\frac{1}{\alpha} + \frac{1}{2} \right) e^{-\alpha} - \frac{1}{e^{\alpha} - 1} \right] \frac{d\alpha}{\alpha} \quad (7)$$

Cor. (iii). Using 2α for α in (5) in Cor. (i), and $n = 1$, we get,

$$P(1) = \int_0^{\infty} \left[\frac{1}{2} \left(\frac{1}{\alpha} + 1 \right) e^{-2\alpha} - \frac{1}{e^{2\alpha} - 1} \right] \frac{d\alpha}{\alpha} \quad (8)$$

$$\therefore 2P(1) = \int_0^{\infty} \left[\left(\frac{1}{\alpha} + 1 \right) e^{-2\alpha} - \frac{2}{e^{2\alpha} - 1} \right] \frac{d\alpha}{\alpha} \quad (9)$$

Subtracting (7) from (9),

$$\begin{aligned}
 P(1) &= \int_0^\infty \left[\left(\frac{1}{\alpha} + 1 \right) e^{-2\alpha} - \frac{2}{e^{2\alpha} - 1} - \left(\frac{1}{\alpha} + \frac{1}{2} \right) e^{-\alpha} + \frac{1}{e^{\alpha} - 1} \right] \frac{d\alpha}{\alpha} \\
 &= \int_0^\infty \left[\frac{e^{-2\alpha} - e^{-\alpha}}{\alpha} + e^{-2\alpha} - \frac{1}{2} e^{-\alpha} + \frac{1}{e^{\alpha} + 1} \right] \frac{d\alpha}{\alpha} \\
 &\quad \because \frac{1}{e^{\alpha} - 1} - \frac{2}{e^{2\alpha} - 1} = \frac{1}{e^{\alpha} + 1} \\
 &= \int_0^\infty \left(\frac{1}{e^{\alpha} + 1} - \frac{1}{2} e^{-2\alpha} \right) \frac{d\alpha}{\alpha} \\
 &\quad + \int_0^\infty \left(\frac{e^{-2\alpha} - e^{-\alpha}}{\alpha} + \frac{3}{2} e^{-2\alpha} - \frac{1}{2} e^{-\alpha} \right) \frac{d\alpha}{\alpha} \\
 &= \int_0^\infty \left(\frac{1}{e^{\alpha} + 1} - \frac{1}{2} e^{-2\alpha} \right) \frac{d\alpha}{\alpha} - 1 + \frac{1}{2} \log 2 \quad (10)
 \end{aligned}$$

[Ref. w.o. Ex. 9]

Again using $n = \frac{1}{2}$ in (4),

$$\begin{aligned}
 \log \Gamma(1/2) &= \int_0^\infty \left[-\frac{1}{2} e^{-\alpha} - \frac{e^{-\alpha} - e^{-\frac{1}{2}\alpha}}{1 - e^{-\alpha}} \right] \frac{d\alpha}{\alpha} \\
 \text{or } \log \sqrt{\pi} &= \int_0^\infty \left[\frac{e^{-\alpha} - e^{-2\alpha}}{1 - e^{-2\alpha}} - \frac{1}{2} e^{-2\alpha} \right] \frac{d\alpha}{\alpha}, \text{ using } 2\alpha \text{ for } \alpha. \\
 \text{or } \frac{1}{2} \log \pi &= \int_0^\infty \left[\frac{1}{e^{\alpha} + 1} - \frac{1}{2} e^{-2\alpha} \right] \frac{d\alpha}{\alpha} \quad (11) \\
 &\quad \because 1 - e^{-2\alpha} = (e^{\alpha} + 1)(e^{-\alpha} - e^{-2\alpha})
 \end{aligned}$$

$\therefore$ Using (11) in (10), $P(1) = \frac{1}{2} \log \pi - 1 + \frac{1}{2} \log 2 = -1 + \log \sqrt{2\pi}$.

Cor. (iv). In (7) using $(a\alpha)$ and $(b\alpha)$ for α ,

$$P(1) = \int_0^\infty \left[\left(\frac{1}{a\alpha} + \frac{1}{2} \right) e^{-a\alpha} - \frac{1}{e^{a\alpha} - 1} \right] \frac{d\alpha}{\alpha} \quad (12)$$

and

$$P(1) = \int_0^\infty \left[\left(\frac{1}{b\alpha} + \frac{1}{2} \right) e^{-b\alpha} - \frac{1}{e^{b\alpha} - 1} \right] \frac{d\alpha}{\alpha} \quad (13)$$

 $\therefore (13) \times b - (12) \times a$ gives

$$(b-a)P(1) = \int_0^\infty \left[\frac{a}{e^{a\alpha} - 1} - \frac{b}{e^{b\alpha} - 1} + \frac{1}{2} (be^{-b\alpha} - ae^{-a\alpha}) + \frac{e^{-b\alpha} - e^{-a\alpha}}{\alpha} \right] \frac{d\alpha}{\alpha} \quad (14)$$

when $a > 0, b > 0$.

$$\therefore (b-a)P(1) = \int_0^\infty \left[\frac{a}{e^{a\alpha} - 1} - \frac{b}{e^{b\alpha} - 1} + \frac{1}{2} (be^{-b\alpha} - ae^{-a\alpha}) \right] \frac{d\alpha}{\alpha} + \int_0^\infty \frac{e^{-b\alpha} - e^{-a\alpha}}{\alpha^2} d\alpha$$

$$\begin{aligned} \text{Now, } \int_0^\infty \frac{e^{-b\alpha} - e^{-a\alpha}}{\alpha^2} d\alpha &= \lim_{\lambda \rightarrow 0} \int_\lambda^\infty \frac{e^{-b\alpha} - e^{-a\alpha}}{\alpha^2} d\alpha \\ &= \lim_{\lambda \rightarrow 0} \left\{ \left[\frac{e^{-b\alpha} - e^{-a\alpha}}{-\alpha} \right]_\lambda^\infty - \int_\lambda^\infty \frac{-be^{-b\alpha} + ae^{-a\alpha}}{-\alpha} d\alpha \right\}, \text{ by I.b.p.} \\ &= \lim_{\lambda \rightarrow 0} \frac{e^{-b\lambda} - e^{-a\lambda}}{\lambda} + \int_0^\infty \frac{ae^{-a\alpha} - be^{-b\alpha}}{\alpha} d\alpha \\ &= \lim_{\lambda \rightarrow 0} \frac{-be^{-b\lambda} + ae^{-a\lambda}}{1} + \int_0^\infty \frac{ae^{-a\alpha} - be^{-b\alpha}}{\alpha} d\alpha, \text{ by L'Hospital's rule.} \\ &= -(b-a) + \int_0^\infty \frac{ae^{-a\alpha} - be^{-b\alpha}}{\alpha} d\alpha \end{aligned}$$

$$\therefore (b-a)P(1) = \int_0^\infty \left[\frac{a}{e^{a\alpha} - 1} - \frac{b}{e^{b\alpha} - 1} + \frac{1}{2} (ae^{-a\alpha} - be^{-b\alpha}) \right] \frac{d\alpha}{\alpha} - (b-a)$$

$$\text{or, } (b-a) \left[-1 + \log \sqrt{2\pi} \right] = D_0$$

$$\left[\because P(1) = -1 + \log \sqrt{2\pi} \quad \text{by Cor. (iii)} \right]$$

$$\text{or, } \int_0^\infty \left[\frac{a}{e^{a\alpha}-1} - \frac{b}{e^{b\alpha}-1} + \frac{1}{2} (ae^{-a\alpha} - be^{-b\alpha}) \right] \frac{d\alpha}{\alpha} \\ = (b-a) \log \sqrt{2\pi} \quad (15)$$

Cor. (v). Using $a = 1$, $b = 2$ in (15), so

$$\int_0^\infty \left[\frac{1}{e^\alpha-1} - \frac{2}{e^{2\alpha}-1} + \frac{1}{2} (e^{-\alpha} - 2e^{-2\alpha}) \right] \frac{d\alpha}{\alpha} = \log \sqrt{2\pi}$$

$$\text{or } \int_0^\infty \left[\frac{1}{e^\alpha-1} \left(1 - \frac{2}{e^\alpha+1} \right) + \frac{1}{2} (e^{-\alpha} - 2e^{-2\alpha}) \right] \frac{d\alpha}{\alpha} = \log \sqrt{2\pi}$$

$$\text{or } \int_0^\infty \left[\frac{1}{e^\alpha+1} + \frac{1}{2} (e^{-\alpha} - 2e^{-2\alpha}) \right] \frac{d\alpha}{\alpha} = \log \sqrt{2\pi} \quad (16)$$

10.2.11. GAUSS'S FORMULA FOR $\Gamma(mn)$

If m is a positive integer, then Gauss's formula for $\Gamma(mn)$ states that

$$\Gamma_n \left[\left(n + \frac{1}{m} \right) \right] \Gamma_n \left[\left(n + \frac{2}{m} \right) \right] \dots \Gamma_n \left[\left(n + \frac{m-1}{m} \right) \right] = \frac{(2\pi)^{(m-1)/2} \Gamma(mn)}{m^{mn-1/2}}$$

Proof: Let $P = \Gamma_n \left[\left(n + \frac{1}{m} \right) \right] \Gamma_n \left[\left(n + \frac{2}{m} \right) \right] \dots \Gamma_n \left[\left(n + \frac{m-1}{m} \right) \right]$

Taking log on both sides,

$$\log P = \log \Gamma_n + \log \Gamma_n \left[\left(n + \frac{1}{m} \right) \right] + \log \Gamma_n \left[\left(n + \frac{2}{m} \right) \right] + \dots + \log \Gamma_n \left[\left(n + \frac{m-1}{m} \right) \right]$$

$$= \int_0^\infty \left[(n-1) e^{-\alpha} - \frac{e^{-\alpha} - e^{-n\alpha}}{1 - e^{-\alpha}} \right] \frac{d\alpha}{\alpha} + \int_0^\infty \left[\left(n + \frac{1}{m} - 1 \right) e^{-\alpha} - \frac{e^{-\alpha} - e^{-\left(n + \frac{1}{m} \right) \alpha}}{1 - e^{-\alpha}} \right] \frac{d\alpha}{\alpha} +$$

$$\int_0^\infty \left[\left(n + \frac{2}{m} - 1 \right) e^{-\alpha} - \frac{e^{-\alpha} - e^{-\left(n + \frac{2}{m} \right) \alpha}}{1 - e^{-\alpha}} \right] \frac{d\alpha}{\alpha} +$$

$$\dots + \int_0^\infty \left[\left(n + \frac{m-1}{m} - 1 \right) e^{-\alpha} - \frac{e^{-\alpha} - e^{-\left(n + \frac{m-1}{m} \right) \alpha}}{1 - e^{-\alpha}} \right] \frac{d\alpha}{\alpha}, \text{ by (4) of 10.2.10.}$$

$$\begin{aligned}
&= \int_0^\infty \left[\left(mn - \frac{m+1}{2} \right) e^{-\alpha} - \frac{me^{-\alpha}}{1-e^{-\alpha}} + \frac{e^{-n\alpha} \frac{1-e^{-\alpha}}{1-e^{-\alpha/m}}}{1-e^{-\alpha}} \right] \frac{d\alpha}{\alpha} \\
&= \int_0^\infty \left[\left(mn - \frac{m+1}{2} \right) e^{-\alpha} - \frac{me^{-\alpha}}{1-e^{-\alpha}} + \frac{e^{-n\alpha}}{1-e^{-\alpha/m}} \right] \frac{d\alpha}{\alpha} \\
&= \int_0^\infty \left[\left(mn - \frac{m+1}{2} \right) e^{-m\alpha} - \frac{me^{-m\alpha}}{1-e^{-m\alpha}} + \frac{e^{-mn\alpha}}{1-e^{-\alpha}} \right] \frac{d\alpha}{\alpha}, \quad (1)
\end{aligned}$$

replacing α by $(m\alpha)$.

Again using mn for n in the eqn. (4) of 10.2.10,

$$\log \Gamma(mn) = \int_0^\infty \left[(mn-1) e^{-\alpha} - \frac{e^{-\alpha} - e^{-mn\alpha}}{1-e^{-\alpha}} \right] \frac{d\alpha}{\alpha} \quad (2)$$

$\therefore$ (1)-(2) gives,

$$\begin{aligned}
\log \left(\frac{P}{\Gamma(mn)} \right) &= \int_0^\infty \left[\left(mn - \frac{m+1}{2} \right) e^{-m\alpha} - \frac{me^{-m\alpha}}{1-e^{-m\alpha}} + \frac{e^{-mn\alpha}}{1-e^{-\alpha}} \right. \\
&\quad \left. - (mn-1) e^{-\alpha} + \frac{e^{-\alpha}}{1-e^{-\alpha}} - \frac{e^{-mn\alpha}}{1-e^{-\alpha}} \right] \frac{d\alpha}{\alpha} \\
&= \int_0^\infty \left[\frac{1}{e^\alpha - 1} - \frac{m}{e^{m\alpha} - 1} + \frac{1}{2} (e^{-\alpha} - m e^{-m\alpha}) \right] \frac{d\alpha}{\alpha} \\
&\quad + \left(\frac{1}{2} - mn \right) \int_0^\infty \frac{e^{-\alpha} - e^{-mn\alpha}}{\alpha} d\alpha \\
&= (m-1) \log \sqrt{2\pi} + \left(\frac{1}{2} - mn \right) \log m, \text{ [by eqn. 15 cor. (iv)} \\
&\quad \text{and eqn. (1) of 10.2.10].}
\end{aligned}$$

$$= \log \left\{ \left[(2\pi)^{(m-1)/2} \right] / m^{(mn - \frac{1}{2})} \right\}$$

$$\therefore \frac{P}{\Gamma(mn)} = (2\pi)^{\frac{m-1}{2}} / m^{(mn - \frac{1}{2})}$$

$$\text{or, } P = \frac{(2\pi)^{(m-1)/2} \Gamma(mn)}{m^{(mn - \frac{1}{2})}}.$$

Cor : This Gauss's formula enables to get the gamma of integral multiple of n such as $\Gamma(2n)$, $\Gamma(3n)$ etc. as follows :

Let $m = 2$, so

$$\Gamma(2n) = \Gamma(n) \Gamma\left(n + \frac{1}{2}\right) \frac{2^{(2n-1/2)}}{(2\pi)^{1/2}} = \Gamma(n) \Gamma\left(n + \frac{1}{2}\right) \frac{2^{2n-1}}{\sqrt{\pi}}$$

or, $\Gamma(2n) \sqrt{\pi} = \Gamma(n) \Gamma\left(n + \frac{1}{2}\right) 2^{2n-1}$. This is known as the **Duplicating formula**.

Let $m = 3$. So,

$$\Gamma(3n) = \Gamma(n) \Gamma\left(n + \frac{1}{3}\right) \Gamma\left(n + \frac{2}{3}\right) \frac{3^{3n-1/2}}{2\pi} = \frac{3^{3n-1/2}}{2\pi} \Gamma(n) \Gamma\left(n + \frac{1}{3}\right) \Gamma\left(n + \frac{2}{3}\right).$$

10.3. BETA FUNCTION

Definition : The definite integral $\int_0^1 x^{m-1} (1-x)^{n-1} dx$, known as first Eulerian integral, is called the Beta function of m and n and is denoted by $B(m, n)$ or $\beta(m, n)$ i.e.

$$B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx, \quad m > 0, n > 0.$$

10.3.1. A FUNDAMENTAL PROPERTY, $B(m, n) = B(n, m)$.

By def., $B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx, m > 0, n > 0.$

Using substitution, $1-x=y$, we have, $dx = -dy$ and

Therefore,

x	0	1
y	1	0

$$\begin{aligned} B(m, n) &= \int_0^1 (1-y)^{m-1} y^{n-1} dy \\ &= \int_0^1 y^{n-1} (1-y)^{m-1} dy = \int_0^1 x^{n-1} (1-x)^{m-1} dx \\ &= B(n, m) \end{aligned}$$

i.e. the interchange of the position of m and n does not change the value of beta function. This is the fundamental property of beta function. Also called the **symmetry** property of beta function.

10.3.2. DIFFERENT FORMS OF BETA FUNCTION

Different substitutions transform the form of the original integral. Some of them are as follows :

(i) Let $x = \frac{y}{1+y}$. Therefore, $dx = \frac{1}{(1+y)^2} dy$ and

x	0	1
y	0	∞

$\left[\because y = \frac{x}{1-x} \right]$. So,

$$\begin{aligned} \therefore B(m, n) &= \int_0^1 x^{m-1} (1-x)^{n-1} dx = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}} dy \\ &= \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx. \end{aligned}$$

(ii) Let $x = \sin^2 \theta$. So, $dx = 2 \sin \theta \cos \theta d\theta$ and

x	0	1
θ	0	$\pi/2$

$$\begin{aligned} \therefore B(m, n) &= 2 \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta \\ &= 2 \int_0^{\pi/2} \cos^{2m-1} \theta \sin^{2n-1} \theta d\theta. \end{aligned}$$

(iii) Let $x = \frac{y}{a}$. Then $dx = \frac{1}{a} dy$ and

x	0	1
y	0	a

So,

$$\begin{aligned} B(m, n) &= \frac{1}{a^{m+n-1}} \int_0^a y^{m-1} (a-y)^{n-1} dy \\ &= \frac{1}{a^{m+n-1}} \int_0^a x^{m-1} (a-x)^{n-1} dx. \end{aligned}$$

10.3.3. CONVERGENCE OF BETA FUNCTION

The integrand of beta function is continuous for both $m \geq 1$ and $n \geq 1$ and the integral is a definite integral. So, the beta function is convergent for both $m \geq 1$ and $n \geq 1$. But the integrand has s.p. at $x = 0$ and $x = 1$ for $m < 1, n \geq 1$ and $m \geq 1, n < 1$ respectively and in these cases the integral becomes an improper integral of the second kind. This is why, it is necessary to test the convergence in these two cases.

Case (i) : $m < 1, n \geq 1$

As $x = 0$ is a point of singularity of the integrand, the integral can be expressed into the sum of two integrals as

$$\begin{aligned}
 B(m, n) &= \int_0^1 x^{m-1} (1-x)^{n-1} dx \\
 &= \int_0^{1/2} x^{m-1} (1-x)^{n-1} dx + \int_{1/2}^1 x^{m-1} (1-x)^{n-1} dx
 \end{aligned}$$

The second integral on the right is a definite integral and is convergent automatically. The first integral is an improper integral of the second kind with singularity at $x = 0$. To test its convergence, let

$$f(x) = x^{m-1} (1-x)^{n-1} = \frac{(1-x)^{n-1}}{x^{1-m}} \quad \text{and} \quad g(x) = \frac{1}{x^{1-m}}. \quad \text{So,}$$

$$\lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow 0} (1-x)^{n-1} = 1 \quad \text{and}$$

$$\begin{aligned}
 \int_0^{1/2} g(x) dx &= \int_0^{1/2} \frac{1}{x^{1-m}} dx = \lim_{\delta \rightarrow 0} \int_{\delta}^{1/2} \frac{1}{x^{1-m}} dx \\
 &= \lim_{\delta \rightarrow 0} \left[\frac{1}{m} x^m \right]_{\delta}^{1/2} = \frac{1}{m 2^m}, \quad m > 0 \text{ is convergent.}
 \end{aligned}$$

$\therefore$ by TS_3 of 9.7.2, $\int_0^{1/2} x^{m-1} (1-x)^{n-1} dx$ for $m > 0$ is convergent. So, $B(m, n)$ is convergent for $m > 0$.

Case (ii). $m \geq 1, n < 1$.

As $x = 1$ is a point of singularity, the integral can be written as

$$\begin{aligned}
 B(m, n) &= \int_0^1 x^{m-1} (1-x)^{n-1} dx \\
 &= \int_0^{1/2} x^{m-1} (1-x)^{n-1} dx + \int_{1/2}^1 x^{m-1} (1-x)^{n-1} dx.
 \end{aligned}$$

Of the two integrals on the right, the first one is a definite one and is convergent. To test the convergence of the second integral, let

$$f(x) = x^{m-1} (1-x)^{n-1} = \frac{x^{m-1}}{(1-x)^{1-n}} \quad \text{and} \quad g(x) = \frac{1}{(1-x)^{1-n}} \quad \text{and it}$$

can be proved as in case (i) that the integral $\int_{1/2}^1 x^{m-1} (1-x)^{n-1} dx$ is

convergent for $n > 0$. Therefore $B(m, n)$ is convergent for $n > 0$ also. So, it is convergent for both $m > 0$ and $n > 0$.

10.4. RELATION BETWEEN GAMMA AND BETA FUNCTIONS.

Uptil now no ways of evaluating a beta function is shown. This is evaluated with the help of gamma function. The relationship of gamma and beta functions is derived below by two methods :

Method 1.

In this method multiple integral is used as 10.2.8 with polar co-ordinate transformation. From 10.2.1 (ii) $\Gamma(m+2) \int_0^\infty e^{-x^2} x^{2m-1} dx$

and $\Gamma(n+2) \int_0^\infty e^{-y^2} y^{2n-1} dy$. So, on multiplication,

$$\begin{aligned}\Gamma(m) \Gamma(n) &= 4 \int_0^\infty e^{-x^2} x^{2m-1} dx \cdot \int_0^\infty e^{-y^2} y^{2n-1} dy \\ &= 4 \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} x^{2m-1} y^{2n-1} dx dy.\end{aligned}$$

Using polar co-ordinates $x = r \cos \theta$ and $y = r \sin \theta$, the element of area $dx dy = r dr d\theta$ (Ref. ch. 16) and r and θ separately varies from 0 to ∞ and 0 to $\pi/2$ respectively as x and y both vary from 0 to ∞ . Therefore,

$$\begin{aligned}\therefore \Gamma(m) \Gamma(n) &= 4 \int_0^{\pi/2} \int_0^\infty e^{-r^2} \cos^{2m-1} \theta \sin^{2n-1} \theta \cdot r^{2(m+n)-1} dr d\theta \\ &= 2 \int_0^{\pi/2} \cos^{2m-1} \theta \sin^{2n-1} \theta \left(2 \int_0^\infty e^{-r^2} \cdot r^{2(m+n)-1} dr \right) d\theta \\ &= B(m, n) \cdot \Gamma(m+n), \text{ by 10.3.2 (ii) and 10.2.1 (ii)} \\ \therefore B(m, n) &= \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}.\end{aligned}\tag{1}$$

Method 2.

This method uses integration under the integral sign and the property of interchanging the order of multiple integral (Ref. ch. 16). From 10.2.1.

(i), $\Gamma(n) = a^n \int_0^\infty x^{n-1} e^{-ax} dx$ in which 'a' is a parameter. Multiplying

both sides by $a^{m-1} e^{-a}$ and integrating w.r.t. 'a' from 0 to ∞ , we have,

$$\Gamma(n) \int_0^\infty a^{m-1} e^{-a} da = \int_0^\infty \left[a^{m+n-1} e^{-a} \int_0^\infty x^{n-1} e^{-ax} dx \right] da$$

$$\begin{aligned}
 \text{or } \Gamma(n) \Gamma(m) &= \int_0^\infty \int_0^\infty a^{m+n-1} x^{n-1} e^{-a(1+x)} dx da \\
 &= \int_0^\infty \int_0^\infty a^{m+n-1} x^{n-1} e^{-a(1+x)} da dx, \\
 &\quad \text{(interchanging order)} \\
 &= \int_0^\infty x^{n-1} \left(\int_0^\infty a^{m+n-1} e^{-a(1+x)} da \right) dx \\
 &= \int_0^\infty x^{n-1} \frac{\Gamma(m+n)}{(1+x)^{m+n}} dx, \quad [\text{Comparing 'a' with} \\
 &\quad (1+x) \text{ in 10.2.1. (i)}] \\
 &= \Gamma(m+n) \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx \\
 &= \Gamma(m+n) B(m, n) \quad \text{by 10.3.2 (i)}
 \end{aligned}$$

$$\therefore B(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}.$$

Cor. (i). From 10.3.2 (ii),

$$B(m, n) = 2 \int_0^{\pi/2} \cos^{2m-1} \theta \sin^{2n-1} \theta d\theta$$

$$\therefore \int_0^{\pi/2} \cos^{2m-1} \theta \sin^{2n-1} \theta d\theta = \frac{1}{2} B(m, n) = \frac{\Gamma(m) \Gamma(n)}{2 \Gamma(m+n)}$$

Let $2m-1=p$ and $2n-1=q$. So, $m = \frac{p+1}{2}$ and $n = \frac{q+1}{2}$.

$$\begin{aligned}
 &\therefore \int_0^{\pi/2} \cos^p \theta \sin^q \theta d\theta \\
 &= \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = \frac{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}{2 \Gamma\left(\frac{p+q+2}{2}\right)}. \quad (2)
 \end{aligned}$$

Cor. (ii). Let, in cor (i), $p=0$. Then $\int_0^{\pi/2} \sin^q \theta d\theta = \frac{1}{2} B\left(\frac{1}{2}, \frac{q+1}{2}\right)$

$$= \frac{\Gamma(1/2) \Gamma\left(\frac{q+1}{2}\right)}{2 \Gamma\left(\frac{q+2}{2}\right)} = \frac{\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{q+2}{2}\right)} \cdot \frac{\sqrt{\pi}}{2}. \text{ Similarly, if } q=0, \text{ then}$$

$$\int_0^{\pi/2} \cos^p \theta \, d\theta$$

$$= \frac{1}{2} B\left(\frac{p+1}{2}, \frac{1}{2}\right) = \frac{\Gamma\left(\frac{p+1}{2}\right) \Gamma(1/2)}{2 \Gamma\left(\frac{p+2}{2}\right)} = \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p+2}{2}\right)} \cdot \frac{\sqrt{\pi}}{2}. \quad (3)$$

From these two results and prop. 10 of ch. 8, it follows that

$$\int_0^{\pi/2} \sin^r \theta \, d\theta = \int_0^{\pi/2} \cos^r \theta \, d\theta = \frac{1}{2} B\left(\frac{1}{2}, \frac{r+1}{2}\right)$$

$$= \frac{\Gamma(1/2) \Gamma\left(\frac{r+1}{2}\right)}{2 \Gamma\left(\frac{r+2}{2}\right)}, r > -1$$

$$= \frac{\sqrt{\pi} \Gamma\left(\frac{r+1}{2}\right)}{2 \Gamma\left(\frac{r+2}{2}\right)}, r > -1. \quad (4)$$

Cor (iii). If in cor (i) $p = q = 0$ or in cor (ii) $r = 0$, then

$$\int_0^{\pi/2} d\theta = \frac{1}{2} B\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{\Gamma(1/2) \Gamma(1/2)}{2 \Gamma 1}$$

or
$$\frac{\pi}{2} = \frac{(\Gamma(1/2))^2}{2} \text{ or } \Gamma(1/2) = \sqrt{\pi}. \quad (5)$$

Cor. (iv). From 10.3.2 (i), $B(m, n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx$.

Let $n = 1 - m$. As $m > 0, n > 0$ from 10.3.3, so $1 - m > 0$ i.e. $1 > m$ i.e. $m < 1$. Therefore,

$$B(m, 1 - m) = \int_0^{\infty} \frac{x^{m-1}}{1+x} dx \quad (6)$$

converges for $0 < m < 1$. The integral on the right cannot be integrated with the knowledge of this book. This requires the knowledge of contour integrals in Complex Variables or Laplace Transformation or Fourier Series which are beyond the scope of this book. So, to meet the necessity, we have

to quote the useful result as
$$\int_0^{\infty} \frac{x^{m-1}}{1+x} dx = \frac{\pi}{\sin m\pi}, 0 < m < 1.$$

$$\therefore B(m, 1-m) = \frac{\pi}{\sin m\pi}, 0 < m < 1$$

$$\text{or, } \Gamma(m) \Gamma(1-m) = \frac{\pi}{\sin m\pi}, 0 < m < 1 \quad (7)$$

Replacing m by n we have the property 10.2.2 (iv) as

$$\Gamma(n) \Gamma(1-n) = \frac{\pi}{\sin n\pi}, 0 < n < 1 \quad (8)$$

N.B. W.o. example 6 for another method.

10.5. ERROR FUNCTION

In applications particularly in probability statistics there arises another important and useful special function called error function. It is

defined as $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-r^2} dr$. For different values of x , the table of

$\text{erf}(x)$ is available. The integral is also known as probability integral in statistics. Some particular values can be obtained from definition and integration. For example,

$$\text{erf}(0) = \frac{2}{\sqrt{\pi}} \int_0^0 e^{-r^2} dr = 0$$

$$\text{erf}(-x) = \frac{2}{\sqrt{\pi}} \int_0^{-x} e^{-r^2} dr = -\text{erf}(x)$$

$\therefore \text{erf}(x)$ is an odd function.

$$\begin{aligned} \text{Also, } \text{erf}(\infty) &= \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-r^2} dr = \frac{2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} \text{ by 10.2.8} \\ &= 1. \end{aligned}$$

This shows that the maximum probability is 1 and so the complementary error function of x is defined as

$$\text{erfc}(x) = 1 - \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-r^2} dr.$$

$\therefore \text{erfc}(0) = 1, \text{erfc}(\infty) = 0, \text{erfc}(-x) = 2 - \text{erfc}(x)$ and so $\text{erfc}(x)$ is neither even nor odd.

10.6. INCOMPLETE GAMMA AND BETA FUNCTIONS

The incomplete gamma function is defined by

$$P(a, x) \equiv \frac{\gamma(a, x)}{\Gamma(a)} \equiv \frac{1}{\Gamma(a)} \int_0^x e^{-t} t^{a-1} dt, \quad a > 0. \quad (1)$$

It is evident that $P(a, 0) = 0$ and $P(a, \infty) = \frac{1}{\Gamma(a)} \Gamma(a) = 1$. (2)

The complement of $P(a, x)$ is denoted by (in this book)

$$Q(a, x) \equiv 1 - P(a, x) \equiv \frac{\Gamma(a, x)}{\Gamma(a)} \equiv \frac{1}{\Gamma(a)} \int_x^\infty e^{-t} t^{a-1} dt, \quad a > 0. \quad (3)$$

Here is also $Q(a, 0) = 1$ and $Q(a, \infty) = 0$. (4)

The incomplete gamma function & its complement are related to error function & its complement as follows :

$$\operatorname{erf}(x) = P\left(\frac{1}{2}, x^2\right), \quad x \geq 0 \quad (5)$$

and $\operatorname{erfc}(x) = Q\left(\frac{1}{2}, x^2\right), \quad x \geq 0. \quad (6)$

The incomplete beta function is defined as

$$\begin{aligned} I_x(a, b) &\equiv \frac{B_x(a, b)}{B(a, b)} \\ &\equiv \frac{1}{B(a, b)} \int_0^x t^{a-1} (1-t)^{b-1} dt, \quad (a, b > 0). \end{aligned} \quad (7)$$

Clearly, $I_0(a, b) = 0$ and $I_1(a, b) = 1$ (8)

Let $1-t = u$ in (7), so, the R.H.S. (7) of becomes

$$\begin{aligned} &= \frac{1}{B(a, b)} \int_1^{1-x} (1-u)^{a-1} u^{b-1} du \\ &= \frac{1}{B(a, b)} \left[\int_0^1 u^{b-1} (1-u)^{a-1} du - \int_0^{1-x} u^{b-1} (1-u)^{a-1} du \right] \\ &= 1 - I_{1-x}(b, a) \quad \because B(a, b) = B(b, a) \end{aligned}$$

$\therefore I_x(a, b) = 1 - I_{1-x}(b, a)$. This is the symmetry property of incomplete beta function.

10.7. PAI FUNCTION OF GAUSS

Gauss denoted $\Gamma(n+1)$ by $\pi(n)$, read as pai n , so, $\pi(n)$ is known as Gauss's pai function. Therefore, $\pi(n) = \Gamma(n+1) = n!$ for n is a +ve integer. The following two properties of pai function are analogous to gamma function.

$$(i) \pi(n) \pi(-n) = \Gamma(1+n) \Gamma(1-n) = n \Gamma(n) \Gamma(1-n) = \frac{n\pi}{\sin n\pi}, 0 < n < 1$$

$$(ii) \pi\left(-\frac{1}{2}\right) = \Gamma\left(1 - \frac{1}{2}\right) = \Gamma(1/2) = \sqrt{\pi}$$

10.7.1. FUNCTION $\psi(n)$ OF GAUSS

$\frac{d}{dn} (\log \Gamma(1+n))$ is represented by $\psi(n)$ i.e.,

$\psi(n) = \frac{d}{dn} (\log \Gamma(1+n))$ and is called the ψ function of Gauss. This is also,

known as di-gamma function and $\frac{d}{dn} \psi(n)$ is called the tri-gamma function. Therefore, by def.

$\psi(n) = \frac{\Gamma'(1+n)}{\Gamma(1+n)}$. Some of the properties of $\psi(n)$ are as follows :

$$(i) \psi(an) = \frac{1}{a} \frac{d}{dn} \log \Gamma(1+an), 'a' \text{ is a constant};$$

$$(ii) \psi(-n) = -\frac{d}{dn} \log \Gamma(1-n);$$

$$\begin{aligned} (iii) \psi(n+1) &= \frac{d}{dn} \log \Gamma(2+n) = \frac{d}{dn} \log \Gamma(1+(1+n)) = \frac{d}{dn} \{\log(1+n) \Gamma(1+n)\} \\ &= \frac{d}{dn} \{\log(1+n) + \log \Gamma(1+n)\} = \frac{1}{1+n} + \psi(n) \end{aligned}$$

$$(iv) \psi(-n-1) - \psi(n) = \pi \cot \pi n, 0 < n < 1$$

$$\therefore \psi(-n) = -\frac{d}{dn} \log \Gamma(1-n).$$

$$\therefore \psi(-n-1) = -\frac{d}{dn} \log \Gamma(-n).$$

$$\begin{aligned} \therefore \psi(-n-1) - \psi(n) &= -\frac{d}{dn} \log \Gamma(-n) - \frac{d}{dn} \log \Gamma(1+n) \\ &= -\frac{d}{dn} [\log \Gamma(-n) + \log \Gamma(1+n)] \\ &= -\frac{d}{dn} (\log \Gamma(-n) \Gamma(1+n)) = -\frac{d}{dn} \left(\log \frac{\pi}{\sin \pi(-n)} \right), 0 < n < 1 \\ &= -\frac{d}{dn} [\log \pi - \log \sin(-\pi n)], 0 < n < 1 \end{aligned}$$

$$= -\left[-\frac{-\pi}{\sin(-\pi n)} \cos(-\pi n) \right] = \pi \cot \pi n, 0 < n < 1$$

Examples

1. Evaluate $\frac{\Gamma(8/3)}{\Gamma(2/3)}$.

Here

$$\Gamma(8/3) = \Gamma(5/3 + 1) = \frac{5}{3} \Gamma(5/3) = \frac{5}{3} \Gamma(2/3 + 1) = \frac{5}{3} \cdot \frac{2}{3} \Gamma(2/3)$$

$$\therefore \frac{\Gamma(8/3)}{\Gamma(2/3)} = \frac{\frac{5}{3} \cdot \frac{2}{3} \cdot \Gamma(2/3)}{\Gamma(2/3)} = \frac{5}{3} \cdot \frac{2}{3} = \frac{10}{9}.$$

2. Evaluate $\int_0^\infty x^m e^{-ax^n} dx$ when m, n and a are all +ve constant.

Let $ax^n = y$. So, $nax^{n-1} dx = dy$ and

x	0	∞
y	0	∞

$$\begin{aligned} \therefore \int_0^\infty x^m e^{-ax^n} dx &= \frac{1}{na} \int_0^\infty \left(\frac{y}{a}\right)^{m/n} \left(\frac{y}{a}\right)^{\frac{1}{n}-1} e^{-y} dy \\ &= \frac{1}{na^{(m+1)/n}} \int_0^\infty y^{\left(\frac{m+1}{n}\right)-1} e^{-y} dy \\ &= \frac{1}{na^{(m+1)/n}} \Gamma\left(\frac{m+1}{n}\right). \end{aligned}$$

3. Prove that $\int_0^1 x^m (\log x)^n dx = \frac{(-1)^n n!}{(m+1)^{n+1}}, m > -1$ and n is

a +ve integer.

Let $x = e^{-y}$. So, $dx = -e^{-y} dy$ and

x	0	1
y	∞	0

$$\begin{aligned} \therefore \int_0^1 x^m (\log x)^n dx &= - \int_\infty^0 e^{-my} (-y)^n e^{-y} dy \\ &= \int_0^\infty (-1)^n e^{-(m+1)y} y^n dy \end{aligned}$$

Let $(m+1)y = t$. So, $dy = \frac{1}{m+1} dt$ and

y	0	∞
t	0	∞

$$\begin{aligned}
 \therefore \int_0^1 x^m (\log x)^n dx &= (-1)^n \int_0^\infty e^{-t} \left(\frac{t}{m+1} \right)^n \frac{1}{m+1} \cdot dt \\
 &= \frac{(-1)^n}{(m+1)^{n+1}} \int_0^\infty e^{-t} t^n dt = \frac{(-1)^n}{(m+1)^{n+1}} \Gamma(n+1) \\
 &= \frac{(-1)^n n!}{(m+1)^{n+1}} \quad \because n \text{ is a +ve integer.}
 \end{aligned}$$

4. Prove that $\int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - \frac{1}{2} \sin^2 \varphi}} = \frac{(\Gamma(1/4))^2}{4\sqrt{\pi}}.$

$$\begin{aligned}
 \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - \frac{1}{2} \sin^2 \varphi}} &= \sqrt{2} \int_0^{\pi/2} \frac{d\varphi}{\sqrt{2 - \sin^2 \varphi}} = \sqrt{2} \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 + \cos^2 \varphi}} \\
 &= \sqrt{2} \int_0^{\pi/2} \frac{\sec \varphi d\varphi}{\sqrt{1 + \sec^2 \varphi}} = \sqrt{2} \int_0^{\pi/2} \frac{\sec \varphi}{\sqrt{2 + \tan^2 \varphi}} d\varphi \\
 &= 2 \int_0^\infty \frac{du}{\sqrt{1 + 2u^2} \sqrt{1 + (1 + 2u^2)}} \text{ when } \tan \varphi = \sqrt{2}u. \\
 &= \frac{1}{\sqrt{2}} \int_1^\infty \frac{dp}{\sqrt{p} \sqrt{p^2 - 1}} \text{ when } 1 + 2u^2 = p \\
 &= \frac{1}{\sqrt{2}} \int_0^{\pi/2} \sqrt{\sec \theta} d\theta, \text{ when } p = \sec \theta \\
 &= \frac{1}{\sqrt{2}} \int_0^{\pi/2} \cos^{-1/2} \theta d\theta = \frac{1}{\sqrt{2}} \frac{\Gamma(1/4) \Gamma(1/2)}{2 \Gamma(3/4)} \\
 &= \frac{1}{2\sqrt{2}} \frac{\Gamma(1/4) \sqrt{\pi}}{\left(\frac{2\sqrt{2}\pi}{\Gamma(1/4)} \right)}, [\because \Gamma(1/4) \Gamma(3/4) = \frac{\pi}{\sin \pi/4} = \sqrt{2}\pi \text{ and} \\
 &\hspace{15em} \Gamma(3/4) = \sqrt{2}\pi/(\pi/4)] \\
 &= (\Gamma(1/4))^2 / (4\sqrt{\pi}).
 \end{aligned}$$

5. Let $F(\theta) = \int_0^\infty e^{-t \cos \theta} t^{a-1} \cos(t \sin \theta) dt,$

and $G(\theta) = \int_0^\infty e^{-t \cos \theta} t^{a-1} \sin(t \sin \theta) dt,$

where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and $a > 0$. Prove that $F(\theta) = [a \cos a\theta]$ and $G(\theta) = [a \sin a\theta]$. Hence show that $F'(\theta) = -aG(\theta)$, $G'(\theta) = aF(\theta)$ and $F''(\theta) + a^2 F(\theta) = 0$.

First Part. Find $F(\theta) + iG(\theta)$. Differentiate $F(\theta) + iG(\theta)$ for second part.

OR

Second part. Let $F(\theta) = e^{m\theta}$. So, the auxiliary equation is $m^2 + a^2 = 0$. $\therefore m = \pm ia$. Therefore, $F(\theta) = A \cos a\theta + B \sin a\theta$. But $F(0) = A = [a$ and $F'(0) = 0 = B$. So, $F(\theta) = [a \cos a\theta$ and (Ref. O.D.E. 2nd order ch. 21)

6. Show that $\int_1^\infty \frac{y^{a-1}}{1+y} dy = \int_0^1 \frac{x^{-a}}{1+x} dx$, $0 < a < 1$. Hence

show that $B(a, 1-a) = \int_0^1 \frac{x^{a-1} + x^{-a}}{1+x} dx$. Using binomial series for

$(1+x)^{-1}$ in the last integral prove that

$$[a \int_0^1 (1-x) = \frac{1}{a} + 2a \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2 - a^2} = \frac{\pi}{\sin a\pi}, 0 < a < 1 \text{ when}$$

$$\frac{\pi}{\sin a\pi} = \frac{1}{a} + 2a \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2 - a^2}, 0 < a < 1,$$

Ist part. Let $y = \frac{1}{x}$. Therefore $\int_1^\infty \frac{y^{a-1}}{1+y} dy = - \int_1^0 \frac{(1/x)^{a-1}}{1+(1/x)} \cdot \frac{dx}{x^2}$

$$= \int_0^1 \frac{x^{-a}}{1+x} dx.$$

2nd part.

$$B(a, 1-a) = \int_0^\infty \frac{u^{a-1}}{1+u} du = \int_0^1 \frac{u^{a-1}}{1+u} du + \int_1^\infty \frac{u^{a-1}}{1+u} du$$

$$= \int_0^1 \frac{u^{a-1}}{1+u} du + \int_0^1 \frac{u^{-a}}{1+u} du, \text{ by Ist part.}$$

$$= \int_0^1 \frac{u^{a-1} + u^{-a}}{1+u} du = \int_0^1 \frac{x^{a-1} + x^{-a}}{1+x} dx.$$

3rd part.

$$\begin{aligned}
B(a, 1-a) &= \int_0^1 (x^{a-1} + x^{-a}) (1+x)^{-1} dx \\
&= \int_0^1 (x^{a-1} + x^{-a}) (1-x+x^2-x^3+x^4-\dots\infty) dx \\
&= \int_0^1 (x^{a-1} + x^{-a}) \sum_{r=0}^{\infty} (-1)^r x^r dx \\
&= \sum_{r=0}^{\infty} (-1)^r \int_0^1 (x^{a+r-1} + x^{r-a}) dx \\
&= \sum_{r=0}^{\infty} (-1)^r \left[\frac{x^{a+r}}{a+r} + \frac{x^{r-a+1}}{r-a+1} \right]_0^1 = \sum_{r=0}^{\infty} (-1)^r \left[\frac{1}{a+r} + \frac{1}{r-a+1} \right] \\
&= \left(\frac{1}{a} + \frac{1}{1-a} \right) - \left(\frac{1}{1+a} + \frac{1}{2-a} \right) + \left(\frac{1}{2+a} + \frac{1}{3-a} \right) \\
&\quad - \left(\frac{1}{3+a} + \frac{1}{4-a} \right) + \dots\infty \\
&= \frac{1}{a} + \frac{2a}{1^2-a^2} - \frac{2a}{2^2-a^2} + \frac{2a}{3^2-a^2} - \dots\infty \\
&= \frac{1}{a} + 2a \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2-a^2}.
\end{aligned}$$

$$\therefore \Gamma(a) \Gamma(1-a) = \frac{\pi}{\sin a\pi} \text{ when } \frac{\pi}{\sin a\pi} = \frac{1}{a} + 2a \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2-a^2}, 0 < a < 1.$$

$$7. \text{ Prove that } \int_0^{\infty} \frac{\cos bx}{x^n} dx = \frac{b^{n-1}}{\Gamma(n)} \cdot \frac{\pi}{2 \cos \frac{n\pi}{2}}, 0 < n < 1.$$

$$\text{By def. } \Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x} dx \text{ and } \Gamma(n) = a^n \int_0^{\infty} y^{n-1} e^{-ay} dy$$

when $x = ay$. Therefore,

$$\frac{1}{a^n} = \frac{1}{\Gamma(n)} \int_0^{\infty} y^{n-1} e^{-ay} dy$$

$$\text{or } \frac{1}{x^n} = \frac{1}{\Gamma(n)} \int_0^{\infty} y^{n-1} e^{-xy} dy, \text{ replacing 'a' by } x.$$

$$\begin{aligned}
\therefore \int_0^{\infty} \frac{\cos bx}{x^n} dx &= \int_0^{\infty} \frac{\cos bx}{\Gamma n} \left(\int_0^{\infty} y^{n-1} e^{-xy} dy \right) dx \\
&= \frac{1}{\Gamma n} \int_0^{\infty} \int_0^{\infty} \cos bx \cdot (y^{n-1} e^{-xy}) dy dx \\
&= \frac{1}{\Gamma n} \int_0^{\infty} \int_0^{\infty} \cos bx \cdot (y^{n-1} e^{-xy}) dx dy, \\
&\quad \text{[interchanging the order of integration]} \\
&= \frac{1}{\Gamma n} \int_0^{\infty} y^{n-1} \left(\int_0^{\infty} e^{-xy} \cos bx dx \right) dy \\
&= \frac{1}{\Gamma n} \int_0^{\infty} y^{n-1} \left[\frac{e^{-xy} (-y \cos bx + b \sin bx)}{y^2 + b^2} \right]_{x=0}^{\infty} dy \\
&= \frac{1}{\Gamma n} \int_0^{\infty} \frac{y^n}{y^2 + b^2} dy = \frac{1}{b^2 \Gamma n} \int_0^{\infty} \frac{y^n}{\left(\frac{y}{b}\right)^2 + 1} dy \\
&= \frac{b^{n-1}}{\Gamma n} \int_0^{\infty} \frac{u^n}{1 + u^2} du \text{ when } \frac{y}{b} = u \\
&= \frac{b^{n-1}}{2 \Gamma n} \int_0^{\infty} \frac{p^{\frac{1}{2}(n-1)}}{1 + p} dp \text{ when } p = u^2 \\
&= \frac{b^{n-1}}{2 \Gamma n} \int_0^{\infty} \frac{p^{\frac{1}{2}(n+1)-1}}{1 + p} dp \\
&= \frac{b^{n-1}}{2 \Gamma n} \cdot \frac{\pi}{\sin \frac{1}{2}(n+1)\pi} \\
&= \frac{b^{n-1}}{\Gamma n} \cdot \frac{\pi}{2 \cos \frac{n\pi}{2}}, 0 < n < 1
\end{aligned}$$

8. Prove that $\gamma(a, x) + \Gamma(a, x) = \Gamma a$. Also derive the formulae

$$\gamma(a+1, x) = a \gamma(a, x) - e^{-x} x^a$$

and

$$\Gamma(a+1, x) = a \Gamma(a, x) + e^{-x} x^a.$$

Since $\gamma(a, x) = \int_0^x e^{-t} t^{a-1} dt, a > 0$ by 10.6

$$\text{and} \quad \Gamma(a, x) = \int_x^\infty e^{-t} t^{a-1} dt, \quad a > 0.$$

$\therefore$ Adding, we have,

$$\begin{aligned} \gamma(a, x) + \Gamma(a, x) &= \int_0^x e^{-t} t^{a-1} dt + \int_x^\infty e^{-t} t^{a-1} dt, \quad a > 0 \\ &= \int_0^\infty e^{-t} t^{a-1} dt, \quad a > 0 = \Gamma(a). \end{aligned}$$

$$\text{Again, by def. } \gamma(a+1, x) = \int_0^x e^{-t} t^a dt$$

$$= \left[-e^{-t} t^a \right]_{t=0}^x + \int_0^x e^{-t} a t^{a-1} dt, \quad (\text{l.b.p.})$$

$$= a \gamma(a, x) - e^{-x} x^a.$$

$$\text{Similarly, } \Gamma(a+1, x) = \int_x^\infty e^{-t} t^a dt$$

$$= \left[-e^{-t} t^a \right]_{t=x}^\infty + \int_x^\infty e^{-t} a t^{a-1} dt$$

$$= a \Gamma(a, x) + e^{-x} x^a.$$

$$9. \text{ Prove that } \int_0^\infty \left(\frac{e^{-2x} - e^{-x}}{x} + \frac{3}{2} e^{-2x} - \frac{1}{2} e^{-x} \right) \frac{dx}{x} = -1 + \frac{1}{2} \log 2.$$

The integrand has a s.p. at $x=0$ and its upper limit is infinite. So, it is an improper integral of the mixed type. Therefore,

$$\begin{aligned} &\int_0^\infty \left(\frac{e^{-2x} - e^{-x}}{x} + \frac{3}{2} e^{-2x} - \frac{1}{2} e^{-x} \right) \frac{dx}{x} \\ &= \int_0^\infty \frac{e^{-2x} - e^{-x}}{x^2} dx + \int_0^\infty \left(\frac{3}{2} e^{-2x} - \frac{1}{2} e^{-x} \right) \frac{dx}{x} \\ &= \lim_{\delta \rightarrow 0} \int_\delta^\infty \frac{e^{-2x} - e^{-x}}{x^2} dx + \int_0^\infty \left(\frac{3}{2} e^{-2x} - \frac{1}{2} e^{-x} \right) \frac{dx}{x} \\ &= \lim_{\delta \rightarrow 0} \left\{ \left[\frac{e^{-2x} - e^{-x}}{-x} \right]_\delta^\infty - \int_\delta^\infty \frac{-2e^{-2x} + e^{-x}}{-x} dx \right\} \\ &\quad + \int_0^\infty \left(\frac{3}{2} e^{-2x} - \frac{1}{2} e^{-x} \right) \frac{dx}{x} \end{aligned}$$

$$= -\lim_{\delta \rightarrow 0} \frac{e^{-2\delta} - e^{-\delta}}{-\delta} + \int_0^{\infty} \left(-\frac{1}{2}e^{-2x} + \frac{1}{2}e^{-x} \right) \frac{dx}{x},$$

[Combining the last two integrals]

$= -1 + \frac{1}{2} \log 2$. Using L'Hospital's rule and the integral def. of $\log x$ (Sec. 10.2.9).

Exercises 10

Prove the following :—

$$1. \int_0^{\infty} x^6 e^{-2x} dx = \frac{45}{8}; \quad 2. \int_0^{\infty} \sqrt{y} e^{-y^3} dy = \frac{\sqrt{\pi}}{3};$$

$$3. \int_0^2 \frac{x^2 dx}{\sqrt{2-x}} = \frac{64\sqrt{2}}{15}; \quad 4. \int_0^a y^4 \sqrt{a^2 - y^2} dy = \frac{\pi a^6}{32};$$

$$5. \int \left(\frac{2n+1}{2} \right) = \frac{1.3.5 \dots (2n-1)}{2^n} \sqrt{\pi}$$

$$6. \int_0^1 \frac{x^n dx}{\sqrt{1-x^2}} = \frac{1.3.5 \dots (n-1)}{2.4.6 \dots n} \cdot \frac{\pi}{2}, \text{ when } n \text{ is even integer}$$

$$= \frac{2.4.6 \dots (n-1)}{1.3.5 \dots n}, \text{ when } n \text{ is odd integer.}$$

$$7. \int_0^{\infty} \frac{e^{-2x}}{\sqrt{x}} dx = \sqrt{\frac{\pi}{2}}; \quad 8. \int_0^{\infty} x^{3/2} e^{-4x} dx = \frac{3\sqrt{\pi}}{128};$$

$$9. \int_0^{\infty} x^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{4}; \quad 10. \int_0^{\infty} x^4 e^{-x^2} dx = \frac{3}{8} \sqrt{\pi};$$

$$11. \int_0^{\infty} \frac{x^{2m}}{1+x^{2n}} dx = \frac{\pi}{2n} \operatorname{cosec} \left(\frac{2m+1}{2n} \right) \pi;$$

$$12. \int_0^{\infty} \frac{x^{2m}}{1-x^{2n}} dx = \frac{\pi}{2n} \cot \left(\frac{2m+1}{2n} \right) \pi;$$

$$13. \int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{\sqrt{\pi}}{n} \cdot \frac{\Gamma(1/n)}{\Gamma\left(\frac{1}{n} + \frac{1}{2}\right)};$$

$$14. \int_0^{\pi/2} \sqrt{\tan \theta} = \frac{\Gamma(1/4) \Gamma(3/4)}{2} = \frac{\pi}{\sqrt{2}};$$

$$15. \int_0^1 x^m (1-x^n)^p dx = \frac{\Gamma(p+1) \Gamma\left(\frac{m+1}{n}\right)}{n \Gamma\left(p+1+\frac{m+1}{n}\right)};$$

$$16. \int_0^1 \frac{x^{m-1} (1-x)^{n-1}}{(a+x)^{m+n}} dx = \frac{\Gamma m \Gamma n}{a^n (1+a)^m \Gamma(m+n)};$$

[Hints: put $\frac{x(1+a)}{a+x} = y$]

$$17. \int_0^1 \frac{x^2 dx}{\sqrt{1-x^2}} \times \int_0^1 \frac{dx}{\sqrt{1+x^4}} = \frac{\pi}{4\sqrt{2}};$$

$$18. \int_0^\infty \cos\left(bx^{\frac{1}{n}}\right) dx = \frac{\Gamma(n+1) \cos(n\pi/2)}{b^n}, \text{ [See W.Q.7]}$$

$$19. \int_0^\infty \left(\frac{\log \frac{1}{t}}{t}\right)^{1/2} dt = 2\sqrt{\pi}; \quad 20. (i) \int_0^1 \left(\frac{t}{\log \frac{1}{t}}\right)^{1/2} dt = \sqrt{\left(\frac{2\pi}{3}\right)}.$$

$$(ii) \frac{B(p, q+1)}{q} = \frac{B(p+1, q)}{p} = \frac{B(p, q)}{p+q}.$$

$$(iii) B(p, q) = B(p+1, q) + B(p, q+1).$$

$$21. \int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{\Gamma(1/4) \Gamma(1/2)}{4 \Gamma(3/4)};$$

$$22. \int_0^{\pi/2} \sqrt{\sin \theta} d\theta = \int_0^1 \frac{v^{1/4}}{2v^{1/2}(1-v)^{1/2}} dv \text{ when } v = \sin^2 \theta \\ = \frac{2 \Gamma(3/4) \Gamma(1/2)}{\Gamma(1/4)}$$

$$23. \int_0^1 x^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{4} \operatorname{erf}(1) - \frac{1}{2e}$$

$$24. \int_0^1 x^r \left(\log \frac{1}{x}\right)^s dx = \frac{\Gamma(s+1)}{(r+1)^{s+1}}$$

$$25. \int_0^\infty \frac{u^{x-1}}{1+u} du = \Gamma x \Gamma(1-x);$$

26. Prove that the Euler-Poisson integral $\int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2$

without using double integration. [Hints : Use $x=\sqrt{t}$ then $\int_0^\infty e^{-x^2} dx$

simplifies to $\frac{1}{2} \int_0^\infty e^{-t} t^{\frac{1}{2}-1} dt = \frac{1}{2} \Gamma(1/2) = \frac{\sqrt{\pi}}{2}$].

27. Prove that $\Gamma\left(\frac{m+1}{2}\right) = \frac{(2m)!}{m! 2^{2m}} \Gamma(1/2) = \frac{(2m)!}{m! 2^{2m}} \sqrt{\pi}$, for

some integer $m > 0$. Hence show that $\Gamma(5/2) = \frac{3}{4}\sqrt{\pi}$ [Hints: $\Gamma\left(m+\frac{1}{2}\right) = \left(m-\frac{1}{2}\right) \left(m-\frac{3}{2}\right) \dots \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma(1/2) = \frac{(2m-1)(2m-3)\dots 5 \cdot 3 \cdot 1}{2^m} \Gamma(1/2)$]

28. Show that

$$(i) \Gamma(-9/2) = \frac{-32\sqrt{\pi}}{945} \left[\text{Ref. eqn. 6 sec. 10.2.5, } q = 1/2, p = 5 \right]$$

$$(ii) \Gamma(5/3) \Gamma(-5/3) = \frac{2\pi\sqrt{3}}{5} \left[\text{Ref. eqn. (1), (3) sec. 10.2.5 then eqn. (7) sec. 10.2.5} \right]$$

$$(iii) \Gamma\left((1/2)+p\right) \Gamma\left((1/2)-p\right) = \frac{\pi}{\cos p\pi} \left[\text{Ref. eqn. (8) sec. 10.4, use } n=p+1/2 \right]$$

$$(iv) \Gamma(-4/3) = \frac{27}{8} \Gamma(5/3) = 3 \cdot 0486 \left[\text{Ref. eqn. (3) sec. 10.2.5 and table} \right]$$

$$(v) (-1/2)! = \sqrt{\pi}; (1/2)! = \sqrt{\pi}/2, (3/2)! = \frac{3\sqrt{\pi}}{4} \text{ and } (0.21)! = .9156.$$

$$\left[\text{Ref. eqn. (7) } \Gamma(n+1) = n! \text{ of sec. 10.2.5} \right]$$

$$(vi) \Gamma(10/3) \cdot \Gamma(-10/3) = \pi \sqrt{3/5}$$

$$(vii) \Gamma(5/4) \cdot \Gamma(-5/4) = 4\pi\sqrt{2}/5$$

$$(viii) \Gamma\left(-m + \frac{1}{2}\right) = \frac{(-1)^m 2^{2m}}{m! (2m)!} \Gamma(1/2)$$

$$\text{and } \Gamma\left(m + \frac{1}{2}\right) \Gamma\left(-m + \frac{1}{2}\right) = (-1)^m \pi, m = 1, 2, 3, \dots$$

29. Compute the following integrals :

(i) $\int_0^\pi \frac{dt}{\sqrt{3 - \cos t}}$ [Hints: Put $\cos t = 1 - 2\sqrt{u}$, then use Beta function and eqn. 8 sec. 10.4].

(ii) $\int_0^\infty \frac{x^{a-1}}{1+x^b} dx$ [Hints: Put $(1+x^b)/x^b = \frac{1}{y}$, eqn. 8. sec. 10.4].

(iii) $\int_0^\infty \frac{dx}{x^a(1+x)}$, $0 < a < 1$ [Hints: Put $x = \frac{u}{1-u}$].

30. The integral definition of logarithmic function is

$L(x) = \int_1^x \frac{dt}{t}$, $x > 0$. Show that $L(xy) = L(x) + L(y)$ and $L(1/x) = -L(x)$.

31. Prove that $\int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx = B(m, n)$ [Hints: Given integral $= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx - \int_1^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$. Now use $x = \frac{1}{t}$ in the 1st. integral]

32. Prove that (i) $B(x, x) = 2^{1-2x} B(x, 1/2)$ and also show that

$\sqrt{\pi} \sqrt{2x} = 2^{2x-1} \Gamma(x) \Gamma(x + 1/2)$ and

$\sqrt{\pi} \sqrt{x} = 2^{x-1} \Gamma(x/2) \Gamma((x+1)/2)$; (ii) $B(x, 1) = \frac{1}{x}$;

(iii) $B(m, m) B(m + 1/2, m + 1/2) = \pi m^{-1} 2^{1-4m}$

33. Prove that $\int_0^\infty \frac{\sin bx}{x^n} dx = \frac{b^{n-1}}{\Gamma(n)} \cdot \frac{\pi}{2 \sin \frac{n\pi}{2}}$, $0 < n < 1$

(Ref. W.O.7)

34. Show that $\int_0^\infty \frac{x^{p-1} \log x}{1+x} dx = -\pi^2 \operatorname{cosec} p\pi \cot p\pi$, $0 < p < 1$.

[Hints: Diff. $\int_0^\infty \frac{x^{p-1}}{1+x} dx = \frac{\pi}{\sin p\pi}$, $0 < p < 1$ w.r.t. p]

35. From W.O. Ex. 5, prove that $\int_0^\infty \frac{\cos t}{t^{1-a}} dt = \Gamma(a) \cos \frac{\pi a}{2}$ and

$$\int_0^\infty \frac{\sin t}{t^{1-a}} dt = \Gamma(a) \sin \frac{\pi a}{2}. \text{ Also show that the first formula is valid for}$$

$0 < a < 1$ and second for $-1 < a < 1, a \neq 0$.

36. Prove that $\psi(-1/4) - \psi(-3/4) = \pi$ and if $\psi_1(x) = \frac{d}{dx} \log \Gamma(x)$ then show that $\psi_1(1-x) - \psi_1(x) = \pi \cot \pi x$.

Answer

29. (i) $\frac{1}{2} \sqrt{\pi/2}$, (ii) $\frac{\pi}{b \sin(a\pi/b)}$, (iii) $\pi/(\sin a\pi)$.

DETERMINATION OF AREA

11.1. HISTORICAL BACKGROUND

The man has been in constant effort since ancient times for acquiring something new to quench the thirst for learning and knowledge. These efforts may be once a source of invention or it may be an essential part of life and applications. For example in 1850 B.C., an unknown Egyptian mathematician developed the formula, $\frac{1}{3}h(a^2 + ab + b^2)$ for the volume of the frustrum of a truncated square pyramid whose height is h and the lengths of lower and upper bases are a and b respectively. The latter mathematician calls his method as "The method of exhaustion" and Cavalieri called it the method of indivisibles. Now a days it is the "Theory of limits". Similarly, in 287-212 B.C. Archimedes determined the area of a parabolic segment. Both these examples are the present days' integration as the process of summation. In sec. 8.2, definite integral is defined as a process of summation and this definite integral is used to find area, volume, arc length, work, population etc. This chapter uses the fundamental theorem of integral calculus and devotes on area only.

Definition of Area: The amount of a plane or a surface lying in a plane region bounded by on all sides is called an *area*. If any one of the bounding side is a curve, then these area is known as *quadrature*. This is why *area* is sometimes named as *quadrature*.

11.2. AREA UNDER A PLANE CURVE : CARTESIAN

In this section, the area under the cartesian curve $y = f(x)$, x -axis, the ordinates $x = a$ and $x = b$ will be determined. Let $f(x)$ be a single valued continuous function in the interval $[a, b]$, $a < b$ and does not change sign (i.e. does not cross x -axis). Let AB denotes a portion of the monotonically increasing (fig. 10) or monotonically decreasing (fig. 11) curve from A to B . Let HA and KB be the ordinates erected at $x = a$ and $x = b$ respectively.

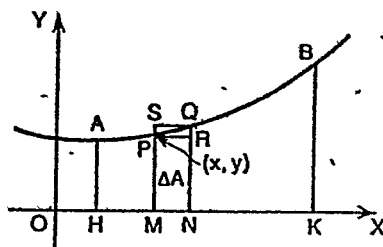


Fig. 10

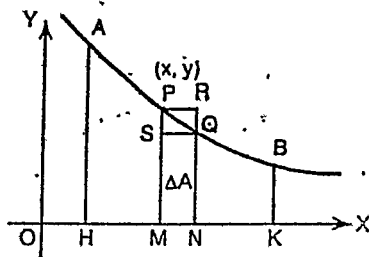


Fig. 11

Now, the area $HKBA$ is to be determined. Let $P(x, y)$ be a variable point on AB . Let the ordinate MP be drawn and it describes the $HKBA$ as it traverses from $x = a$ to $x = b$. Let ' A ' be the variable area $HMPA$. Clearly the area A is a function of the abscissa x of P i.e. $HMPA = A(x)$. Let x increase to $x + \Delta x$ where $\Delta x = MN$ and the corresponding increase of A be $\Delta A = MNPQ$. Now complete the rectangle $PRQS$. Therefore in fig. 10, $NQ > MP$ and in fig. 11, $NQ < MP$ in the interval M to N . In both figures the region $MNPQ$ lies between the rectangles $MNRP$ and $MNQS$. So, its area lies between the areas of these two rectangles. i.e.

ΔA lies between $y \cdot \Delta x$ & $(y + \Delta y) \Delta x$

$\therefore \frac{\Delta A}{\Delta x}$ lies between y & $y + \Delta y$

As $\Delta x \rightarrow 0$, so $\Delta y \rightarrow 0$. Therefore in the limit

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta A}{\Delta x} = \frac{dA}{dx} = y$$

$\therefore$ the derivative of A w.r.t. x is y or in differential form

$$dA = y \, dx$$

$$\begin{aligned} \therefore \text{integrating, } A &= \int y \, dx + C \\ &= \int f(x) \, dx + C \end{aligned}$$

Let $f(x) = F'(x)$. So,

$$A = F(x) + C \quad (1)$$

When $x = a$, then $A = 0$, so, from (1)

$$0 = F(a) + C$$

i.e.

$$C = -F(a)$$

$\therefore$ from (1), $A = F(x) - F(a)$

Again when $x = b$, then $A = \text{area } HKBA$,

$$\therefore \text{Area } HKBA = F(b) - F(a) = \int_a^b f(x) \, dx$$

[Fundamental Th. of integral calculus]

$\therefore$ the area enclosed by the curve $y = f(x)$, the x -axis, the ordinates at $x = a$ and $x = b$ is denoted by the definite integral

$$\int_a^b f(x) \, dx.$$

Cor. The area enclosed by $x = f(y)$, the y -axis and the abscissa $y = c$ & $y = d$ is given by

$$\int_c^d x \, dy = \int_c^d f(y) \, dy.$$

This idea can be extended to the case when the curve rises and falls many a times within a given interval instead of only monotonic. The above

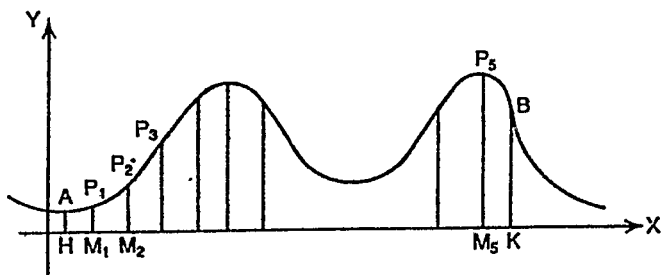


Fig. 12

discussion is valid for the smaller areas HM_1P_1A , $M_1M_2P_2P_1$, M_5KBP_5 etc. (fig. 12). So, sum of these areas is the area $HKBA$ i.e. the area enclosed by $y = f(x)$, x -axis, the ordinates at $x = a$ & $x = b$ irrespective of strict monotonic.

11.3. SIGN CONVENTION

In general the area described by a moving point on the boundary of a region in the anticlockwise direction is *positive* while that in clockwise direction is *negative*, (fig. 13 & fig. 14).

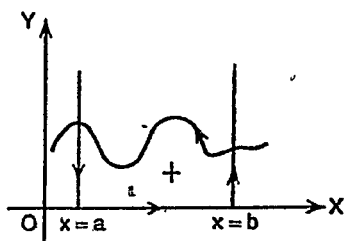


Fig. 13

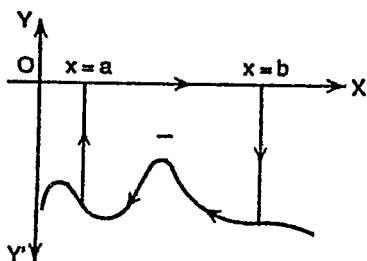


Fig. 14

The anticlockwise direction is determined by keeping the area *left* of the moving point on the boundary as x increases from minimum to maximum and back.

Now sign of the area defined by $\int_a^b f(x) \, dx = \int_a^b y \, dx$ is regarded as positive (negative) if the moving point moves along increasing x and

keeps the area in the *left (right)* i.e. moves anticlockwisely (clockwisely). If the curve $y = f(x)$ crosses the x -axis, then the moving point with increasing x describes area sometimes positively and sometimes negatively. In that case total area can be calculated either by taking absolute values of the negative areas or for areas below x -axis by taking negative y 's i.e. negative $f(x)$'s with increasing x 's.

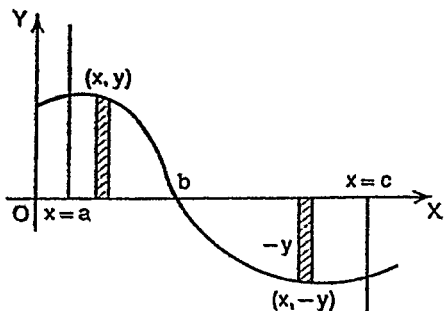


Fig. 15

From fig. 15 if $y = f(x)$ crosses x -axis at $x = b$ then area enclosed by $y = f(x)$, x -axis, the ordinates $x = a$ & $x = c$ is given by

$$\int_a^c f(x) dx = \left[\text{either } \int_a^b f(x) dx + \left| \int_b^c f(x) dx \right| \text{ or } \int_a^b f(x) dx + \int_b^c \{-f(x)\} dx \right].$$

Similar extension for any numbers of crossings can be made.

N.B. Drawing of the required area is a *must*, otherwise total area may be -ve and taking of absolute value of that area may give a wrong result.

11.4. PROCEDURE FOR DETERMINING AREA

To determine the area at *first* draw the graph $y = f(x)$ and *then* decide the lower and the upper limits $x = a$ & $x = b$ respectively. For these purposes some important information from curve tracing are collected below for ready reference.

(i) A clear idea of tracing the curve is necessary. Most simple form of sketching graph is tabulation of the values of the variables as close as possible and plotting on-graph paper. This will give an idea of the curve.

(ii) Use of the methods of curve tracing is an upgraded procedure. This requires the following :

- if $f'(x) > 0$ curve increases monotonically
- if $f'(x) < 0$ curve decreases monotonically
- if $f''(x) < 0$ curve is concave downward
- if $f''(x) > 0$ curve is concave upward
- if $f''(x) = 0$ neither concave upward or downward. It is a point of inflection
- coeffs. (const.) of the lowest degree terms gives the eqn. of the asymptotes

- (g) solving for x or y is needed to know the position of the curve in different quadrants. The signs of the variables for which the eqn. changes the form do not have existence for those signs. For example $x^4 + y^4 = 4a^2xy$ changes with opposite signs of x and y , so the curve does not exist in second and fourth quadrants.
- (h) It is sometimes convenient to express a cartesian eqn. in polar or parametric form to determine the limits.

(iii) Symmetry is another important point to note. If the area of one symmetrical part can be calculated, thus total area is some multiple of the symmetrical part. The following points may be noted for symmetry of a curve :

- (a) In algebraic eqn. if x or y or both are even powered then the curve is symmetrical about y or x or both axes respectively. e.g. $y^2 = 4ax$, $x^2 = 4ay$ and $x^2 + y^2 = a^2$ curves are symmetrical about x , y and both axes respectively.
- (b) Interchange of x and y , keeps the eqn. unchanged, then the curve is symmetrical about the line $y = x$, e.g. $x^3 + y^3 = 3axy$ and $x^2 + y^2 = a^2$.
- (c) Replacement of x by $(-y)$ and y by $(-x)$; keeps eqn. unchanged, then the curve is symmetrical about the line $y = -x$, e.g. $x^3 - y^3 = 3axy$.
- (d) Replacement of both x and y by $(-x)$ and $(-y)$ keep the eqn. unchanged then the curve is symmetrical in opposite quadrants. e.g. $y = x^3$, $3y = -x^3$ are symmetrical in "first and third" and "second and fourth" quadrants respectively.

(iv) Parametric curve $x = \phi(t)$, $y = \psi(t)$ will be symmetrical about x -axis (y -axis) if the replacement of t by a function of t changes the sign of y (or x) but not of x (or y). e.g. $x = at^2$, $y = 2at$ is symmetrical about x -axis as the replacement of t by $(-t)$ changes y but not x . Similarly, $x = a \cos^3 t$, $y = a \sin^3 t$ is symmetrical about both axes because replacement of t by $(-t)$ changes y but not x and that of t by $(\pi - t)$ changes x but not y .

(v) Shifting and reflection properties described below provide an easy means of sketching from the knowledge of a known graph. Thus if the graph of $y = f(x)$ is known (fig. 16(i)) then the graph of

(a) $y = f(x) + c$ shifts up or down the graph of $y = f(x)$ by $|c|$ units if $c > 0$ or $c < 0$ respectively (fig. 16 (ii) & (iii));

(b) $y = f(x - c)$ shifts the graph of $y = f(x)$ to right or left by $|c|$ units if $c > 0$ or $c < 0$ respectively (fig. 16 (iv) & (v));

(c) $y = -f(x)$ reflects $y = f(x)$ through x -axis i.e. the $y = f(x)$ turns upside down by $y = -f(x)$ (fig. 16 (vi))

(d) $y = f(-x)$ reflects $y = f(x)$ through y -axis (fig. 16 (vii)).

(e) If the graph of $y = f(x)$ is known then the graph of $y = f(|x|)$ is obtained (i) by retaining the part of $y = f(x)$ for $x \geq 0$ and (ii) reflecting the retained part about y -axis (fig. 16 (viii)).

(f) If the graph of $y = f(x)$ is known then the graph of $y = |f(x)|$ is obtained by retaining the part of $y = f(x)$ above x -axis and reflecting about x -axis the part of $y = f(x)$ lying below x -axis (fig. 16 (ix)).

(g) The graph of $y = \lambda f(kx + a) + b$ is obtained from the knowledge of $y = f(x)$ and using previous rules sequentially.

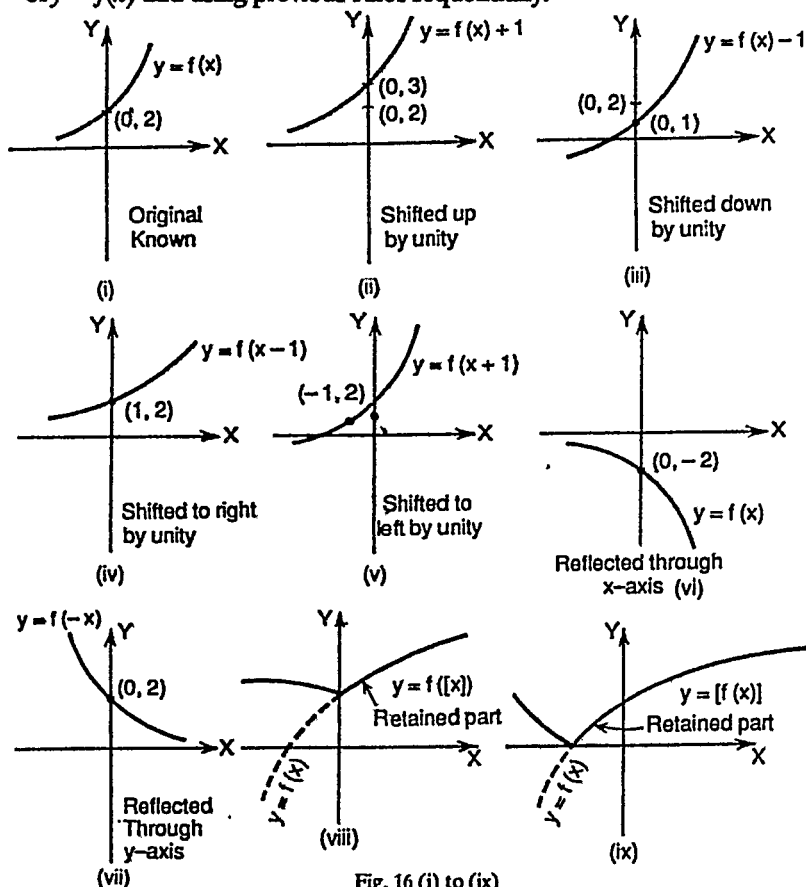


Fig. 16 (i) to (ix)

(vi) Positive and negative areas are to be calculated as in sec. 11.3 for total area.

(vii) Elementary area ΔA is to be shown in every case and is taken in such a way that when it rolls from one end to other it covers the whole area irrespective of its size. Normally it is taken perpendicular to one of the axes.

11.5. MEAN VALUE OF $y = f(x)$.

The average value of the ordinate, y of the area enclosed by $y = f(x)$, x -axis, the ordinates at $x = a$ & $x = b$ is to be determined. Let the area (fig. 17) be divided into $(n - 1)$ smaller areas of equal breadth Δx by erecting n ordinates. Let the ordinates be y_1 at $x = a$, y_2 at $x = a + \Delta x$ and in this way y_n at $x = a + (n - 1) \Delta x = b$. Therefore, the average value of

$$\begin{aligned}
 y &= \lim_{n \rightarrow \infty} \frac{y_1 + y_2 + \dots + y_n}{n} \\
 &= \lim_{n \rightarrow \infty} \frac{(y_1 + y_2 + \dots + y_n) \Delta x}{n \Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\sum_{i=1}^n y_i \Delta x}{b - a + \Delta x} \quad \because a + (n - 1) \Delta x = b \\
 &= \frac{\lim_{\Delta x \rightarrow 0} \sum_{x=a}^b y \Delta x}{\lim_{\Delta x \rightarrow 0} (b - a + \Delta x)} = \frac{\int_a^b f(x) dx}{b - a} \\
 &= \frac{(\text{area under the curve})}{(\text{base of the area under the curve})}.
 \end{aligned}$$

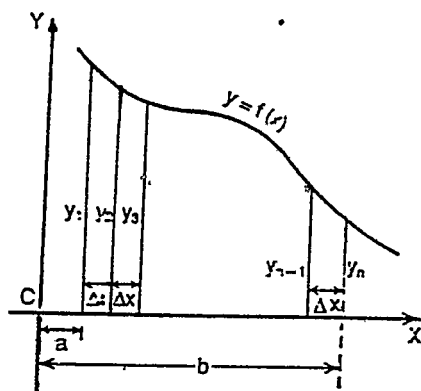


Fig. 17

Examples

1. Find the area of the circle $x^2 + y^2 = a^2$.

The powers of x and y are even in the given eqn. So, the curve is symmetrical about both axes. Hence the area is equally divided into four equal parts.

$$\text{From the eqn., } y^2 = a^2 - x^2, \\ y = \pm \sqrt{a^2 - x^2}$$

$\therefore y$ is maximum and minimum at $B(x=0)$ and $C(x=a)$ respectively (fig. 18).

Therefore, area of

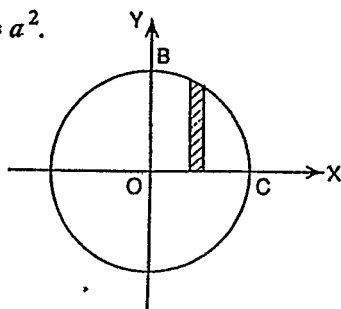


Fig. 18

$$BOC = \int_0^a y \, dx = \int_0^a \sqrt{a^2 - x^2} \, dx, [\because y \text{ is +ve in } BOC \text{ part}]$$

$$= \int_0^{\pi/2} a \cos \theta \cdot a \cos \theta \, d\theta, \text{ when } x = a \sin \theta$$

$$= \frac{\pi}{4} a^2 \text{ square units (sq. units).}$$

$$\therefore \text{total area} = 4 \cdot (\text{area } BOC) = \pi a^2 \text{ sq. units.}$$

2. Prove that the area of the parabola $y^2 = 4ax$ bounded by its double ordinate is $(2/3)$ of the area of the rectangle inscribing the area contained by them.

The power of y in the given eqn. is even. So the curve is symmetrical about the x -axis. Let ABC be the double ordinate & let abscissa of B be x_1 (fig. 19).

$$\therefore \text{Area } AOB = \int_0^{x_1} y \, dx \\ = \int_0^{x_1} \sqrt{4ax} \, dx, \because y \text{ is +ve in the} \\ \text{first quadrant}$$

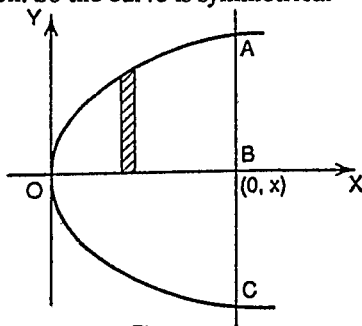


Fig. 19

$$= \frac{2}{3} x_1 \sqrt{4ax_1} = \frac{2}{3} x_1 y_1 \text{ sq. units.}$$

$$\therefore \text{required total area} = 2 \cdot \frac{2}{3} x_1 y_1 = \frac{2}{3} \cdot x_1 (2y_1)$$

$$= \frac{2}{3} (OB \cdot AC)$$

$$= \frac{2}{3} (\text{Area of the rectangle contained by } OB \text{ and } AC)$$

$$= \frac{2}{3} (\text{the area of the rectangle inscribing the area contained by double ordinate } ABC \text{ and } y^2 = 4ax)$$

3. (a) Find the area under the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, x -axis and the ordinates $x = c$ & $x = d$;

(b) find the total area of the ellipse;

(c) find the area between its latus recta.

The powers of both x and y are even. So, the ellipse is symmetrical about both axes. Also,

$$y^2 = \frac{b^2}{a^2} (a^2 - x^2)$$

$$\therefore y = \pm \frac{b}{a} \sqrt{a^2 - x^2}$$

(a) Required area

$$= \int_c^d y \, dx = \frac{b}{a} \int_c^d \sqrt{a^2 - x^2} \, dx,$$

y is +ve in XOY quadrant (fig. 20)

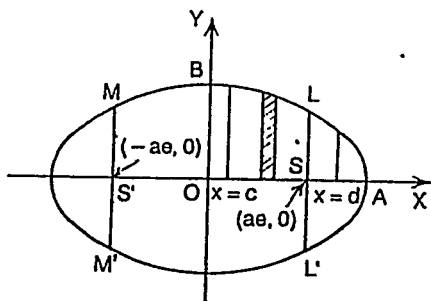


Fig. 20

$$= \frac{b}{2a} \left[d \sqrt{a^2 - d^2} - c \sqrt{a^2 - c^2} + a^2 \left(\sin^{-1} \frac{d}{a} - \sin^{-1} \frac{c}{a} \right) \right] \text{ sq. units}$$

(b) Here ordinates are $x = c = 0$ and $x = d = a$. So, from (a), total area of the ellipse = 4 (area BOAB)

$$= 4 \cdot \frac{b}{2a} \left[a \sqrt{a^2 - a^2} - 0 \sqrt{a^2 - 0} + a^2 \left(\sin^{-1} \frac{a}{a} - \sin^{-1} \frac{0}{a} \right) \right]$$

$$= \pi ab \text{ sq. units}$$

(c) Since the co-ordinates of the foci are $(\pm ae, 0)$. So, from (a), using $x = c = -ae$ and $x = d = ae$, the area

$$\begin{aligned} MS'OSLBM &= \frac{b}{2a} \left[ae \sqrt{a^2 - a^2 e^2} + ae \sqrt{a^2 - a^2 e^2} \right. \\ &\quad \left. + a^2 \left(\sin^{-1} \frac{ae}{a} - \sin^{-1} \frac{(-ae)}{a} \right) \right] \\ &= ab \left(e \sqrt{1 - e^2} + \sin^{-1} e \right) \text{ sq. units.} \end{aligned}$$

$$\begin{aligned}\therefore \text{the reqd. area} &= 2 (\text{area of } MS'OSLBM) \\ &= 2ab \left(e \sqrt{1 - e^2} + \sin^{-1} e \right) \text{ sq. units}\end{aligned}$$

4. Find the area bounded by the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the x -axis

and by an ordinate.

Let $x = x$ be the an ordinate of the hyperbola (fig. 21).

From the given eqn.

$$y^2 = \frac{b^2}{a^2} (x^2 - a^2)$$

$$y = \pm \frac{b}{a} \sqrt{x^2 - a^2}$$

Here y is minimum when $x = \pm a$.

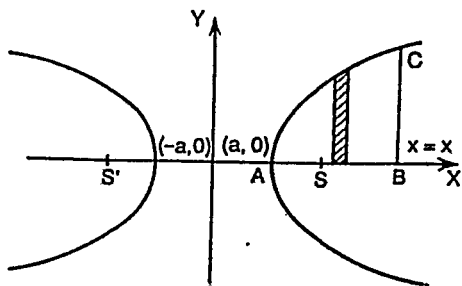


Fig. 21

Again the powers of both x & y are even; so the hyperbola is symmetrical about both axes. Therefore, the area $ASBCA$ of the hyperbola

$$= \int_a^x y \, dx = \frac{b}{a} \int_a^x \sqrt{x^2 - a^2} \, dx, \text{ } y \text{ is +ve in } XOY \text{ quadrant.}$$

$$= \frac{b}{a} \left[\frac{x \sqrt{x^2 - a^2}}{2} - \frac{a^2}{2} \log (x + \sqrt{x^2 - a^2}) \right]_a^x$$

$$= \frac{b}{2a} \left[x \sqrt{x^2 - a^2} - a^2 \log \frac{x + \sqrt{x^2 - a^2}}{a} \right] \text{ sq. units.}$$

5. Determine the area between the cycloid and its base.

Two types of cycloids are considered here – one with base as the x -axis and the other with base parallel to x -axis.

(a) The cycloid with base as the x -axis is

$$x = a (\theta - \sin \theta)$$

$$y = a (1 - \cos \theta)$$

Here y gets minimum when $\theta = 0$ & 2π . So, correspondingly $x = 0, 2\pi a$ and hence the two cusps of the cycloid are $(0, 0)$ and $(2\pi a, 0)$. Therefore, the area under the curve with $x = 0$ and $x = 2\pi a$ is the area $ODCAO$ (fig. 22)

$$= \int_0^{2\pi a} y \, dx$$

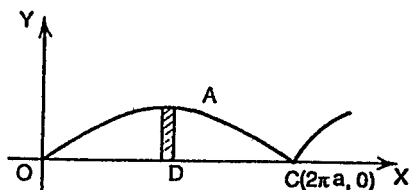


Fig. 22

$$\begin{aligned}
 &= \int_0^{2\pi} a(1 - \cos \theta) \cdot a(1 - \cos \theta) d\theta, \left[\because \theta \text{ varies from } 0 \text{ to } 2\pi \right] \\
 &= a^2 \int_0^{2\pi} (1 - \cos \theta)^2 d\theta \\
 &= a^2 \int_0^{2\pi} \left[1 + \frac{1 + \cos 2\theta}{2} - 2 \cos \theta \right] d\theta = 3\pi a^2 \text{ sq. units}
 \end{aligned}$$

(b) The eqn. of the cycloid with base parallel to the x -axis is

$$x = a(\theta + \sin \theta)$$

$$y = a(1 - \cos \theta)$$

Its base is BCA (fig. 23). Here y gets its minimum and maximum value corresponding to $\theta = 0$ and $\theta = 2\pi$. The curve is symmetrical about y -axis because y does not change sign when θ is replaced by $(-\theta)$. Therefore, the required area = 2 (area of $OACO$)

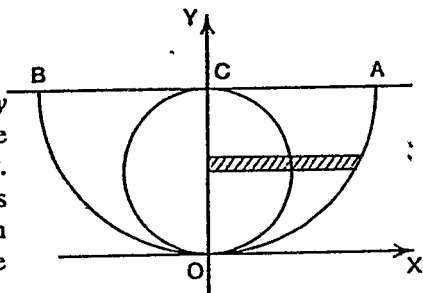


Fig. 23

$$\begin{aligned}
 &= 2 \int_0^{2a} x dy = 2 \int_0^{\pi} a(\theta + \sin \theta) \cdot a \sin \theta d\theta \\
 &= 3\pi a^2 \text{ sq. units.}
 \end{aligned}$$

6. Find the area of a loop of the curve $x(x^2 + y^2) = a(x^2 - y^2)$ and that bounded by its asymptote.

Def. Loop is formed by the self-intersection of the curve about some line and normally this line is the line of symmetry.

The equation is not of normal second degree curve. So, to draw the graph the following should be discussed.

- (i) The power of y is even. So the curve is symmetrical about x -axis.
- (ii) Again the equation can be written as

$$y^2 = x^2 \frac{a-x}{a+x}$$

It is evident that y is imaginary when $x > a$ and $x < (-a)$. So, the curve has no real part for those x 's. (iii) At $x = -a$, $y = \infty$. So the curve must lie in $-a < x < a$. (iv) At $x = 0$, $y = 0$. So, the curve crosses itself at the origin and the x -axis at $(a, 0)$. Hence there forms a loop between $x = 0$ and $x = a$. (v) The coeff. of the highest power of x is $1 \neq 0$. The next higher power is of y and its coeff. $a + x = 0$ is the only eqn. of the asymptote. (vi) Now the eqn. of the tangent to the curve at the origin can be obtained by equating the lowest degree parts equal to zero i.e. $x^2 - y^2 = 0$, i.e. $y = \pm x$. (vii) The curve intersect the x -axis at $x = a$ and

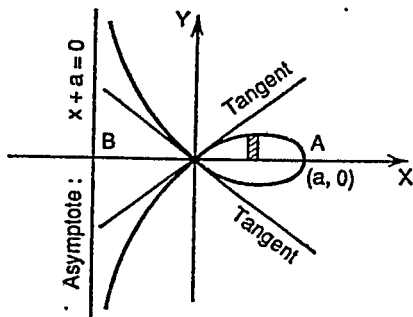


Fig. 24

$$\frac{dy}{dx} = \frac{2ax - 3x^2 - y^2}{2y(x+a)} = \infty \text{ at } (a, 0).$$

From these discussions the figure of the curve stands as fig. 24.

(a) As the loop forms between $x = 0$ to $x = a$ and is symmetrical about x -axis. Therefore, the area of the loop

$$\begin{aligned} &= 2 \int_0^a y \, dx = 2 \int_0^a x \sqrt{\frac{a-x}{a+x}} \, dx. \\ &= 2a^2 \int_0^{\pi/2} \cos \theta \sqrt{\frac{(1-\cos \theta)^2}{1-\cos^2 \theta}} \cdot \sin \theta \, d\theta, \text{ when } x = a \cos \theta. \\ &= 2a^2 \int_0^{\pi/2} (\cos \theta - \cos^2 \theta) \, d\theta = 2a^2 (1 - \pi/4) \text{ sq. units.} \end{aligned}$$

(b) Since the asymptote is at $x = -a$, so, the area between the curve and its asymptote

$$\begin{aligned} &= 2 \int_0^{-a} y \, dx = 2 \int_0^{-a} x \sqrt{\frac{a-x}{a+x}} \, dx = 2a^2 \int_{\pi}^{\pi/2} (\cos \theta - \cos^2 \theta) \, d\theta \\ &\quad \text{when } x = a \cos \theta \\ &= 2a^2 (1 + \pi/4) \text{ sq. units.} \end{aligned}$$

7. Find the area enclosed by $y = \tan x$, x -axis and the ordinates at $x = -\pi/3$ and $x = \pi/3$.

The function $y = \tan x$ is periodic with period π . Its portion between $x = 0^\circ$ to 90° lies above the x -axis and that between $x = 90^\circ$ to 180° lies under the x -axis. Also $y = 0$ when $x = 0$. So, the curve crosses x -axis at the origin (fig. 25).

When x and y are of opposite signs the eqn. changes. So, the curve does not lie in second and fourth quadrant. Hence the sketch of the curve is as in fig. 25. Therefore, the reqd. area

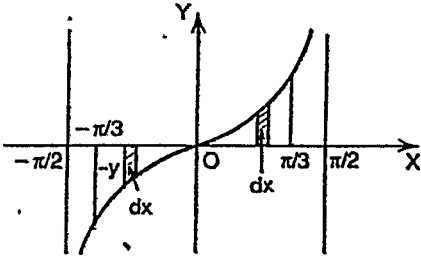
$$= \int_{-\pi/3}^{\pi/3} y \, dx = \left| \int_{-\pi/3}^0 \tan x \, dx \right| + \int_0^{\pi/3} \tan x \, dx$$


Fig. 25

$$= \left| \left[\log \sec x \right]_{-\pi/3}^0 \right| + \left[\log \sec x \right]_0^{\pi/3}$$

$$= | -\log 2 | + \log 2 = 2 \log 2 \text{ sq. units.}$$

Alter.

$$\begin{aligned} \text{Reqd. area} &= \int_{-\pi/3}^0 (-\tan x) \, dx + \int_0^{\pi/3} \tan x \, dx \\ &= \left[-\log \sec x \right]_{-\pi/3}^0 + \left[\log \sec x \right]_0^{\pi/3} \\ &= 2 \log 2 \text{ sq. units.} \end{aligned}$$

8. Find the area under the curve $xy^2 = 1$, co-ordinate axes and the line $x = 2$.

The curve is symmetrical about x -axis as the power of y is even in the equation. Again the coeff. of the highest power of x is $y^2 = 0$ i.e. $y = 0$ and that of y is $x = 0$ are the two asymptotes of the curve; so the curve never passes through the origin. Also,

$$\frac{d^2y}{dx^2} = \frac{3}{4} \cdot \frac{y}{x^2}$$

$\therefore$ the curve is concave upwards if $y > 0$ and concave downwards if $y < 0$. The sketch of the curve is like fig. 26.

The reqd. area

$$= \int_0^2 y \, dx = \int_0^2 \frac{1}{\sqrt{x}} \, dx.$$

The integral here is an improper integral of the second kind having a s.p. at $x = 0$. So, the area

$$= \lim_{\delta \rightarrow 0} \int_{0+\delta}^2 \frac{1}{\sqrt{x}} \, dx$$

$$= \lim_{\delta \rightarrow 0} [2\sqrt{x}]_{\delta}^2 = \lim_{\delta \rightarrow 0} (2\sqrt{2} - 2\sqrt{\delta})$$

$$= 2\sqrt{2} \text{ sq. units.}$$

The given area is not closed, yet it can be said closed because of its finite value.

9. Determine the area between the cissoid $y^2(2a - x) = x^3$ and its asymptotes.

(i) The curve is symmetrical about x -axis as the power of y is even. (ii) The equation of the asymptote is obtained by equating the coeff. of y^2 to zero i.e. $x = 2a$. (iii) The curve has no real part when $x > 2a$ and $x < 0$. So, the curve does not have any existence in the left of $x = 0$ and in the right of $x = 2a$ i.e. the curve lies fully between $x = 0$ and $x = 2a$.

(iv) Also

$$\frac{dy}{dx} = \frac{y}{2} \left(\frac{3}{x} + \frac{1}{2a-x} \right)$$

$\therefore$ the curve increases and decreases monotonically according as $y > 0$ and $y < 0$. (v) And

$$\frac{d^2y}{dx^2} = \frac{1}{2} \left(\frac{3}{x} + \frac{1}{2a-x} \right) \frac{dy}{dx} + \frac{y}{2} \left(-\frac{3}{x^2} + \frac{1}{(2a-x)^2} \right).$$

The curve is concave upwards and downwards according as $y > 0$ and $y < 0$. So, the sketch of the curve is as follows (fig. 27) :

The reqd. area

$$= 2 \int_0^{2a} y \, dx = 2 \int_0^{2a} \frac{x^{3/2}}{\sqrt{2a-x}} \, dx$$

The integral is an improper integral of second kind as the integrand has a s.p. at $x = 2a$.

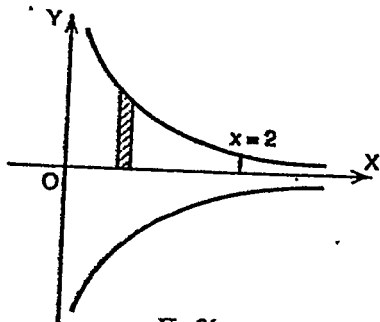


Fig. 26

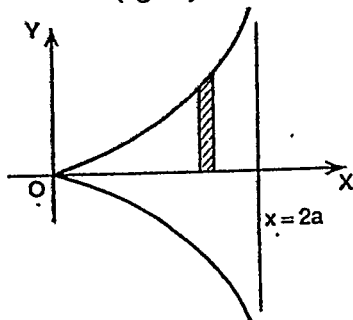


Fig. 27

$$\begin{aligned}
 \therefore \text{area} &= 2 \left[\lim_{\delta \rightarrow 0} \int_0^{2a-\delta} \frac{x^{3/2}}{\sqrt{2a-x}} dx \right] \\
 &= 2 \left[\lim_{\delta \rightarrow 0} \int_0^{2a-\delta} \frac{x^2}{\sqrt{2ax-x^2}} dx \right] \\
 &= 2 \left[\lim_{\delta \rightarrow 0} \left[3a^2 \sin^{-1} \sqrt{\frac{x}{2a}} - \sqrt{2ax-x^2} \left(\frac{x}{2} + \frac{3a}{2} \right) \right]_0^{2a-\delta} \right] \\
 &= 3\pi a^2 \text{ sq. units.}
 \end{aligned}$$

Exercises 11 (1)

1. (a) Determine the area under the following curves, the x -axis or the y -axis as applicable and the given lines :

- (i) $xy = c^2$, $x = a$, $x = b$;
 (ii) $xy = a^2$, $x = a$, $x = 2a$;
 (iii) $y = \cos x$, $x = 0$, $x = \pi$;
 (iv) $y = 2 \cos x$, $x = -\pi/2$, $x = \pi/2$;

(v) $y = \cosh \frac{x}{c}$, $x = -c$, $x = c$;

(vi) $y = \log x$, $x = a$, $x = b$, when (a) $0 < a < b < 1$ and

(b) $0 < a < 1 < b$;

(vii) Catenary $y = 6 \cosh 2x$, $x = 2$, $x = 9$;

(viii) $y = e^x$, $x = 1$, $x = 5$;

(ix) Parabola $y = x^2 - 2x + 5$, $x = 0$, $x = 3$;

(x) Parabola $y = 3x^2$, $x = 2$, $x = 4$;

(xi) Rectangular hyperbola $xy = 1$, $x = 1$, $x = 2$;

(xii) $(1+x^2)y = 1$, $x = \pm 1$;

(xiii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $x = 0$, $x = c$;

(xiv) $y = \frac{b}{a} \sqrt{a^2 - x^2}$, $x = \sqrt{a^2 - b^2}$, $x = a$;

(xv) $y = x^3 + 3x^2 + 2x$, $x = -3$, $x = 3$;

(xvi) $y = xe^{x^2}$, $x = 0$, $x = h$;

(xvii) $y = x^3$, $x = 0$, $x = 4$;

(xviii) $x = 3y^2 - 9$, $y = 0$, $y = 1$;

$$\begin{aligned}
 &\int \sqrt{a^2 - x^2} \\
 &= \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}
 \end{aligned}$$

$$(xix) y = \tan x, x = 0, x = \pi/4;$$

$$(xx) x = 8 + 2y - y^2, y = -1, y = 3;$$

$$(xxi) x = 3y^2 + 2, y = -1, y = 2;$$

$$(xxii) y = 4x - x^2, y = 0;$$

$$(xxiii) y = x^3 - 6x^2 + 8x, y = 0;$$

$$(xxiv) x = 4 - y^2, x = 0;$$

$$(xxv) x = 1 + y^2, x = 10;$$

$$(xxvi) x = y^2 + 4y, x = 0;$$

$$(xxvii) y = \sin x^2, x = 0, x = 1;$$

(Hints. Expand $\sin x^2$ into a power series)

(b) Let $f(x) = x^2$. Find the area enclosed by $y = \{f(x-1) + 4\}$, the x -axis, $x = 0$ and $x = 3$.

(c) Let $f(x) = (x-1)x(x+1)$. Find the area enclosed by $y = f(x+1)$, the x -axis, $x = -3$ and $x = 3$.

(d) Let $f(x) = (x+1)x(1-x)$. Find the area enclosed by x -axis, $x = -3$, $x = 3$ and the reflection of $y = f(x)$ through y -axis.

2. Find the area enclosed by the following curves :

$$(i) x^{2/3} + y^{2/3} = a^{2/3};$$

$$(ii) x = a \cos^3 t, y = b \sin^3 t;$$

$$(iii) \text{Hypocycloid } \left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1;$$

$$(iv) a^2 y^2 = a^2 x^2 - x^4;$$

$$(v) a^2 y^2 = x^3 (2a - x);$$

$$(vi) x = 3 \cos t, y = 2 \sin t;$$

$$(vii) y^2 = x^2 - x^4;$$

$$(viii) x(1+t^2) = 1-t^2, y(1+t^2) = 2t;$$

$$(ix) x = 3 + \cos \theta, y = 4 \sin \theta.$$

3. Find the area of the loop/loops of the following curves : (Ref. Ex. 6 worked out)

$$(i) y^2 = x^2 (4 - x^2);$$

$$(ii) ay^2 = x^2 (a - x);$$

$$(iii) 3ay^2 = x(x-a)^2;$$

$$(iv) xy^2 + (x+a)^2 (x+2a) = 0;$$

$$(v) y^2 = x^4 (4 + x);$$

4. Determine the average ordinates :

- (i) of a semi-circle of radius r ;
- (ii) of the parabola between the ordinates $x = -2$ and $x = 2$;
- (iii) of the cycloid $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$;

5. Find the area of one of the loops of :

(i) $y^2 = \frac{x^2(a^2 - x^2)}{a^2 + x^2}$;

(ii) $y^2 = \frac{x^2(a+x)}{a-x}$;

(iii) $x = a \sin 2t$, $y = a \sin t$;

(iv) $x(x^2 + y^2) = a(x^2 - y^2)$;

(v) $y^2(a^2 + x^2) = x^2(a^2 - x^2)$;

6. Determine the area between the following curves and the concerned asymptote :

(i) $x^2y^2 = a^2(y^2 - x^2)$;

(ii) $y^2(2a - x) = x^3$;

(iii) $y^2(a + x) = (a - x)^3$;

(iv) $xy^2 = (2a - x)4a^2$;

(v) $x^2(x^2 + y^2) = a^2(y^2 - x^2)$;

(vi) $x^3 = y^2(a - x)$;

(vii) $x = a \sin^3 t$, $y = \frac{a \sin^3 t}{\cos t}$.

7. Find the area enclosed by the curve $y = \tan^2 x$, x -axis and the line $x = \pi/2$.

8. Prove that the area enclosed by $a^2x^2 = y^3(2a - y)$, is equal to that of a circle of radius ' a ' .

9. Show that the ratio of the area enclosed by $a^4y^2 = x^5(2a - x)$ and that by a circle of radius ' a ' is 5 : 4 .

10. Prove that the area enclosed by the co-ordinate axes and the parabola $\sqrt{x} + \sqrt{y} = \sqrt{a}$ is $a^2/6$ sq. units.

[Hints : $x = a \sin^4 t$, $y = a \cos^4 t$ may be used for eliminating surds.

The eqn. of the parabola becomes $(x - y)^2 = a(2x - 2y - a)$. Its axis is $x - y = 0$, vertex is at $(a/4, a/4)$ and it touches the co-ordinate axes at $(a, 0)$ and $(0, a)$ respectively.]

11. Find the area of an arch of $y = \sin x$ and the x -axis.

12. Find the area of an arch of $y = \cos x$ and the x -axis.

13. Find the area of an arch of $y = \cos 3x$ and the x -axis.

Answers

1. (i) $c^2 \log \frac{b}{a}$ sq. units. (ii) $a^2 \log 2$ sq. units. (iii) 2 sq. units.
 (iv) 4 sq. units. (v) $c^2 \left(c - \frac{1}{c}\right)$ sq. units.
 (vi) (a) $(a \log a - b \log b - a + b)$ sq. units.
 (b) $(a \log a + b \log b - a - b + 2)$ sq. units.
 (vii) $\frac{3}{2} (e^{18} - e^4 + e^{-4} - e^{-18}) = 3 (\sinh 18 - \sinh 4)$ sq. units.
 (viii) $(e^5 - e)$ sq. units. (ix) 15 sq. units. (x) 56 sq. units.
 (xi) $\log 2$ sq. units. (xii) $(\pi/2)$ sq. units.
 (xiii) $\frac{b}{2a} (c \sqrt{a^2 - c^2} + a^2 \sin^{-1} \frac{c}{a})$ sq. units.
 (xiv) $\frac{\pi ab}{4} - \frac{b^2}{2a} \sqrt{a^2 - b^2} - \frac{ab}{2} \sin^{-1} \frac{\sqrt{a^2 - b^2}}{a}$ sq. units (xv) 54 sq. units.
 (xvi) $\frac{1}{2} (e^{1/2} - 1)$ sq. units. (xvii) 64 sq. units. (xviii) 8 sq. units.
 (xix) $\frac{1}{2} \log 2$ sq. units. (xx) $\frac{92}{3}$ sq. units. (xxi) 15 sq. units.
 (xxii) $\frac{32}{3}$ sq. units. (xxiii) 8 sq. units. (xxiv) $\frac{32}{3}$ sq. units.
 (xxv) 36 sq. units. (xxvi) $\frac{32}{3}$ sq. units. (xxvii) 3103 sq. units.
 (b) 15 sq. units. (c) 54 sq. units. (d) 54 sq. units. 2. (i) $3\pi a^2/8$ sq. units.
 (ii) $3\pi ab/8$ sq. units. (iii) $3\pi ab/8$ sq. units. (iv) $4a^2/3$ sq. units.
 (v) πa^2 sq. units. (vi) 6π sq. units. (vii) $4/3$ sq. units. (viii) π sq. units.
 (ix) 4π sq. units. 3. (i) $32/3$ sq. units. (ii) $8a^2/15$ sq. units.
 (iii) $\frac{8a^2}{15\sqrt{3}}$ sq. units. (iv) $2a^2(1 - \pi/4)$ sq. units. (v) $\frac{4096}{105}$ sq. units.
 4. (i) $\pi r/4$ units. (ii) $8/3$ units. (iii) $3a/2$ units.
 5. (i) $(\pi - 2)a^2/2$ sq. units. (ii) $2a^2(1 - \pi/4)$ sq. units.
 (iii) $4a^2/3$ sq. units. (iv) $2a^2(1 - \pi/4)$ sq. units.
 (v) $(\pi - 2)a^2/2$ sq. units. 6. (i) $4a^2$ sq. units. (ii) $3\pi a^2$ sq. units.

(iii) $3\pi a^2$ sq. units. (iv) $4\pi a^2$ sq. units. (v) $a^2(\pi + 2)$ sq. units.

(vi) a^2 sq. units. (vii) $3\pi a^2/4$ sq. units. 7. not defined 11. 2 sq. units.

12. 2 sq. units. 13. $2/3$ sq. units.

11.6. AREA ENCLOSED BY TWO CURVES

Let the equation of the curves AB and CD be $y = f(x)$ and $y = \phi(x)$ respectively and ACL and BDM be the ordinates parallel to y -axis at $x = a$ and $x = b$ respectively (fig. 28).

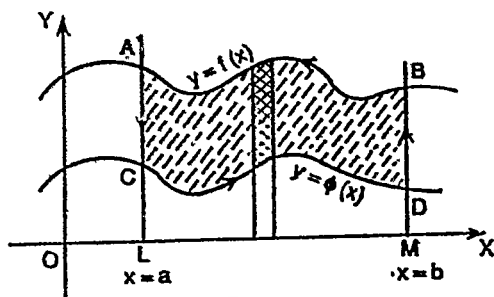


Fig. 28

The area $CDBA$ is to be determined. Therefore,

Area $CDBA$ = (Area $LMBA$) - (Area $LMDC$)

$$\begin{aligned} &= \int_a^b f(x) dx - \int_a^b \phi(x) dx, \text{ by Sec. 11.2} \\ &= \int_a^b [f(x) - \phi(x)] dx. \end{aligned}$$

If $AC = BD = 0$ i.e. AB and CD intersect at A and B , then a and b becomes the abscissa of A and B . So, (a, c) and (b, d) be the points of intersection of the curves and the included area

$$= \int_a^b [f(x) - \phi(x)] dx.$$

If the eqn. of the curves are $x = f_1(y)$ and $x = \phi_1(y)$, the area will be

$$\int_c^d [f_1(y) - \phi_1(y)] dy.$$

Hence to determine the area between two intersecting curves, the point of intersection of the curves is to be determined first and either of the above two formulae is to be used according as the interval of x or y .

N.B. 11.4 (vii) is to be considered carefully, so that, with minimum labour the integration can be performed. Note that integrand is (outer curve - inner curve).

Examples

1. Find the area interior to $y^2 = 2ax - x^2$ and exterior to $y^2 = ax$ lying in the first quadrant. Hence find the corresponding total area.

The first curve is a circle with centre at $(a, 0)$ and radius ' a '. The second curve is a parabola with vertex at $(0, 0)$ and focus at $(a/4, 0)$. Both the circle and the parabola are symmetrical about x -axis. They intersect at $(0, 0)$ and $(a, \pm a)$. So, the interval of x is $[0, a]$ (Fig. 29).

Therefore, the area included between the curves in the first quadrant

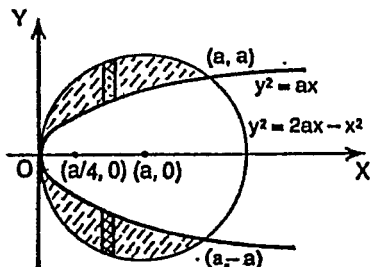


Fig. 29

$$= \int_0^a [f(x) - \varphi(x)] dx$$

$$= \int_0^a [\sqrt{2ax - x^2} - \sqrt{ax}] dx$$

$$= \int_0^a \sqrt{2ax - x^2} dx - \int_0^a \sqrt{ax} dx$$

$$= I_1 - I_2$$

Now,
$$I_1 = \int_0^a \sqrt{2ax - x^2} dx = \int_0^a \sqrt{a^2 - (a - x)^2} dx$$

$$= a^2 \int_0^{\pi/2} \cos^2 \theta d\theta \text{ when } a - x = a \sin \theta$$

$$= \pi a^2 / 4 \text{ sq. units.}$$

$$I_2 = \int_0^a \sqrt{ax} dx = \frac{2}{3} a^2 \text{ sq. units.}$$

The reqd. area $= I_1 - I_2 = a^2 (\pi/4 - 2/3)$ sq. units. And the total area $= 2a^2 \left(\frac{\pi}{4} - \frac{2}{3} \right)$ sq. units.

2. Find the area enclosed by the parabola $y^2 = 8x$ and the straight line $4x - y - 4 = 0$.

The parabola and the straight line intersect at $(2, 4)$ and $(1/2, -2)$.

Method 1. If elementary area or strip is taken perpendicular to y -axis, it covers the whole area as it rolls from the minimum point $(1/2, -2)$ to the maximum points $(2, 4)$ (fig. 30). Required area

$$= \int_{-2}^4 \left[\frac{1}{4}(y+4) - \frac{1}{8}y^2 \right] dy$$

[$\because$ outer curve is the straight line. 11.6 N.B.]

$$= \frac{9}{2} \text{ sq. units.}$$

Method 2. If the elementary area is taken perpendicular to x -axis, its rolling from $(0, 0)$ to $(2, 4)$ divides the area into two by the line $x = 1/2$, instead of covering the whole area (fig. 31). The area in left of the line $x = 1/2$ is the area under the parabola only from $x = 0$ to $x = 1/2$ while the other to the right of $x = 1/2$ is the area between the given parabola & the given line from $x = 1/2$ to $x = 2$ and it involves two integration. So, it is more laborious than method-1 and so method-1 is preferable. Now the area here

$$\begin{aligned} &= \frac{1}{2} \int_0^{1/2} \sqrt{8x} \, dx \\ &+ \int_{1/2}^2 (\sqrt{8x} - 4(x-1)) \, dx \\ &= \frac{9}{2} \text{ sq. units.} \end{aligned}$$

3. Find area enclosed by the semi-cubical parabola $y^3 = x^2$ and its chord joining $(-1, 1)$ and $(8, 4)$.

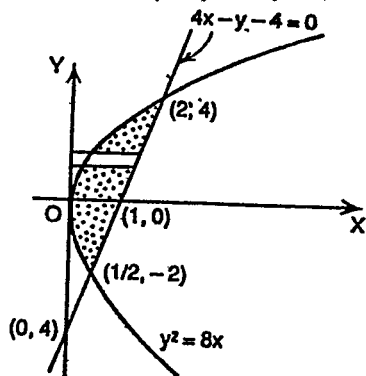


Fig. 30

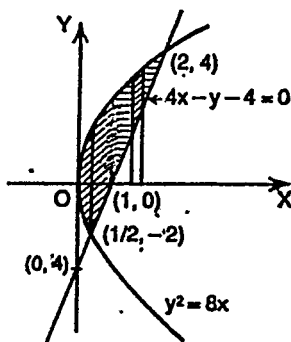


Fig. 31

The semi-cubical parabola is symmetrical about y -axis with vertex at origin and lies wholly above the x -axis. The eqn. of the chord is $y - 1 = \frac{1}{3}(x + 1)$ or $y = \frac{1}{3}(x + 4)$

(Fig. 32)

Reqd. area = Area AOB

$$= \int_{-1}^8 \left[\frac{1}{3}(x + 4) - x^{2/3} \right] dx$$

$$= 22.5 - 19.8 = 2.7 \text{ sq. units.}$$

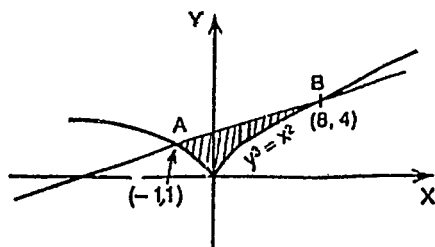


Fig. 32

Exercises 11 (2)

1. Find the area enclosed by $y^2 = 4ax$ and $x^2 = 4ay$.

2. Find the area of the lower portion of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

intercepted by the line $y = c$.

3. Find the larger area enclosed by the circle $x^2 + y^2 = 64a^2$ and the parabola $y^2 = 12ax$.

4. Determine the area between the ellipses $x^2 + 2y^2 = a^2$ and $2x^2 + y^2 = a^2$.

5. Find the area interior to the curves $y = x^2$ and $y = 4x - x^2$.

6. Determine the area included between $y = ax$ and $x^2 + y^2 = 4ax$.

7. Determine the area enclosed by the parabola $x^2 = 4ay$ and the witch of agnesi $y = 8a^3/(x^2 + 4a^2)$.

8. Determine the area of the parabola $y^2 = 2x$ intercepted by the line $y = 4x - 1$.

9. What is the area included between the parabolas $y = 6x - x^2$ and $y = x^2 - 2x$.

10. Find the common area between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 4x$.

11. Find the area of the region bounded by y -axis, the lines $y = a$, $y = 2a$ and the semi-cubical parabola $ay^2 = x^3$.

12. Determine the area bounded by the curves $y\sqrt{4x - x^2 + 1} = x^3$ and $y\sqrt{4x - x^4 + 1} = 1$ in the first quadrant.

13. Determine the area bounded by $y\sqrt{x^3 - x} = 3x^2$ and $y\sqrt{x^3 - x} = 1$ in the first quadrant from $x = 2$ to $x = 3$.

14. Determine the area bounded by $y(e^x - 1) = e^x + 1$ and $y(e^x - 1) = 1$ in the first quadrant from $x = 1$ to $x = 2$.

15. Find the area of the parabola $y = 6 + 4x - x^2$ intercepted by its chord joining the points $(-2, -6)$ and $(4, 6)$.

16. Find the area enclosed by $x^2 + 4y^2 = 5$ and $xy = 1$ in the first quadrant.

17. Find the area bounded by the curves $y = 8 + 4x - 2x^2 - x^3$ and $y = x + 8$.

18. Find the area bounded by $y = x^2$, $y = 2x$ and $y = 2$ lying below the third curve.

19. Find the area between $y = \sin x$ and $y = \cos x$ from the first intersection to the left of y -axis to that in the right.

$$[\text{Hints : } \int_{-3\pi/4}^{\pi/4} (\cos x - \sin x) dx]$$

20. Determine the area bounded by $y\sqrt{x^4 - x^2 + 1} = 4x^3$ and $y\sqrt{x^4 - x^2 + 1} = 2x$ in the first quadrant.

21. Find the area of a portion of the ellipse $13x^2 + 10xy + 13y^2 = 72$ divided by the y -axis.

[Hints : Decide the two roots of y as y_1 & y_2 . Then find the area under $(y_1 - y_2)$ from $x = +\sqrt{(13/2)}$ to $x = -\sqrt{(13/2)}$.

22. Find the area interior to $y = \operatorname{cosec} x$ and $y = \cot x$ from $x = 0$ to $x = \pi/2$.

Answers

1. $16a^2/3$ sq. units.

2. $\left\{ \pi ab \mp \left(\frac{ac}{b} \sqrt{b^2 - c^2} + ab \sin^{-1} \left(\frac{\sqrt{b^2 - c^2} \cdot c}{b} \right) \right\}$ sq. units.

3. $\frac{16a^2}{3} (8\pi - \sqrt{3})$ sq. units. 4. $2a^2 \sqrt{2} \sin^{-1} \frac{1}{\sqrt{3}}$ sq. units. 5. $8/3$ sq. units.

6. $a^2(3\sqrt{3} + 4\pi/3)$ sq. units. 7. $2a^2(3\pi - 2)/3$ sq. units.

8. $9/32$ sq. units. 9. $64/3$ sq. units. 10. $(8\pi - 6\sqrt{3})/3$ sq. units.

11. $1.304a^2$ sq. units. 12. $\frac{1}{2}$ sq. units. 13. $2(\sqrt{3} - \sqrt{2})$ sq. units.

14. $\log(1 + e)$ sq. units. 15. 36 sq. units. 16. $\left(\frac{5}{4} \sin^{-1} \frac{3}{5} - \log 2 \right)$ sq. units.

17. $71/6$ sq. units. 18. $(4\sqrt{2} - 3)/3$ sq. units. 19. $2\sqrt{2}$ sq. units.

20. $(4 - 2\sqrt{3})$ sq. units. 21. 3π sq. units. 22. $\log 2$ sq. units.

11.7. AREA UNDER CURVES IN POLAR CO-ORDINATES.

Theorem. If $r = f(\theta)$ is a single valued continuous function of θ in the interval $[\alpha, \beta]$, then area enclosed by $r = f(\theta)$ and $\theta = \alpha$ to $\theta = \beta$, ($\alpha > \beta$)

is $\frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta$.

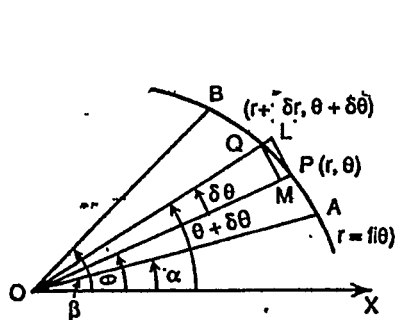


Fig. 33

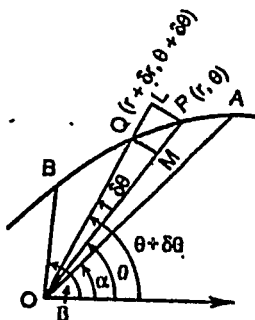


Fig. 34

Proof : Let $r = f(\theta)$, the lines OA and OB at $\theta = \alpha$ and $\theta = \beta$ enclose the area $OABO$. Let $P(r, \theta)$ be a point on monotonically increasing (fig. 33) or decreasing (fig. 34) curve $r = f(\theta)$. Then area $OAP O$ is a function of θ and let it be $A(\theta)$. Let $Q(r + \delta r, \theta + \delta \theta)$ be a near-by point of $P(r, \theta)$. Then area $OPQO = \delta A$. Let the arcs with radii OP and OQ be drawn such that they intersect OQ or extended OQ and OP or extended OP at L and M respectively.

Since θ is measured in radian. So, by def. of radian,

$$\frac{\text{sector area } OPL}{\text{area of the circle, } \pi r^2} = \frac{\delta \theta}{2\pi}$$

$$\text{sector area } OPL = \frac{1}{2} r^2 \delta \theta$$

Similarly, sector area $OMQ = \frac{1}{2} (r + \delta r)^2 \delta \theta$

The area $OPQO$ lies between the sector areas OPL and OMQ . i.e.

δA lies between $\frac{1}{2} r^2 \delta \theta$ and $\frac{1}{2} (r + \delta r)^2 \delta \theta$

i.e. $\frac{\delta A}{\delta \theta}$ lies between $\frac{1}{2} r^2$ and $\frac{1}{2} (r + \delta r)^2$

As $\delta \theta \rightarrow 0$, δr also $\rightarrow 0$. Therefore,

$$\lim_{\delta\theta \rightarrow 0} \frac{\delta A}{\delta\theta} = \frac{dA}{d\theta} = \frac{1}{2}r^2, \therefore \lim_{\delta\theta \rightarrow 0} \frac{1}{2}(r + \delta r)^2 = \frac{1}{2}r^2$$

$$\begin{aligned} \therefore \int_{\alpha}^{\beta} \frac{1}{2}r^2 d\theta &= \int_{\alpha}^{\beta} \frac{dA}{d\theta} \cdot d\theta = [A]_{\theta=\alpha}^{\theta=\beta} \\ &= A \Big|_{\theta=\beta} - A \Big|_{\theta=\alpha} \\ &= \text{area } OABO - 0 = \text{area } OABO. \end{aligned}$$

$$\therefore \text{area of } OABO = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta.$$

Extension :

If $r = f(\theta)$ is wavy instead of monotonically increasing or decreasing as in fig. 35, then the above theorem is also applicable after dividing the area into smaller areas satisfying the above conditions e.g. area $OABO = \text{area } OAP_1O + \dots + \text{area } OP_nBO$. This formula for area is valid for any single valued function $r = f(\theta)$.

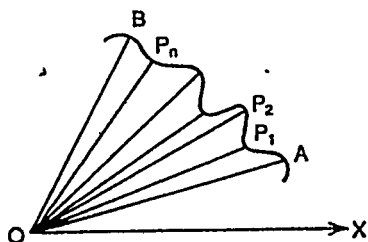


Fig. 35

11.8. PROCEDURE FOR DETERMINING AREA (POLAR)

As in sec. 11.4, the curve $r = f(\theta)$ is to be drawn and limits $\theta = \alpha$ and $\theta = \beta$ are to be decided for minimum and maximum value of $|r|$. It is not necessary that lower and upper limits should correspond to minimum & maximum value of $|r|$ respectively. In drawing the graph the following should be carefully noted:

(i) In polar co-ordinates the initial line OX rotates from 0° to 360° in the anticlockwise direction about the pole (initial point), O with different radii giving different polar points (r_i, θ_i) , $i = 1, 2, 3, \dots$. If $\theta = \alpha$ produces a positive r then the radius vector is measured $|r|$ units along the direction of the line $\theta = \alpha$ from the pole whereas if it produces a negative r then it is measured $|r|$ units in the opposite direction of the line $\theta = \alpha$ from the pole as in fig. 36.

(ii) Equation of some standard curves are given below for ready reference.

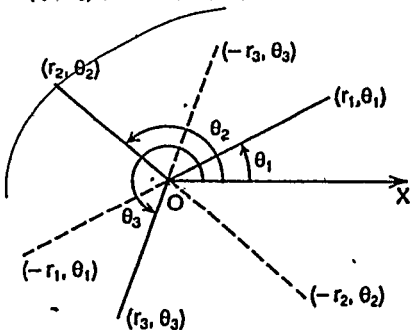


Fig. 36

(a) $r = a$ is eqn. of a circle with centre at the origin and radius 'a'.

(b) $\theta = a\pi$ represents the eqn. of a line inclined at an angle ' $a\pi$ ' with the initial line,

(c) $r = \theta, \theta \geq 0$ represents the eqn. of a spiral.

(d) $r = a(1 + \sin \theta), a(1 - \cos \theta)$ are the eqns. of cardioids.

(e) $r = a \cos n\theta$ or $a \sin n\theta$ is the eqn. of rose petal

(f) $r = a + b \sin \theta$ or $a + b \cos \theta$ is the eqn. of a limaçon.

(g) $r = \frac{ed}{1 - e \cos \theta}$ represents ellipse if $e < 1$, parabola if $e = 1$ and hyperbola if $e > 1$ and $x = -d, d > 0$ is the equation of directrix, perpendicular to x -axis. If directrix is perpendicular to y -axis, then

$$r = \frac{ed}{1 - e \sin \theta}.$$

(h) $r^2 = 2a^2 \cos 2\theta$ is the eqn. of cylinder, generating lines are parallel to z -axis.

(iii) Symmetry is again very important to calculate area as in cartesian co-ordinates. Some tests of symmetry are the following :

(a) If θ is replaced by $(-\theta)$ in $r = f(\theta)$ and the eqn. remains unchanged, then the curve is symmetrical about the initial line.

(b) Either θ is replaced by $(\pi - \theta)$ or r by $(-r)$ and θ by $(-\theta)$ simultaneously and the eqn. remains unchanged then the curve is symmetrical about $\theta = \pi/2$. Similarly if θ is replaced by $(n\pi - \theta)$ and the eqn. remains unchanged then the curve is symmetrical about $\theta = \frac{n\pi}{2}$.

(c) Either θ is replaced by $(\theta + \pi)$ or r by $(-r)$ and the eqn. remains unchanged then the curve is symmetrical about the pole.

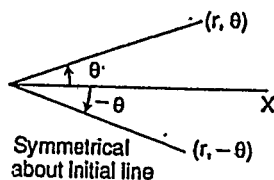


Fig. 37 (i)

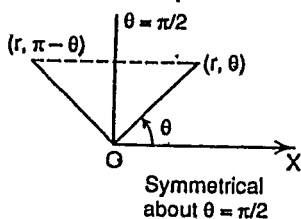


Fig. 37 (ii)

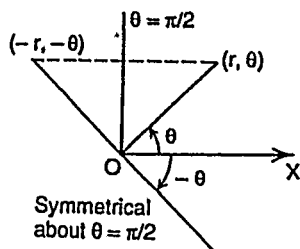


Fig. 37 (iii)

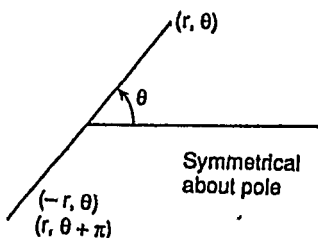


Fig. 37 (iv)

These symmetries are shown pictorially in figs. 37 (i)—37 (iv).

(iv) Elementary sector area is to be shown and selected suitably.

(v) It is sometimes convenient to study the nature of the curve by expressing polar into cartesian and parametric and vice-versa. The relationships of cartesian and polar are

$$x = r \cos \theta, y = r \sin \theta \text{ and } r = \sqrt{x^2 + y^2}, \theta = \tan^{-1} \frac{y}{x}.$$

11.9. AREA BETWEEN TWO POLAR CURVES

Position of the pole plays an important role in the determination of the area between two curves in polar co-ordinates. This is not necessary in cartesian co-ordinates.

(i) Pole is interior to area : If the pole is within the area $ARBSA$ (fig. 38) where $r_1 = f_1(\theta)$ denotes the curve ASB and $r_2 = f_2(\theta)$ denotes the curve ARB intersecting at $A : \theta = \alpha$ and $B : \theta = \beta$, then the two areas $ARBOA$ and $AOBSA$ are calculated

separately using $\frac{1}{2} \int_{\alpha}^{\beta} r_2^2 d\theta$ and

$\frac{1}{2} \int_{\beta}^{\alpha} r_1^2 d\theta$ and added together.

This is because no suitable common elementary sector area can be constructed, so that its movement from $\theta = \alpha$ to $\theta = \beta$ covers the whole area.

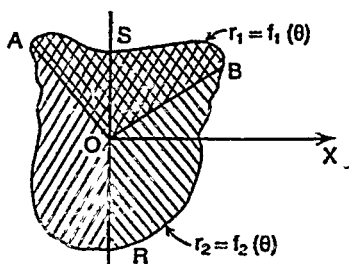


Fig. 38

(ii) Pole is exterior to area : In this case a common elementary sector area can be constructed (fig. 39)

and the required area is $\frac{1}{2} \int_{\alpha}^{\beta} (r_1^2 - r_2^2) d\theta$ as in cartesian co-ordinates.

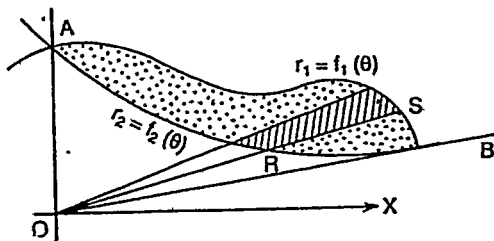


Fig. 39

11.10. AREA OF A SINGLE CLOSED CURVE

Area of a single closed curve is determined as in sec. 11.9 after careful observation of the position of the pole. There arises two cases :

(i) **Pole interior to the curve :** If pole lies within curve (fig. 40)

$$\text{then the area} = \frac{1}{2} \int_0^{2\pi} r^2 d\theta.$$

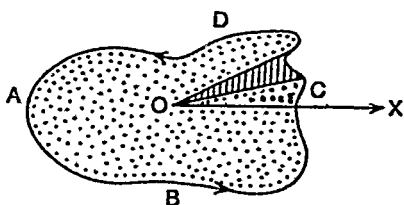


Fig. 40

(ii) **Pole exterior to the curve :** If the pole lies outside the curve (fig. 41) then the area

$$= \frac{1}{2} \int_{\alpha}^{\beta} (r_1^2 - r_2^2) d\theta, \text{ where } r_1 = f_1(\theta)$$

and $r_2 = f_2(\theta)$ are the two portions ABC and CDA of the closed curve $ABCD$ corresponding to the minimum and maximum variation of θ .

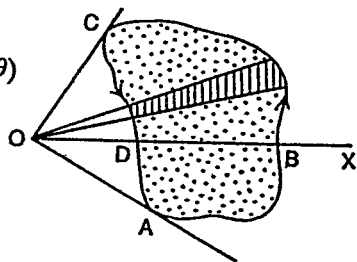


Fig. 41

Parametric form of the area of a single closed non-self-intersecting curve can be obtained easily. Let $x = f_1(t)$, $y = f_2(t)$ be the parametric form of the equation of the curve. From the relation

$$x = r \cos \theta, \quad y = r \sin \theta,$$

$$\frac{dx}{dt} = \frac{dr}{dt} \cos \theta - r \sin \theta \frac{d\theta}{dt}$$

$$\frac{dy}{dt} = \frac{dr}{dt} \sin \theta + r \cos \theta \frac{d\theta}{dt}$$

$$\therefore x \frac{dy}{dt} - y \frac{dx}{dt} = r^2 \frac{d\theta}{dt}$$

$$\therefore \text{area} = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta = \frac{1}{2} \int_{t_1}^{t_2} \left(x \frac{dy}{dt} - y \frac{dx}{dt} \right) dt$$

$$= \frac{1}{2} \int_{t_1}^{t_2} (x dy - y dx).$$

If the curve is self-intersecting (fig. 42) then care should be taken as in sec. 11.3, because of the appearance of both negative and positive area in a single case. So, area of each loop (area enclosed by self-intersection) is to be calculated separately and absolute values are to be added to get the total area.

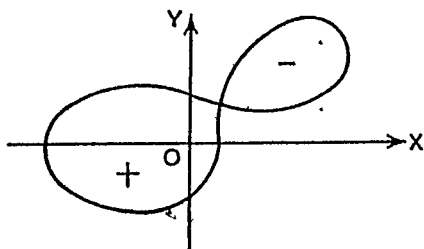


Fig. 42

11.11. APPROXIMATE INTEGRATION

Sometimes evaluation of the definite integral $\int_a^b f(x) dx$ in closed form becomes difficult or even impossible. Then approximate values are calculated by approximation procedures. Of these procedures only two common procedures—(i) trapezoidal rule and (ii) Simpson's rule will be discussed here.

(i) **Trapezoidal rule :** This rule divides the interval $(b - a)$ into n equal segments of length Δx by the points $x_0 (=a), x_1, x_2, \dots, x_n (=b)$. At these points ordinates $y_0 = f(x_0), \dots, y_n = f(x_n)$ are erected and joining the neighbouring ordinates pair by pair trapeziums are constructed (fig. 43). As the area of a trapezium $= \frac{1}{2} (\text{sum of parallel sides}) \times (\text{distance between parallel sides})$.

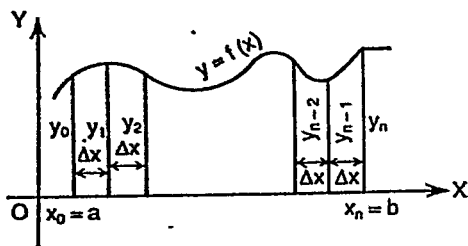


Fig. 43

Therefore, area of the first trapezium $= \frac{1}{2} (y_0 + y_1) \Delta x$

Area of the second trapezium $= \frac{1}{2} (y_1 + y_2) \Delta x$

" " " third " " $= \frac{1}{2} (y_2 + y_3) \Delta x$

... ..

... ..

$$\text{---} \quad \text{---} \quad (n-1) \text{th} \quad \text{---} \quad \text{---} = \frac{1}{2} (y_{n-2} + y_{n-1}) \Delta x$$

$$\text{---} \quad \text{---} \quad n\text{th} \quad \text{---} \quad \text{---} = \frac{1}{2} (y_{n-1} + y_n) \Delta x.$$

$$\begin{aligned} \text{So, their sum} &= (\text{Approximate value of}) \int_a^b f(x) dx \\ &= (\text{Approx. value of the area enclosed by} \\ &\quad y = f(x), x = a, x = b \text{ and } x\text{-axis}) \\ &= \frac{1}{2} (y_0 + 2y_1 + 2y_2 + \dots + 2y_{n-1} + y_n) \Delta x \end{aligned}$$

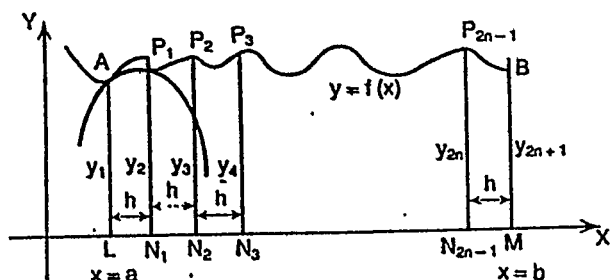
This approximate value will be more and more accurate as the number of these trapeziums increases. Since the approx. value is determined by means of the area of trapezium, hence the name trapezoidal rule.

(ii) **Simpson's or parabolic rule:** If the interval $[a, b]$ is divided into n pairs of subintervals and ordinates are erected at those points of divisions including the end points 'a' and 'b', then the approximate value

of $\int_a^b f(x) dx = \frac{h}{3} \{(\text{sum of first \& last ordinates}) + 2(\text{sum of the odd ordinates}) + 4(\text{sum of even ordinates})\}$ i.e. if the interval $(b-a)$ is divided into $2n$ equal parts of length $h = \frac{b-a}{2n}$ by the points $x = a, a+h, \dots, a+2nh = b$ and the corresponding ordinates are $y_1, y_2, \dots, y_{2n+1}$, then approx. value of

$$\int_a^b f(x) dx = \frac{h}{3} \left[(y_1 + y_{2n+1}) + 2(y_3 + y_5 + \dots + y_{2n-1}) + 4(y_2 + y_4 + \dots + y_{2n}) \right]$$

Proof. Let the equation $y = f(x)$ represents the curve AB and the ordinates at $x = a$ and $x = b$ be LA and MB respectively. Let the length LM be divided into $2n$ equal parts of length $h = \frac{b-a}{2n}$ by the points $N_1, N_2, \dots$ etc. Let $N_1P_1, N_2P_2, \dots$ be the ordinates at those points (fig. 44). Then



$$LA=y_1, N_1 P_1=y_2, N_2 P_2=y_3, \dots, N_{2n-1} P_{2n-1}=y_{2n}, BM=y_{2n+1}.$$

Let a parabola be constructed through A, P_1, P_2 such that its axis is parallel to y -axis and let it corresponds to the ordinate at N_1 i.e. $N_1 P_1$.

Considering N_1 as origin and $N_1 P_1$ as y -axis. Let the equation of the parabola be $y = A + Bx + Cx^2$. (1)

Now area under this parabola (1), the x -axis, the ordinates LA & $N_2 P_2$ is

$$\begin{aligned} \int_{-h}^h (A + Bx + Cx^2) dx &= \left[Ax + \frac{Bx^2}{2} + \frac{Cx^3}{3} \right]_{-h}^h \\ &= \left(Ah + \frac{Bh^2}{2} + \frac{Ch^3}{3} \right) - \left(-Ah + \frac{Bh^2}{2} - \frac{Ch^3}{3} \right) \\ &= 2Ah + \frac{2Ch^3}{3} = 2h \left(A + \frac{Ch^2}{3} \right) \end{aligned} \quad (2)$$

$\therefore$ the points $A(-h, y_1), P_1(0, y_2)$ and $P_2(h, y_3)$ lie on parabola (1). So,

$$y_1 = A - Bh + Ch^2 \quad (3)$$

$$y_2 = A \quad (4)$$

$$y_3 = A + Bh + Ch^2 \quad (5)$$

$$\text{Solving (3), (4), (5), } A = y_2, C = \frac{y_1 - 2y_2 + y_3}{2h^2}$$

$$\begin{aligned} \therefore \text{ from (2), area} &= 2h \left(y_2 + \frac{y_1 - 2y_2 + y_3}{6} \right) = \frac{h}{3} (y_1 + 4y_2 + y_3) \\ &= (\text{area of the first two segments between the} \\ &\quad \text{ordinates } y_1 \text{ \& } y_3). \end{aligned}$$

Similarly, area of the second two segments between the ordinates y_3 and $y_5 = \frac{h}{3} (y_3 + 4y_4 + y_5)$,

area of the third two segments between the ordinates y_5 and y_7

$$= \frac{h}{3} (y_5 + 4y_6 + y_7),$$

$$\dots \dots \dots$$

area of the n th segments between the ordinates y_{2n-1} and y_{2n+1}

$$= \frac{h}{3} (y_{2n-1} + 4y_{2n} + y_{2n+1})$$

Adding, (the approximate area under the curve, x-axis and the ordinates at $x = a$ and $x = b$) = (Approx. value of $\int_a^b f(x) dx$)

$$= \frac{h}{3} \left[(y_1 + y_{2n+1}) + 2(y_3 + y_5 + \dots + y_{2n-1}) + 4(y_2 + y_4 + \dots + y_{2n}) \right]$$

As the parabola is used to deduce this result, hence the method is parabolic rule.

Examples

1. Find the area of the limacon $r = a \cos \theta \mp b$ where a and b are positive.

The given eqn. remains unchanged when $(-\theta)$ replaces θ . So, the limacon is symmetrical about the initial line. There arise two cases (i) when $a \leq b$, (ii) when $a > b$.

Case (i) : $a \leq b$. The eqn. $r = a \cos \theta \mp b$ describes the outer curve (fig. 45) when $a \leq b$. For +ve sign, the eqn. describes upper part ACO when θ varies from 0 to π and for -ve sign, it describes the lower part $OC'A$. Again $|r|$ is maximum at $\theta = 0$ and minimum at $\theta = \pi$.

$$\begin{aligned} \therefore \text{reqd. area} &= 2 \cdot \frac{1}{2} \int_0^\pi r^2 d\theta = \int_0^\pi (a \cos \theta \mp b)^2 d\theta \\ &= \frac{\pi}{2} (2b^2 + a^2) \quad \text{sq. units.} \end{aligned}$$

Case (ii) : $a > b$.

When $a > b$, the eqn. describes two loops. The outer loop is described by $r = a \cos \theta + b$ and the inner loop by $r = a \cos \theta - b$. The outer loop

is maximum at $\theta = 0$ and minimum at

$\theta = \pi - \cos^{-1} \frac{b}{a} = \theta - \alpha$ (say) i.e. as

θ varies from 0 to $(\theta - \alpha)$ the eqn.

describes the outer curve ACO . The

inner loop is maximum at $\theta = \alpha$ and

minimum at $\theta = 0$ i.e. as θ varies from

0 to α the eqn. describes the portion

BDO .

$\therefore$ area of the two loops

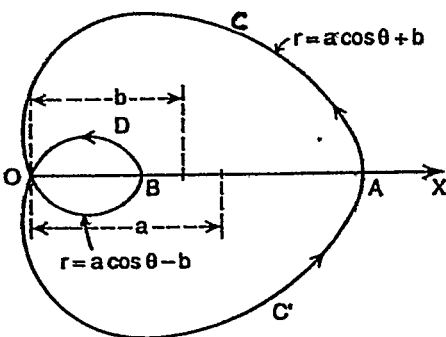


Fig. 45

$$= \int_0^{\pi-\alpha} (a \cos \theta + b)^2 d\theta + \int_0^{\alpha} (a \cos \theta - b)^2 d\theta$$

Let $\theta = \pi - \varphi$ for the second integral.

$$\begin{aligned} \therefore \int_0^{\alpha} (a \cos \theta - b)^2 d\theta &= - \int_{\pi}^{\pi-\alpha} (a \cos \varphi + b)^2 d\varphi \\ &= \int_{\pi-\alpha}^{\pi} (a \cos \theta + b)^2 d\theta \end{aligned}$$

$$\begin{aligned} \therefore \text{the reqd. area} &= \int_0^{\pi-\alpha} (a \cos \theta + b)^2 d\theta + \int_{\pi-\alpha}^{\pi} (a \cos \theta + b)^2 d\theta \\ &= \int_0^{\pi} (a \cos \theta + b)^2 d\theta = \frac{\pi}{2} (2b^2 + a^2) \text{ sq. units.} \end{aligned}$$

2. Find the area interior to the outer and exterior to the inner loops of the limaçon $r = a(1 - 2 \cos \theta)$.

To sketch the curve, let us tabulate some values of r & θ as follows and plot the points in figure 46.

θ	0	$\pi/12$	$\pi/6$	$\pi/4$	$\pi/3$	$5\pi/12$	$\pi/2$	π	$5\pi/3$	$7\pi/4$	2π
r	$-a$	$-9a$	$-7a$	$-5a$	0	$5a$	a	$3a$	0	$-4a$	$-a$

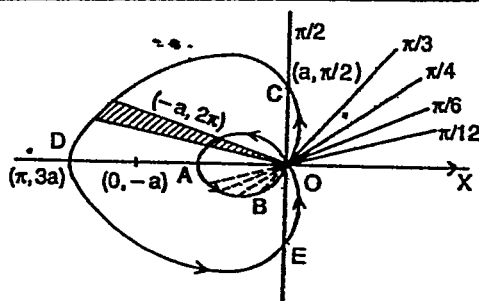


Fig. 46

As θ varies from 0 to $\frac{\pi}{3}$, the eqn. describes the lower part ABO of the inner loop and when θ varies from $\frac{\pi}{3}$ to π , it describes upper part OCD of the outer loop. Similarly, as θ varies from π to $\frac{5\pi}{3}$ the eqn. describes the lower part DEO of the outer loop and lastly it describes the upper part of the inner loop.

Sine the curve is symmetrical about the initial line. Therefore, the area $ABOFA$ of the inner loop,

$$A_1 = 2 \int_0^{\pi/3} \frac{1}{2} r^2 d\theta = a^2 \int_0^{\pi/3} (1 - 2 \cos \theta)^2 d\theta = a^2 \left(\pi - \frac{3\sqrt{3}}{2} \right) \text{ sq. units.}$$

and area $OCDEO$ of the outer loop,

$$\begin{aligned} A_2 &= 2 \int_{\pi/3}^{\pi} \frac{1}{2} r^2 d\theta = a^2 \int_{\pi/3}^{\pi} (1 - 2 \cos \theta)^2 d\theta \\ &= a^2 \left(2\pi + \frac{3\sqrt{3}}{2} \right) \text{ sq. units.} \end{aligned}$$

$\therefore$ the reqd. area, $A = A_2 - A_1$

$$= a^2 (\pi + 3\sqrt{3}) \approx 8.34 a^2 \text{ sq. units.}$$

3. Find the area of a loop of $r = a \cos 2\theta$. Hence find the total area of all the loops.

The given eqn. remains unchanged on the replacement of θ by $(-\theta)$ and $\pi - \theta$. So, the curve is symmetrical about the initial line $\theta = 0$ and the line $\theta = \frac{\pi}{2}$. Also r is minimum when $\theta = \frac{\pi}{4}$ and maximum

when $\theta = 0$. So, as θ increases from 0 to $\frac{\pi}{4}$, r decreases from 'a' to '0' describing the part AP_1O . Similarly, as

θ varies from $\frac{\pi}{4}$ to $\frac{3\pi}{4}$, the eqn. describes $OP_2B'P_3O$ with the negative values of r . Again as θ increases from $\frac{3\pi}{4}$ to $\frac{5\pi}{4}$ the eqn. describes $OP_4A'P_5O$ with positive values of r . In this way θ increases from $\frac{5\pi}{4}$ to $\frac{7\pi}{4}$ and r again becomes negative describing OP_6BP_7O . Ultimately when θ increases from $\frac{7\pi}{4}$ to 2π the eqn. describes OP_8A .

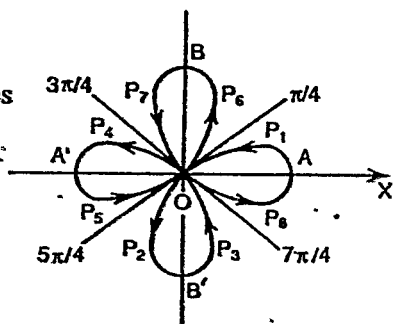


Fig. 47

So, the curve consists of four equal loops (fig. 47). Therefore, area of a loop = 2 (area of AP_1O)

$$= 2 \cdot \frac{1}{2} \int_0^{\pi/4} r^2 d\theta = a^2 \int_0^{\pi/4} \cos^2 2\theta d\theta = \pi a^2 / 8 \text{ sq. units.}$$

$$\therefore \text{total area of 4 loops} = 4 \cdot \frac{\pi a^2}{8} = \frac{\pi a^2}{2} \text{ sq. units.}$$

N.B. In general eqn. $r = a \sin n\theta$, $r = a \cos n\theta$ represents the curves known as **rose petals**. Each of them consists of $2n$ or n loops according as n is even or odd. So, each quadrant is divided into n parts and each petal will be equally spaced starting from the first symmetry for odd n and for even n , the petals will be symmetrical about the dividing lines.

4 Find the area of a loop of the curve $x^4 + y^4 = 2a^2xy$. Hence find the total area enclosed by the curve.

The given eqn. is of 4th degree. It is convenient to study the problem if it is expressed into polar form. Let $x = r \cos \theta$ and $y = r \sin \theta$. Therefore, the equivalent eqn. in polar form is

$$r^2 = \frac{2a^2 \sin \theta \cos \theta}{\cos^4 \theta + \sin^4 \theta}$$

This eqn. remains unchanged if r and θ are replaced by $(-r)$ and $(\frac{\pi}{2} - \theta)$. $\therefore$ the curve is symmetrical about the pole and the line $\theta = \pi/4$. Therefore, the curve lies in the first and the third quadrant. Also when

θ	0	$\pi/4$
r	0	$\pm \sqrt{2} a$

Therefore, curve passes the origin twice and crosses $\theta = \pi/4$ at two different points. Hence the curve forms loops about the line of symmetry $\theta = \pi/4$ and the loops lie in first and third quadrant respectively (fig. 48). Therefore, area of the loop in the first quadrant

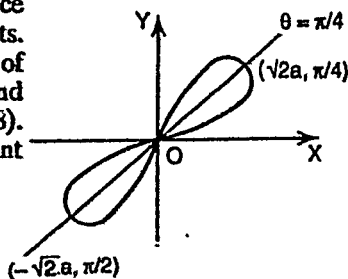


Fig. 48

$$= \frac{1}{2} \int_0^{\pi/2} r^2 d\theta$$

$$= a^2 \int_0^{\pi/2} \frac{\sin \theta \cos \theta}{\cos^4 \theta + \sin^4 \theta} d\theta$$

$$= a^2 \int_0^{\infty} \frac{u du}{1 + u^4}, \text{ when } u = \tan \theta$$

$$= \frac{a^2}{2} \left[\tan^{-1} u^2 \right]_0^{\infty} = \frac{\pi a^2}{4} \text{ sq. units.}$$

5. Determine the area inside the circle $r = \sin \theta$ and outside the cardioid $r = 1 - \cos \theta$.

$$\because r = \sin \theta \text{ and } r = 1 - \cos \theta$$

$$\therefore \sin \theta = 1 - \cos \theta \text{ or } \sin^2 \theta = (1 - \cos \theta)^2$$

$$\text{or } 2 \cos \theta (\cos \theta - 1) = 0$$

$$\therefore \text{either } \cos \theta = 0 \text{ or } \cos \theta - 1 = 0$$

$$\therefore \theta = \frac{\pi}{2} \text{ or } \theta = 0.$$

So, the points of intersection of the given circle and cardioid are $(0, 0)$ and $(1, \pi/2)$ and the reqd. area lies in $0 \leq \theta \leq \pi/2$ (fig. 49).

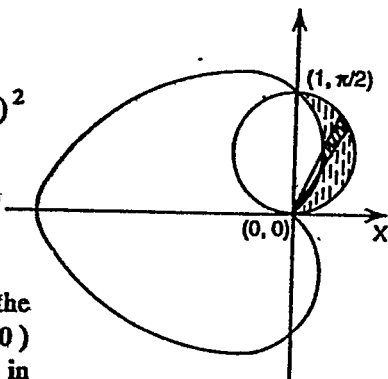


Fig. 49

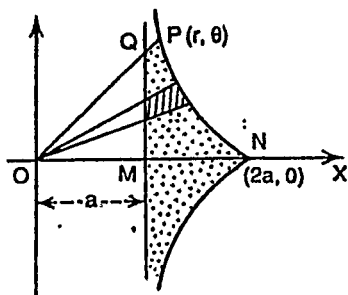
$$\begin{aligned} \text{The reqd. area} &= \int_0^{\pi/2} \frac{1}{2} (r_1^2 - r_2^2) d\theta \\ &= \frac{1}{2} \int_0^{\pi/2} [\sin^2 \theta - (1 - \cos \theta)^2] d\theta \\ &= (1 - \pi/4) \text{ sq. units.} \end{aligned}$$

6. Find the area between the curve $r = a(\sec \theta + \cos \theta)$ and its asymptote.

The eqn. of the asymptote of the given curve is $r \cos \theta = a$. So, the reqd. area is the area between $r \cos \theta = a$ and $r = a(\sec \theta + \cos \theta)$. Both of them are symmetrical about the initial line $\theta = 0$. For the given curve $0 \leq \theta < \frac{\pi}{2}$, let $P(r, \theta)$ be a point on the curve. The radius vector OP intersects the asymptote QM at Q (fig. 50). Therefore, the reqd. area will be equal to

$$\left\{ 2 \lim_{\theta \rightarrow \pi/2} [\text{area } MNPQ] \right\}$$

$$\therefore \text{area } MNPQ = (\text{area } ONPO) - (\text{area } OMQO)$$



$$\begin{aligned} &= \underbrace{\frac{1}{2} \int_0^\theta r^2 d\theta}_{\text{curve}} - \underbrace{\frac{1}{2} \int_0^\theta r^2 d\theta}_{\text{asymptote}} \\ &= \frac{1}{2} \int_0^\theta [a^2 (\sec \theta + \cos \theta)^2 - a^2 \sec^2 \theta] d\theta \\ &= \frac{a^2}{2} \int_0^\theta (\cos^2 \theta + 2) d\theta \end{aligned}$$

$$= \frac{a^2}{2} \left(\frac{5}{2} \theta + \frac{1}{4} \sin 2\theta \right)$$

$$\begin{aligned} \therefore \text{the reqd. area} &= 2 \left[\lim_{\theta \rightarrow \pi/2} \frac{a^2}{2} \left(\frac{5}{2} \theta + \frac{1}{4} \sin 2\theta \right) \right] \\ &= 2 \cdot \frac{a^2}{2} \left(\frac{5}{2} \cdot \frac{\pi}{2} \right) = \frac{5}{4} \pi a^2 \text{ sq. units.} \end{aligned}$$

7. Evaluate $\int_{2.45}^{4.45} \log x \, dx$ using $n = 20$.

Trapezoidal rule :

$$\Delta x = \frac{b - a}{n} = \frac{4.45 - 2.45}{20} = 0.1$$

According to this rule,

$$\int_{2.45}^{4.45} \log x \, dx = \frac{\Delta x}{2} \left[(y_0 + y_n) + 2 (y_1 + y_2 + \dots + y_{n-1}) \right]$$

$\therefore \Delta x = 0.1$, Therefore, abscissa

$$x = 2.45, 2.55, 2.65, \dots, 4.45$$

and ordinates $y = \log 2.45, \log 2.55, \log 2.65, \dots, \log 4.45$

$$\begin{aligned} \therefore \int_{2.45}^{4.45} \log x \, dx &= \frac{0.1}{2} \left[(\log 2.45 + \log 4.45) \right. \\ &\quad \left. + 2 (\log 2.55 + \log 2.65 + \dots + \log 4.35) \right] \\ &= \frac{1}{20} \left[(.8961 + 1.4929) + 2 (.9361 + .9746 + 1.0116 + \dots + \right. \\ &\quad \left. 1.4469 + 1.4702) \right] \text{ from log table or using calculator.} \\ &= \frac{1}{20} (2.3890 + 2 \times 23.2841) \\ &= 2.4479 \text{ sq. units.} \end{aligned}$$

N.B. Log table uses $\log_{10} x$. Here $\log x \equiv \log_e x = \log_{10} x \times \log_e 10$ and $\log_e 10 = 2.3026$. For calculator, press $\ln$ key only.

Simpson's rule : According to this rule,

$$\begin{aligned} \int_{2.45}^{4.45} \log x \, dx &= \frac{h}{3} \left[(y_1 + y_{2n+1}) + 2 (y_3 + y_5 + \dots + y_{2n-1}) \right. \\ &\quad \left. + 4 (y_2 + y_4 + \dots + y_{2n}) \right] \end{aligned}$$

$$\text{Here } h = \frac{b - a}{n} = \frac{4.45 - 2.45}{20} = 0.1$$

$$\therefore y_1 = \log 2.45, y_2 = \log 2.55, \dots, y_{2n-1} = \log 4.25, \\ y_{2n} = \log 4.35, y_{2n+1} = \log 4.45.$$

$$\therefore \int_{2.45}^{4.45} \log x \, dx = \frac{0.1}{3} \left[(\log 2.45 + \log 4.45) \right. \\ \left. + 2 (\log 2.65 + \log 2.85 + \dots + \log 4.25) \right. \\ \left. + 4 (\log 2.55 + \log 2.75 + \dots + \log 4.35) \right] \\ = \frac{1}{30} \left[(.8961 + 1.4929) + 2 (11.0425) + 4 (12.2416) \right] \\ = 2.4480 \text{ sq. units.}$$

Simpson's rule gives better approx. than Trapezoidal rule.

8. Find the area under the curve represented by the following table, x -axis, the ordinates at $x = a$ and $x = 4$.

x	1	1.5	2	2.5	3	3.5	4
y	2	2.4	2.7	2.8	3	2.9	2.1

Here, the abscissa from $x = 1$ to $x = 4$ is divided into 6 equal parts each of length $\Delta x = 0.5$. Therefore, from trapezoidal rule

$$\text{Area} = \int_1^4 y \, dx = \frac{\Delta x}{2} \left[(y_0 + y_n) + 2 (y_1 + y_2 + \dots + y_{n-1}) \right] \\ = \frac{0.5}{2} \left[(2 + 2.1) + 2 (2.4 + 2.7 + 2.8 + 3 + 2.9) \right] \\ = 7.78 \text{ sq. units.}$$

Again, from Simpson's rule,

$$\text{Area} = \int_1^4 y \, dx = \frac{h}{3} \left[(y_1 + y_{2n+1}) + 2 (y_3 + y_5 + \dots + y_{2n-1}) \right. \\ \left. + 4 (y_2 + y_4 + \dots + y_{2n}) \right] \\ = \frac{0.5}{3} \left[(2 + 2.1) + 2 (2.7 + 3) + 4 (2.4 + 2.8 + 2.9) \right] \\ = 7.99 \text{ sq. units.}$$

Exercises 11 (3)

Determine the area bounded by the following curves and the radii vectors:

1. $r(1 + \cos \theta) = a, \theta = 0, \theta = 120^\circ$.
2. $r^2 \cos 2\theta = a^2, \theta = 0, \theta = 30^\circ$.
3. Hyperbolic spiral $r\theta = a, \theta = \alpha, \theta = \beta$.
4. Parabola $r = \sec^2 \frac{\theta}{2}, \theta = 0, \theta = \frac{\pi}{2}$.

5. The line $r = \sec \theta$, $\theta = -\frac{\pi}{4}$, $\theta = \frac{\pi}{4}$.

6. $r = \sqrt{1 - \cos \theta}$, $\theta = \frac{\pi}{2}$, $\theta = \pi$.

7. $r = a \cos \theta$, $\theta = 0$, $\theta = \frac{\pi}{3}$,

8. $r = \theta$, $\theta = 0$, $\theta = \frac{\pi}{2}$.

Find the area enclosed by the following curves :

9. $r^2 = a^2 \cos^2 \theta + b^2 \sin^2 \theta$. 10. Cardioid $r = a(1 - \cos \theta)$.

11. $r = a \sin 3\theta$. 12. $r = 2 - \sec \theta$. 13. Cardioid $r = a(1 + \sin \theta)$.

14. $r = a \cos 3\theta$. 15. $r = 2 - \cos \theta$. 16. $r = \frac{1}{2} + \cos 2\theta$.

Find the area of a loop of the following curves :

17. $x^4 + y^4 = 2a^2xy$. 18. $x(x^2 + y^2) = a(x^2 - y^2)$.

19. $x^3 + y^3 = 3axy$. 20. $r = a \sin 4\theta$. 21. $r = a \sin 2\theta$.

22. $r = a \sin n\theta$. 23. $r^2 = a^2 \cos 2\theta$.

Find the common area between the curves below :

24. $r = 3 \cos \theta$, $r = 1 + \cos \theta$. 25. $r = 1 - \cos \theta$, $r = \sin \theta$.

26. $r^2 = \cos 2\theta$, $r^2 = \sin 2\theta$.

27. $r = \cos \theta$, $r = \sin \theta$.

28. Find the area of the inner loop of

$r = a(-1 + 2 \cos \theta)$. [Hints : in W. O. 1 use $\alpha = \frac{\pi}{3}$.

$$A = a^2 \int_0^{\frac{\pi}{3}} (-1 + 2 \cos \theta)^2 d\theta]$$

29. Prove that the area of a loop of $r = a \cos n\theta$ is $\frac{\pi}{4n} a^2$, when n is an integer. Prove also that this area will be $\frac{\pi}{4} a^2$ or $\frac{\pi}{2} a^2$ sq. units. according as n is odd or even.

30. Prove that the area between the cissoid $r = \frac{a \sin^2 \theta}{\cos \theta}$ and its asymptote is $\frac{3\pi}{4} a^2$ sq. units.

31. Find the area interior to lemniscate $r^2 = 2a^2 \cos 2\theta$ and exterior to the circle $r = a$.

32. Determine the common area of the cardioid $r = a(1 + \cos \theta)$ and circle $r = \frac{3}{2}a$. Also determine the area of the remaining cardioid.

33. Determine any error in the evaluation of $\int_0^1 x^3 dx$ using

Simpson's rule.

34. Determine the error in the evaluation of $\int_0^{\pi/2} \sin x dx$, using 11

ordinates in Simpson's rule.

35. A curve passes through the points tabulated below. Find the area enclosed by this curve, x -axis, the ordinates $x = 1$ and $x = 9$ using Simpson's rule.

x	1	2	3	4	5	6	7	8	9
y	0	0.6	0.9	1.2	1.4	1.5	1.7	1.8	2

36. Find the approx. value of $\int_1^3 \log x dx$.

(a) using trapezoidal rule with $n = 5$ and

(b) Simpson's rule with $n = 8$

Verify by integration.

37. Evaluate $\int_0^1 \sqrt{1+x^3} dx$ using trapezoidal rule with $n = 5$ and

Simpson's rule with $n = 4$.

Answers

1. $.866 a^2$ sq. units. 2. $0.33 a^2$ sq. units. 3. $\frac{a^2}{2} \left(\frac{1}{\alpha} - \frac{1}{\beta} \right)$ sq. units

4. $\frac{1}{2}$ sq. units 5. 1 sq. units. 6. $\frac{1}{2} \left(1 + \frac{\pi}{2} \right)$ sq. units. 7. $0.37 a^2$ sq. units.

8. $\frac{\pi^3}{8}$ sq. units. 9. $\frac{\pi}{2} (a^2 + b^2)$ sq. units. 10. $\frac{3\pi a^2}{2}$ sq. units.

11. $\frac{\pi a^2}{4}$ sq. units. 12. $\left[\frac{4}{3} \pi + \sqrt{3} - 4 \log(2 + \pi) \right]$ sq. units.

13. $\frac{3\pi}{2} a^2$ sq. units. 14. $\frac{\pi}{4} a^2$ sq. units. 15. $\frac{9\pi}{2}$ sq. units.

16. $\frac{3\pi}{2}$ sq. units. 17. $\frac{3}{4} a^2$ sq. units. 18. $2a^2 \left(1 - \frac{\pi}{4} \right)$ sq. units.

$$19. \frac{3}{2} a^2 \text{ sq. units. } 20. \frac{\pi}{16} a^2 \text{ sq. units. } 21. \frac{\pi}{8} a^2 \text{ sq. units.}$$

$$22. \frac{\pi}{4h} a^2 \text{ sq. units. } 23. \frac{a^2}{2} \text{ sq. units. } 24. \frac{5\pi}{4} \text{ sq. units.}$$

$$25. \left(\frac{\pi}{2} - 1 \right) \text{ sq. units. } 26. \left(1 - \frac{1}{\sqrt{2}} \right) \text{ sq. units. } 27. \frac{1}{4} \left(\frac{\pi}{2} - 1 \right) \text{ sq. units.}$$

$$28. a^2 \left(\pi - \frac{3\sqrt{3}}{2} \right) \text{ sq. units. } 31. \frac{1}{3} a^2 (3\sqrt{3} - \pi) \text{ sq. units.}$$

$$32. \left(\frac{7\pi}{4} - \frac{9\sqrt{3}}{8} \right) a^3 \text{ sq. units. and } a^2 \left\{ \frac{9\sqrt{3}}{9} - \frac{\pi}{4} \right\} \text{ sq. units}$$

$$33. \text{ No error } 34. .0004 \quad 35. 10.13 \text{ sq. units. } 36. (a) 1.2870, (b) 1.2958; 1.2954 \\ 37. 1.115 \text{ and } 1.111.$$

RECTIFICATION AND INTRINSIC EQUATIONS

12.1 DEFINITION

Rectification is the determination of length between two points on the arc of a curve.

Let $y = f(x)$ be the equation of a continuous and differentiable curve and $P(a, b)$ and $Q(c, d)$ be two points on it. If P and Q coincide the curve becomes *closed* otherwise *open*. Examples of closed curves are circle, ellipse, rectangle etc. and that of open curves are part of the stated curves, parabola and hyperbola etc.

The curve $y = f(x)$ may have cusp or even be self-intersecting forming loops. Whatever may be the shape of the continuous curve, the arc PQ is divided into n subarcs by the points $P(x_0, y_0), (x_1, y_1), (x_2, y_2), \dots, (x_{n-1}, y_{n-1})$ and $Q(x_n, y_n)$ (fig. 51). Then each of the two consecutive points are joined by chords or straight lines. The **length of the curve** PQ is then the **limiting value** of the **sum of those chords** as $n \rightarrow \infty$ i.e. the length of each chord $\rightarrow 0$. Therefore, the length of the curve depends on the existence of that limiting value and the term **length** is uniquely applicable to those for which this type of limiting value exists. This is the well-known **rectifiable curve** or **curve capable of straightened out**.

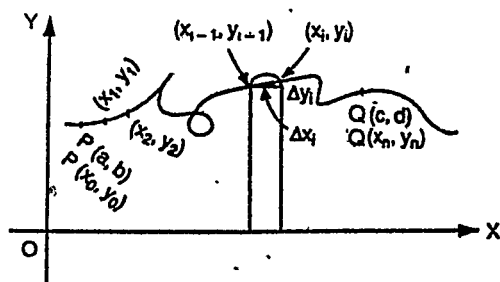


Fig. 51

From fig. 51, $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_i = y_i - y_{i-1}$ and the hypotenuse of the right triangle with sides $|\Delta x_i|$ and $|\Delta y_i|$ is the i th chord or straight line segment. Therefore

$$(\text{ith chord})^2 = (\Delta x_i)^2 + (\Delta y_i)^2$$

$$\therefore \text{ by def., length of the curve, } PQ = \lim_{n \rightarrow \infty} \sum_{i=1}^n \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

$$\therefore \text{ either, } PQ = \lim_{n \rightarrow \infty} \sum_{i=1}^n \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} \Delta x_i, \quad \Delta x_i \neq 0 \quad (1)$$

$$\text{or } PQ = \lim_{n \rightarrow \infty} \sum_{i=1}^n \sqrt{1 + \left(\frac{\Delta x_i}{\Delta y_i}\right)^2} \Delta y_i, \quad \Delta y_i \neq 0 \quad (2)$$

Since the curve is continuous and differentiable, let $(x_i^*, \phi(x_i^*))$ be a point on the i th subarc and by mean value theorem of differential calculus,

$$\phi'(x_i^*) = \frac{y_i - y_{i-1}}{x_i - x_{i-1}} = \frac{\Delta y_i}{\Delta x_i}$$

$\therefore$ from (1), length of the curve PQ

$$\begin{aligned} &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \sqrt{1 + [\phi'(x_i^*)]^2} \Delta x_i = \int_a^b \sqrt{1 + \{\phi'(x)\}^2} dx \\ &= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_a^b ds, \quad \because ds^2 = \left(1 + \left(\frac{dy}{dx}\right)^2\right) dx^2 \end{aligned}$$

$\therefore$ integral of the differential arc length between the ends of the arc gives the length of the arc. Similar conclusion can be drawn from (2).

12.2. DETERMINATION OF ARC LENGTH

The arc-length, s can be determined by integrating differential arc-length, ds corresponding to the curve in explicit or implicit cartesian, parametric, polar or pedal equation as follows :

(i) For cartesian curve, $y = f(x)$:

If $\frac{dy}{dx}$ is continuous and single-valued in the interval $[a, b]$, then arc-length s between two points with abscissa 'a' and 'b' is

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

Proof : Let AB be a portion of the curve $y = f(x)$ and the abscissae for A and B be 'a' and 'b' respectively. Let $P(x, y)$ be any point on the curve AB (fig. 52). So, arc AP is a function of x . Let $AP = s$. Then from diff. calculus,

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

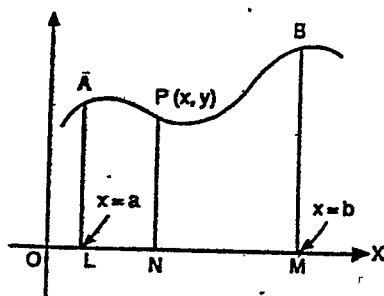


Fig. 52

Integrating w.r.t. 'x' from 'a' to 'b', we have,

$$\begin{aligned}\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx &= \int_a^b \frac{ds}{dx} dx = [s(x)]_{x=a}^b \\ &= (\text{Value of } s \text{ at } x=b) - (\text{Value of } s \text{ at } x=a) \\ &= \text{Length of arc } AB - 0 \\ &= \text{Length of arc } AB, s\end{aligned}$$

$$\therefore s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

(ii) For the cartesian curve $x = f(y)$:

As in (i) it can be proved that

$$s = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy, \quad \therefore \frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}$$

(iii) For the parametric curve $x = f(t)$, $y = \phi(t)$:

As in (i) it can be proved that

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt, \quad \therefore \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

(iv) For the polar curve $r = f(\theta)$:

If $\frac{dr}{d\theta}$ is continuous and single valued in the interval $[\alpha, \beta]$, then the arc-length, s , between two points with vectorial angles α and β , is

$$s = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

Proof : Let $r = f(\theta)$ be the polar equation of the curve AB and the vectorial angles corresponding to the points A and B be α and β respectively. Let $P(r, \theta)$ be a point on AB (fig. 53). Then the arc AP is a function of θ . Therefore, from differential calculus

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

$\therefore$ integrating w.r.t. ' θ ' from α to β , we have,

$$\int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

$$= \int_{\alpha}^{\beta} \frac{ds}{d\theta} d\theta$$

$$= [s(\theta)]_{\alpha}^{\beta}$$

$$= (\text{Value of } s \text{ at } \theta = \beta) - (\text{Value of } s \text{ at } \theta = \alpha)$$

$$= \text{Length of arc } AB - 0$$

$$= \text{Length of arc } AB, s$$

$$\therefore s = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

(v) For the polar curve $\theta = f(r)$:

With similar arguments as in (iv)

$$s = \int_{r_1}^{r_2} \sqrt{1 + r^2 \left(\frac{d\theta}{dr}\right)^2} dr, \therefore \frac{ds}{dr} = \sqrt{1 + r^2 \left(\frac{d\theta}{dr}\right)^2}$$

(vi) For the pedal curve, $p = f(r)$:

Arc length between two points with radius vector $r = a$ and $r = b$ is

$$s = \int_a^b \frac{r dr}{\sqrt{r^2 - p^2}}.$$

Proof : Let the pedal equation of the curve AB be $p = f(r)$ and at the points, A and B the radius vectors be $r = a$ and $r = b$. Let $P(p, r)$ be a point on AB (fig. 54). Then the arc AP is a function of r . Therefore, from differential calculus

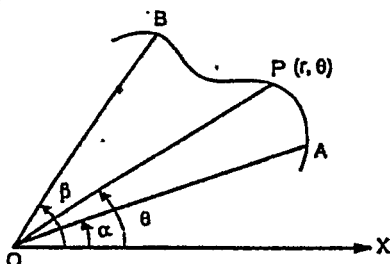


Fig. 53.

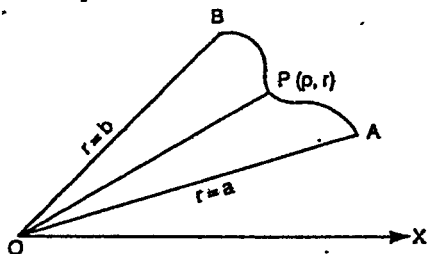


Fig. 54

$$\frac{ds}{dr} = \sec \phi = \frac{1}{\cos \phi} = \frac{1}{\sqrt{1 - \sin^2 \phi}}, \quad \therefore \frac{dr}{ds} = \cos \phi$$

$$\therefore \frac{ds}{dr} = \frac{r}{\sqrt{r^2 - p^2}}, \quad \therefore p = r \sin \phi$$

$\therefore$ Integrating w.r.t. r from $r = a$ to $r = b$, we get,

$$\begin{aligned} \int_a^b \frac{r}{\sqrt{r^2 - p^2}} dr &= \int_a^b \frac{ds}{dr} dr = [s(r)]_a^b \\ &= (\text{Value of } s \text{ at } r = b) - (\text{Value of } s \text{ at } r = a) \\ &= \text{Length of arc } AB, s - 0 \\ &= \text{Length of arc } AB, s \end{aligned}$$

$$\therefore s = \int_a^b \frac{r}{\sqrt{r^2 - p^2}} dr.$$

(vii) Length of an arc of an evolute :

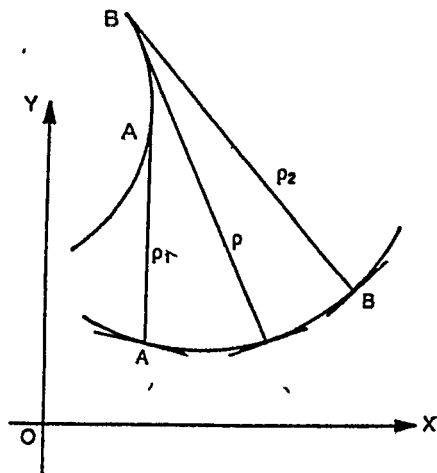


Fig. 55

Evolute is the locus of the centres of curvatures of the curve. It is known from differential calculus that the length between two points of an evolute of a curve is the difference between the radii of curvatures at those points. From fig. 55, let ρ_1 and ρ_2 be the radii of curvatures at A and B of the curve AB and let a and b be the two corresponding points on its evolute (fig. 55). Then the length of arc, ab is $s = \rho_1 - \rho_2 = \rho_2 - \rho_1$ in the case of aforesaid figure.

12.3 INTRINSIC EQUATION OF A CURVE.

Definition : The intrinsic equation of a curve is the relation between s and ψ where s is the arc-length from a fixed point A to a variable point P on the curve and ψ is the angle between the tangents at A and P (fig. 56).

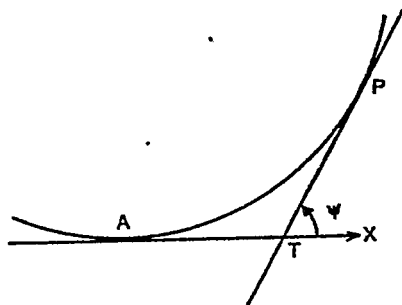


Fig. 56

The intrinsic equation only determines the **shape** of the curve without giving any position in the plane. For example, let us find the intrinsic equation of a circle. Let the centre of the circle be C and its radius be ' a '. Let A be a fixed point on the circle from which the arc-length is measured. Let $P(s, \psi)$ be a point on the circle and ψ be an angle between the tangents at A and P (fig. 57).

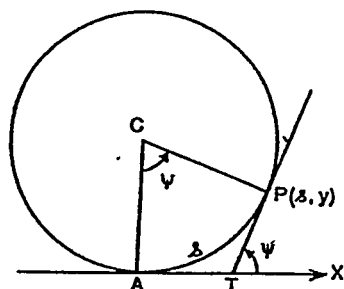


Fig. 57

$\therefore$ arc $AP = s$ and $\angle XTP = \psi$.

Join AC and CP . Then $\angle CAT = \angle CPT = 1$ right $\angle$

$\therefore$ $ATPC$ is an inscribed quadrilateral.

$\therefore \angle ACP = \angle XTP = \psi$. But $\angle ACP = \frac{\text{arc } AP}{\text{radius}}$

$\therefore \psi = \frac{s}{a}$ or $s = a\psi$ is the required intrinsic equation of a circle.

12.3.1 METHOD OF FINDING INTRINSIC EQUATION OF A CURVE

The following are the methods for finding intrinsic equation of the curves in different systems :

(i) **For cartesian equation :** Let the cartesian equation of the curve be $y = f(x)$ and the tangent at the fixed point be parallel to the x -axis. Let $P(x, y)$ be a point on the curve and the tangent at P cuts the x -axis at T (fig. 58). Let $x = a$ at A . If arc $AP = s$, then

$$s = \int_a^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \quad (1)$$

and $\tan \psi = \frac{dy}{dx} \quad (2)$

$\therefore$ the required intrinsic equation is the eliminant of x between (1) and (2).

(ii) **For Parametric equations:** Let the parametric equation of the curve be $x = f(t)$, $y = \phi(t)$. As in case (i), the arc length s is measured from the fixed point A and the tangent at A is parallel to x -axis. The tangent at the variable point P makes an angle ψ with the x -axis at T (fig. 58). If $t = t_0$ at A , then

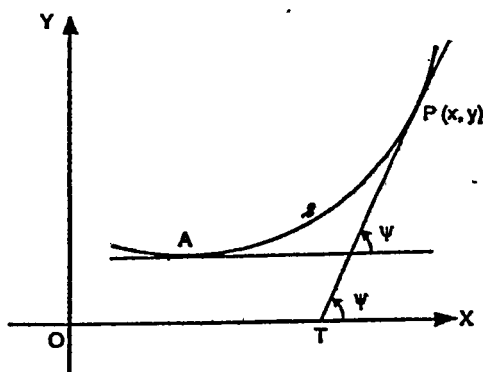


Fig. 58

$$s = \int_{t_0}^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \quad (1)$$

and $\tan \psi = \frac{dy}{dx} = \frac{\phi'(t)}{f'(t)} \quad (2)$

$\therefore$ the required intrinsic equation is the eliminant of t between (1) and (2).

(iii) **For polar equation $r = f(\theta)$:** Let A be the fixed point and the tangent at A be parallel to the initial line. Let $P(r, \theta)$ be a point on the curve and the tangent at P cuts the initial line at T . If $\theta = \alpha$ for A , then

$$s = \int_{\alpha}^{\theta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta = \int_{\alpha}^{\theta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} d\theta \quad (1)$$

Also, $\psi = \theta + \phi \quad (2)$

and $\tan \phi = \frac{r \frac{d\theta}{dr}}{f(\theta)} = \frac{f'(\theta)}{f(\theta)} \quad (3)$

$\therefore$ the required intrinsic equation is the eliminant of θ and ϕ from (1), (2) and (3).

(iv) For pedal equation $p = f(r)$: Let the pedal equation of the curve be $p = f(r)$. With similar reasoning as before,

$$s = \int_a^r \frac{r \, dr}{\sqrt{r^2 - p^2}} = \int_a^r \frac{r \, dr}{\sqrt{r^2 - \{f(r)^2\}}} = F(r), \text{ (say)} \quad (1)$$

and
$$\frac{ds}{d\psi} = \rho = \frac{r \, dr}{dp} = \frac{r}{f'(r)} \quad (2)$$

$\therefore$ eliminating r from (1) and (2) let

$$\frac{ds}{d\psi} = \phi(s) \text{ or } \frac{d\psi}{ds} = \frac{1}{\phi(s)}$$

$$\therefore \psi = \int \frac{ds}{\phi(s)} + c \text{ is the required intrinsic equation.}$$

12.3.2 INTRINSIC EQUATIONS OF EVOLUTE AND INVOLUTE

The intrinsic equation of evolute and involute of a curve can be determined from the intrinsic equation of the curve as follows :

(i) **Intrinsic equation of evolute from the intrinsic equation of the curve :** Let the intrinsic equation of the curve be $s = f(\psi)$ and its fixed point be A . Let A' and P' be the points on the evolute corresponding to A and P of the curve and s' be the arc length $A'P'$ and ψ' be the angle between the tangents at A' and P' (fig. 59).

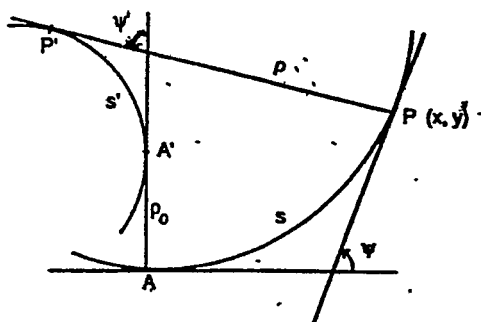


Fig. 59

From differential calculus, the perpendiculars at A and P are tangents at A' and P' of the evolute. So, $\psi' = \psi$: (1)

Let ρ_0 and ρ be the radii of curvature at A and P respectively. So, if

$$\rho > \rho_0, s' = \rho - \rho_0 = \frac{ds}{d\psi} - \rho_0 \quad (2)$$

or, $s' = f'(\psi) - \rho_0 = f'(\psi') - \rho_0$ by (1)

This is the required equation of evolute.

(ii) **Intrinsic equation of the involute from the intrinsic equation of the curve :** Previous fig. 59 case (i). Since the given curve AP in case (i) is the involute of the curve $A'P'$ which is the evolute in case (i) i.e. in this case curve $A'P'$ is considered as the given curve whose involute is AP curve. Let the intrinsic equation of $A'P'$ be $s' = \lambda(\psi')$. Therefore, from equations (1) and (2) of (i),

$$\frac{ds}{d\psi} - \rho_0 = \lambda(\psi)$$

$$\therefore \frac{ds}{d\psi} = \lambda(\psi) + \rho_0$$

$$\therefore s = \int [\lambda(\psi) + \rho_0] d\psi$$

This is the required intrinsic equation of the involute.

Examples

1. Find the perimeter of the four cusped hypocycloid

$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1.$$

To test the **symmetry**, let the parametric form of the curve be $x = a \cos^3 \theta$ and $y = b \sin^3 \theta$. The replacement of θ by its functions $(-\theta)$ and $(\pi - \theta)$ changes the sign of y and x respectively. So, the curve is symmetrical about both axes. Therefore, perimeter of the curve is divided into 4 equal parts.

Also, as θ varies from 0 to $\frac{\pi}{2}$, so x varies from a to 0 and y varies from 0 to a along x - and y -axes respectively. From the concept of curve tracing, the sketch of the equation is as follows (fig. 60).

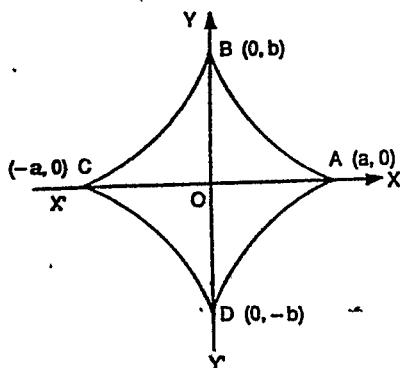


Fig. 60

From fig. 60, perimeter $ABCD = 4 AB$. Form the equations,

$$\frac{dy}{dx} = -\frac{b}{a} \left(\frac{x}{a}\right)^{-1/3} \left(\frac{y}{b}\right)^{1/3}$$

and

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{b^2}{a^2} \left[\left(\frac{x}{a}\right)^{-2/3} - 1 \right]$$

$\therefore s = \text{Length of arc } AB$

$$\begin{aligned} &= \int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_0^a \sqrt{1 + \frac{b^2}{a^2} \left[\left(\frac{x}{a}\right)^{-2/3} - 1 \right]} dx \\ &= \int_0^{\pi/2} \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} \cdot 3 \sin \theta \cos \theta d\theta \text{ when } x = a \cos^3 \theta \\ &= \frac{3}{2} \int_0^1 \sqrt{a^2 (1-u) + b^2 u} \cdot du \text{ when } \sin^2 \theta = u \\ &= \frac{3}{2} \int_0^1 \sqrt{a^2 + (b^2 - a^2)u} du = \frac{3}{2} \cdot \frac{2}{3} \left[(a^2 + (b^2 - a^2)u)^{3/2} / (b^2 - a^2) \right]_0^1 \\ &= \frac{b^3 - a^3}{b^2 - a^2} = \frac{a^2 + ab + b^2}{a + b} \text{ units} \end{aligned}$$

$\therefore$ the required perimeter $= 4 \cdot \frac{a^2 + ab + b^2}{a + b}$.

2. Find the length of the curve $y^2 = -4x$ from $(0, 0)$ to $(-1, 2)$.

The curve is the well-known parabola (fig. 61).

From the equation, $\frac{dx}{dy} = -\frac{1}{2}y$

$$\therefore 1 + \left(\frac{dx}{dy}\right)^2 = \frac{1}{4}(y^2 + 4)$$

$\therefore$ reqd. arc-length, s

$$= \int_0^2 \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

$$= \int_0^2 \sqrt{\frac{1}{4}(y^2 + 4)} dy$$

$$= \frac{1}{2} \left[\frac{y \sqrt{y^2 + 4}}{2} + 2 \log (y + \sqrt{y^2 + 4}) \right]_0^2$$

$$= [\sqrt{2} + \log (1 + \sqrt{2})] \text{ units.}$$

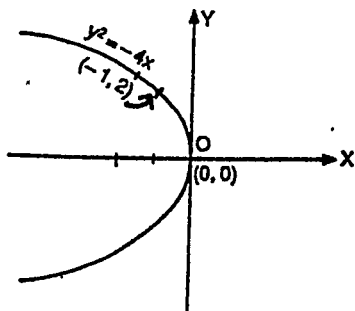


Fig. 61

3. Show that the arc length, s measured from some θ to $\theta = \frac{\pi b}{a}$ for the epicycloid $x = (a+b) \cos \theta - b \cos \frac{a+b}{b} \theta$, $y = (a+b) \sin \theta - b \sin \frac{a+b}{b} \theta$, is $s = \frac{4b(a+b)}{a} \cos \frac{a}{2b} \theta$. Hence show that $s^3 \propto x^2$ where s , the arc length of $x^{2/3} + y^{2/3} = a^{2/3}$ measured from one cusp on the y -axis to some θ , with $b = -a/4$ in the epicycloid.

Definition : When a circle rolls on outside of a fixed circle, the curve traced by the first contact point is called an epicycloid and the same inside the fixed circle is an hypocycloid (fig. 62).

First Part :

$$\therefore x = (a+b) \cos \theta - b \cos \frac{a+b}{b} \theta$$

$$\text{and } y = (a+b) \sin \theta - b \sin \frac{a+b}{b} \theta.$$

$$\therefore \frac{dx}{d\theta} = (a+b) \left[\sin \frac{a+b}{b} \theta - \sin \theta \right]$$

$$\text{and } \frac{dy}{d\theta} = (a+b) \left[\cos \theta - \cos \frac{a+b}{b} \theta \right]$$

$$\therefore \left(\frac{dx}{d\theta} \right)^2 + \left(\frac{dy}{d\theta} \right)^2 = 4(a+b)^2 \sin^2 \frac{a}{2b} \theta$$

$$\therefore s = \int_{\theta}^{\pi b/a} \sqrt{\left(\frac{dx}{d\theta} \right)^2 + \left(\frac{dy}{d\theta} \right)^2} d\theta = \int_{\theta}^{\pi b/a} 2(a+b) \sin \frac{a}{2b} \theta d\theta$$

$$= -\frac{4b(a+b)}{a} \left[\cos \frac{a}{2b} \theta \right]_{\theta}^{\pi b/a} \quad (i)$$

$$= \frac{4b(a+b)}{a} \cos \frac{a}{2b} \theta$$

$$\therefore \text{the reqd., } s = \frac{4b(a+b)}{a} \cos \frac{a}{2b} \theta \text{ units.}$$

Second Part : Since $b = -\frac{a}{4}$, so

$$x = \frac{3}{4} a \cos \theta + \frac{a}{4} \cos 3\theta = a \cos^3 \theta \quad (ii)$$

$$\text{and } y = a \sin^3 \theta$$

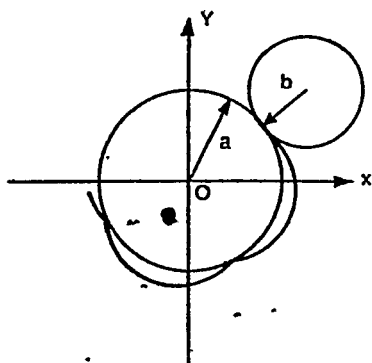


Fig. 62

$$\therefore \left(\frac{x}{a}\right)^{1/3} = \cos \theta \text{ and } \left(\frac{y}{a}\right)^{1/3} = \sin \theta$$

$$\text{i.e. squaring and adding } \left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{a}\right)^{2/3} = 1$$

or, $x^{2/3} + y^{2/3} = a^{2/3}$ is a hypocycloid (Astroid).

Also for cusp on the y-axis, $\theta = \pm \pi/2$. Therefore, from (i)

$$\begin{aligned} s &= \frac{-4(-a/4)(3a/4)}{a} \left[\cos \frac{a}{2\left(-\frac{a}{4}\right)} \theta \right]_{\pi/2}^0 \\ &= \frac{3a}{4} [\cos 2\theta]_{\pi/2}^0 = \frac{3a}{4} [1 + \cos 2\theta] = \frac{3a}{4} \cdot 2 \cos^2 \theta \\ &= \frac{3}{2} a \cos^2 \theta \end{aligned} \quad \text{(iii)}$$

$\therefore$ (ii) and (iii) show that $s^3 \propto x^2$.

4. Show that the length of an arc of the ellipse $x = a \sin t$, $y = b \cos t$ is $a \int_0^t \sqrt{1 - e^2 \sin^2 t} dt$, where e is the eccentricity of the ellipse. Hence show that the perimeter of the ellipse is

$$2\pi a \left\{ 1 - \left(\frac{1}{2}\right)^2 \cdot \frac{e^2}{1} - \left(\frac{1.3}{2.4}\right)^2 \cdot \frac{e^4}{3} - \left(\frac{1.3.5}{2.4.6}\right)^2 \cdot \frac{e^6}{6} - \dots \dots \infty \right\}$$

First Part :

Let s be the arc length from (a, b) to (x, y) . So,

$$\begin{aligned} s &= \int_0^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_0^t \sqrt{a^2 \cos^2 t + b^2 \sin^2 t} dt \\ &= \int_0^t \sqrt{a^2 (1 - \sin^2 t) + b^2 \sin^2 t} dt = \int_0^t \sqrt{a^2 - (a^2 - b^2) \sin^2 t} dt \\ &= \int_0^t \sqrt{a^2 (1 - e^2 \sin^2 t)} dt \text{ where } e^2 = \frac{a^2 - b^2}{a^2} \\ &= a \int_0^t \sqrt{1 - e^2 \sin^2 t} dt \text{ units.} \end{aligned}$$

Second Part :

$$\text{The perimeter of the ellipse} = 4a \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 t} dt$$

$$\begin{aligned}
 &= 4a \int_0^{\pi/2} \left\{ 1 - \frac{1}{2} e^2 \sin^2 t - \frac{1}{2.4} e^4 \sin^4 t - \frac{1.3}{2.4.6} e^6 \sin^6 t - \dots \infty \right\} dt \\
 &= 4a \left\{ \int_0^{\pi/2} dt - \frac{1}{2} e^2 \int_0^{\pi/2} \sin^2 t dt - \frac{1}{2.4} e^4 \int_0^{\pi/2} \sin^4 t dt - \right. \\
 &\quad \left. \frac{1.3}{2.4.6} e^6 \int_0^{\pi/2} \sin^6 t dt - \dots \infty \right\} \\
 &= 4a \left\{ \frac{\pi}{2} - \frac{1}{2} e^2 \cdot \frac{1}{2} \cdot \frac{\pi}{2} - \frac{1}{2.4} e^4 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} - \frac{1.3}{2.4.6} e^6 \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} - \dots \infty \right\} \\
 &= 2\pi a \left\{ 1 - \left(\frac{1}{2}\right)^2 \frac{e^2}{1} - \left(\frac{1.3}{2.4}\right)^2 \frac{e^4}{3} - \left(\frac{1.3.5}{2.4.6}\right)^2 \frac{e^6}{5} - \dots \infty \right\} \text{ units.}
 \end{aligned}$$

5. Prove that the total length of the curve $r = a \sin^3 \frac{\theta}{3}$ is $\frac{3\pi a}{2}$ units. Also show that OA, AB and BC (fig. 63) form an A.P.

The curve is symmetrical about the line $\theta = \frac{\pi}{2}$. The two symmetrical parts are $OABC$ and $CDAO$ and θ varies from 0 to $\frac{3\pi}{2}$ in $OABC$ (fig. 63).

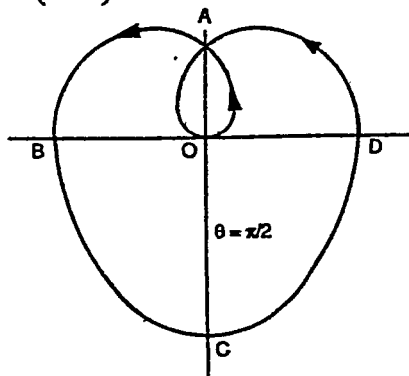


Fig. 63

$$\begin{aligned}
 \therefore \text{ the required total length} &= 2 \int_0^{3\pi/2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta \\
 &= 2 \int_0^{3\pi/2} \sqrt{a^2 \sin^6 \frac{\theta}{3} + a^2 \sin^4 \frac{\theta}{3} \cos^2 \frac{\theta}{3}} d\theta = 2 \int_0^{3\pi/2} a \sin^2 \frac{\theta}{3} d\theta \\
 &= \frac{3\pi a}{2} \text{ units.}
 \end{aligned}$$

Second Part :

$$\text{Length of } OA = \int_0^{\pi/2} a \sin^2 \frac{\theta}{3} d\theta = \frac{a}{8} (2\pi - 3\sqrt{3}) \text{ units} \quad (i)$$

$$\text{Length of } AB = \int_{\pi/2}^{\pi} a \sin^2 \frac{\theta}{3} d\theta = \frac{a}{8} \cdot 2\pi \text{ units} \quad (ii)$$

$$\text{Length of } BC = \int_{\pi}^{3\pi/2} a \sin^2 \frac{\theta}{3} d\theta = \frac{a}{8} (2\pi + 3\sqrt{3}) \text{ units} \quad (\text{iii})$$

(i), (ii) and (iii) show that OA , AB and BC form an $A.P.$

6. Find the arc length of the equi-angular spiral $r = ae^{\theta \cot \alpha}$ from the radius vector r_1 to r_2 .

Differentiating the given curve w.r.t. ' θ ', we have,

$$\frac{dr}{d\theta} = a \cot \alpha e^{\theta \cot \alpha}. \text{ Therefore,}$$

$$\left(r \frac{d\theta}{dr}\right)^2 = \left(\frac{a e^{\theta \cot \alpha}}{a \cot \alpha e^{\theta \cot \alpha}}\right)^2 = \tan^2 \alpha. \text{ So, the required arc length,}$$

$$s = \int_{r_1}^{r_2} \sqrt{1 + \left(r \frac{d\theta}{dr}\right)^2} dr = \int_{r_1}^{r_2} \sqrt{1 + \tan^2 \alpha} dr$$

$$= (r_2 - r_1) \sec \alpha \text{ units.}$$

7. Find the arc length of epicycloid $p^2 = \frac{c^2(r^2 - a^2)}{c^2 - a^2}$ from $r = r_1$ to the vertex (where $r = c$).

$$\begin{aligned} \text{The required length} &= \int_{r_1}^c \frac{r}{\sqrt{r^2 - p^2}} dr \\ &= \int_{r_1}^c \frac{r}{\sqrt{r^2 - \frac{c^2(r^2 - a^2)}{c^2 - a^2}}} dr = \frac{\sqrt{c^2 - a^2}}{a} \int_{r_1}^c \frac{r}{\sqrt{c^2 - r^2}} dr \\ &= \frac{\sqrt{c^2 - a^2}}{a} \sqrt{c^2 - r_1^2} = \frac{c^2 - a^2}{a} \sqrt{\frac{c^2 - r_1^2}{c^2 - a^2}} \\ &= \frac{c^2 - a^2}{a} \sqrt{r_1^2 - \frac{c^2(r_1^2 - a^2)}{c^2 - a^2}} = \frac{c^2 - a^2}{a} \sqrt{r_1^2 - p_1^2} \\ &\quad \text{where } p_1 = \frac{c^2(r_1^2 - a^2)}{c^2 - a^2} \\ &= \frac{c^2 - a^2}{a} \sqrt{r_1^2 - p_1^2} \text{ units.} \end{aligned}$$

8. Find the arc length of the evolute of the parabola $x^2 = 4ay$ from one cusp of the evolute to one point of intersection of the evolute and the parabola.

From differential calculus, the radius of curvature at any point (x, y) of

the parabola $x^2 = 4ay$ is $\rho = \left(1 + \frac{x^2}{4a^2}\right)^{3/2}$ (i)

and the equation of its evolute is

$$27ax^2 = 4(y - 2a)^3 \quad \text{(ii)}$$

The evolute has a cusp $A'(0, 2a)$ and intersects the parabola at $(\pm 4\sqrt{2}a, 8a)$. Corresponding to these two points of evolute, the points on the parabola are $A(0, 0)$ and $(\mp 2\sqrt{2}a, 2a)$ (fig. 64). So, the required

$$\begin{aligned} \text{length } A'B' &= \rho_B - \rho_A = 2a \left(1 + \frac{8a^2}{4a^2}\right)^{3/2} - 2a \left(1 + \frac{0}{4a^2}\right)^{3/2} \\ &= 2a(3\sqrt{3} - 1) \text{ units.} \end{aligned}$$

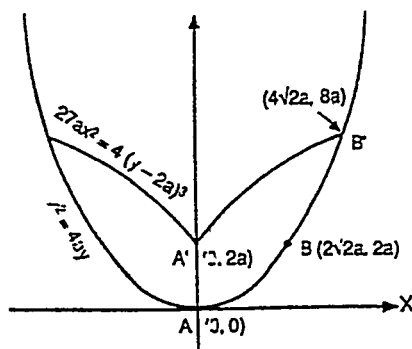


Fig. 64

9. Find the intrinsic equation of the catenary $y = c \cosh \frac{x}{c}$ when the arc length is measured from the vertex, at $x = 0$.

Here $\tan \psi = \frac{dy}{dx} = \sinh \frac{x}{c}$ (i)

and $s = \int_0^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_0^x \sqrt{1 + \sinh^2 \frac{x}{c}} dx$

$$= \int_0^x \cosh \frac{x}{c} dx = c \sinh \frac{x}{c} \quad \text{(ii)}$$

$\therefore$ eliminating x from (i) and (ii), $x = c \tan \psi$ which is the required intrinsic equation.

10. Find the intrinsic equation of the cycloid $x = a(t + \sin t)$, $y = a(1 - \cos t)$ when the arc length is measured from the origin.

$$\text{Here } \tan \psi = \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{a \sin t}{a(1 + \cos t)} = \tan \frac{t}{2}$$

$$\therefore \psi = \frac{t}{2} \quad \text{or} \quad t = 2\psi \quad (i)$$

$$\begin{aligned} \text{Again arc length, } s &= \int_0^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= \int_0^t \sqrt{a^2(1 + \cos t)^2 + a^2 \sin^2 t} dt = 2a \int_0^t \cos \frac{t}{2} dt \\ &= 4a \sin \frac{t}{2} \end{aligned} \quad (ii)$$

$\therefore$ eliminating t from (i) and (ii), the required intrinsic equation is $s = 4a \sin \psi$.

ρ 11. Find the intrinsic equation of the cycloid $r = a(1 - \cos \theta)$ when the arc length is measured from one cusp (where $\theta = 0$).

The arc length,

$$\begin{aligned} s &= \int_0^\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta = \int_0^\theta \sqrt{a^2(1 - \cos \theta)^2 + a^2 \sin^2 \theta} d\theta \\ &= a \int_0^\theta \sqrt{2(1 - \cos \theta)} d\theta = 2a \int_0^\theta \sin \frac{\theta}{2} d\theta = 8a \sin^2 \frac{\theta}{4} \end{aligned} \quad (i)$$

$$\text{Also} \quad \psi = \theta + \phi \quad (ii)$$

$$\text{and} \quad \tan \phi = r \frac{d\theta}{dr} = \frac{a(1 - \cos \theta)}{a \sin \theta} = \tan \frac{\theta}{2}$$

$$\therefore \quad \phi = \frac{\theta}{2} \quad (iii)$$

$$\therefore \text{ from (ii) and (iii), } \psi = \frac{3}{2}\theta \quad \text{or} \quad \theta = \frac{2}{3}\psi$$

$$\therefore \text{ from (i), } s = 8a \sin^2 \frac{\psi}{6} \text{ is the required intrinsic equation.}$$

ρ 12. Find the intrinsic equation of the evolute and involute of the cycloid $s = 4a \sin \psi$.

$$\text{Here } \frac{ds}{d\psi} = 4a \cos \psi, \therefore \rho = \frac{ds}{d\psi} = 4a \cos \psi$$

$$\therefore \rho_0 = 4a \text{ at the vertex.}$$

$$\begin{aligned} \text{Also } \psi &= \psi', \therefore s' = \rho_0 - \rho \quad \because \rho_0 > \rho \text{ for this cycloid.} \\ &= 4a - 4a \cos \psi' \end{aligned}$$

$$\therefore \text{required intrinsic equation of evolute is } s = 4a(1 - \cos \psi) \\ = 8a \sin^2 \psi$$

and that of involute is $s = \int (4a \sin \psi + A) d\psi$ when $\rho_0 = A$

$$= -4a \cos \psi + A \psi + c$$

But at $\psi = 0, s = 0 \therefore c = 4a$

$$\therefore s = 4a(1 - \cos \psi) + A \psi = 8a \sin^2 \frac{\psi}{2} + A \psi \text{ is the intrinsic equation}$$

of the involute.

13. An uniform chain is hanged at two points 20' apart on a horizontal line. The sag of the middle point is 5'. Find its length when it forms a catenary.

Let the y -axis passes through mid-point of the chain and x -axis is c unit below the sag (fig. 65). Then the equation of the catenary is

$$y = c \cosh \frac{x}{c} \quad (i)$$

$$\therefore \frac{dy}{dx} = \sinh \frac{x}{c}$$

and $\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

$$= \sqrt{1 + \sinh^2 \frac{x}{c}} = \cosh \frac{x}{c}$$

$\therefore$ the length of the chain

$$= 2 \int_0^{10} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$= 2 \int_0^{10} \cosh \frac{x}{c} dx = 2c \sinh \frac{10}{c} \text{ units} \quad (ii)$$

Also fig. 65 shows that $y = 5 + c$ when $x = 10$

$$\therefore \text{from (i), } 5 + c = c \cosh \frac{10}{c} = c \left[1 + \left(\frac{10}{c} \right)^2 \cdot \frac{1}{2} \right]$$

$$\left[\text{rejecting higher powers of } \frac{10}{c} \text{ in the series of } \cosh \frac{10}{c} \right]$$

$$\therefore 5 + c = c + \frac{100}{2c} \text{ or } c = 10'$$

$$\therefore \text{from (ii) the required length} = 2 \cdot 10 \sinh \frac{10}{10}$$

$$= 20 \sinh 1 = 20 (1.1752) \text{ using calculator or table}$$

$$= 23.5 \text{ feet.}$$

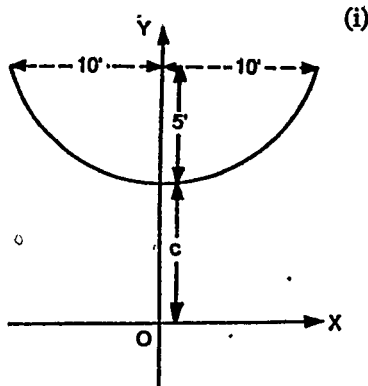


Fig. 65

Exercises 12

- Find the arc length from the vertex of the parabola $y^2 = 4px$ to the point $(b, \sqrt{4pb})$.
- Find the length of the parabola $y^2 = 4ax$ from the vertex to one end of the latus rectum.
- Determine the perimeter of the circle $x^2 + y^2 = a^2$.
- Find the length of the curve $y = \log x$ between $(1, 0)$ and $(e, 1)$.
- Determine the length of the curve $y = x\sqrt{x}$ from $(1, 1)$ to $(4, 8)$.
- Find the arc length of the curve $y = \log \cos x$ from $x = 0$ to $x = \pi/3$.
- Find the arc length of $y = \log \sec x$ from the point $(0, 0)$ to the point $(\frac{\pi}{3}, \log 2)$.
- Find the arc length of $e^y = (e^x + 1)/(e^x - 1)$ from $x = a$ to $x = b$.
- Find the length of the circle $x^2 + y^2 = a^2$ intercepted by the lines $x = a \cos \alpha$ and $x = a \cos \beta$.
- Show that the arc length of $x^2 = a^2 (1 - \exp(y/a))$ from the origin to any point (x, y) is $[a \log \{(a+x)/(a-x)\} - x]$ units.
- Determine the perimeter of the loop of $3ay^2 = x^2(a-x)$.
- Determine the perimeter of the loop of $3ay^2 = x(x-a)^2$.
- Show that the total length of the curve $x^2(a^2 - x^2) = 8a^2 y^2$ is $\pi a/\sqrt{2}$ units.
- Show that the distance of any point on the catenary $y = c \cosh \frac{x}{c}$ from its vertex is $s = c \sinh \frac{x}{c}$. Hence show that $s^2 = y^2 - c^2$.
- Find the length of $y = x^{2/3}$ from $(-1, 1)$ to $(-27, 9)$.
[Hints : Integrate w.r.t. y .]
- Find the length of $3x^2 = y^3$ from $y = 1$ to $y = 20$.
- Find the length of $x = 3y^{3/2} - 1$ from $y = 0$ to $y = 4$.
- Find the arc length of $x = e^t \sin t$, $y = e^t \cos t$ from $t = 0$ to $t = \pi$.
- Find the length of one arch of the cycloid $x = a(\phi - \sin \phi)$, $y = a(1 - \cos \phi)$.
- Find the length of the parabola $x = at^2$, $y = 2at$ from the origin to $t = 1$.
- Prove that the distance of $(0, 0)$ to (x, y) along the curve $x = a \sin 2\theta (1 + \cos 2\theta)$, $y = a \cos 2\theta (1 - \cos 2\theta)$ is $\frac{4}{3} a \sin 3\theta$ units.

22. Find the length of the loop of $x = t^2, y = t - \frac{1}{3}t^3$.
23. Find the perimeter of the astroid $x = a \cos^3 \theta, y = a \sin^3 \theta$.
24. Find the one-fourth length of the curve $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$.
25. Find the total length of $x^{2/3} + y^{2/3} = a^{2/3}$.
26. Prove that the length of an arc of the curve
 $x \sin \theta + y \cos \theta = f'(\theta), x \cos \theta - y \sin \theta = f''(\theta)$,
 is given by $s = f(\theta) + f''(\theta) + c$ where c is a constant.

[Hints :

$$s = \int \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta = \int [f'(\theta) + f''(\theta)] d\theta = f(\theta) + f''(\theta) + c.]$$

27. Find the length of the spiral $r = a\theta$ from $\theta = 0$ to $\theta = \pi$.
28. Determine the perimeter of the cardioid $r = a(1 - \cos \theta)$.
29. Determine the perimeter of the cardioid $r = a(1 + \cos \theta)$.
30. Determine the perimeter of $r = 2a \cos \theta$.
31. Find the length of the cissoid $r = \frac{a \sin^2 \theta}{\cos \theta}$ from the pole to the point (r_1, θ_1) .
32. Find the length of the spiral $r = e^{a\theta}$ from the pole to (r, θ) .
33. Find the length of the curve $r = a \cos^4 \frac{1}{4} \theta$.

[Hints : Required length is twice that of $\theta = 0$ to $\theta = 2\pi$.]

34. Find the length of the curve $r = \cos^2 \frac{1}{2} \theta$.
35. Determine the length of $r = a \sec^2 \frac{\theta}{2}$ from $\theta = 0$ to $\theta = \frac{\pi}{2}$.
36. Find the arc length of the equiangular spiral $p = r \sin \alpha$ between the radii vectors r_1 and r_2 .
37. Prove that the arc length of $p^2(a^4 + r^4) = a^4 r^2$ from $r = b$ to $r = c$ is equal to that of $xy = a^2$ from $x = b$ to $x = c$.
38. Prove that the length of the evolute of the parabola $y^2 = 4ax$ intercepted by the parabola is $4a(3\sqrt{3} - 1)$ units..

39. Prove that the length of the evolute of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$4 \left(\frac{a^2}{b} - \frac{b^2}{a} \right) \text{ units.}$$

40. Find the intrinsic equations of :

- (i) $x^2 = 4ay$ when s is measured from the vertex.
- (ii) $y^2 = 4ax$ when s is measured from the vertex.
- (iii) $y^2 = 4ax$ when s is measured from the vertex and the tangent is inclined at angle ψ with y -axis.
- (iv) $x^{2/3} + y^{2/3} = a^{2/3}$ when s is measured from one cusp on the x -axis and y -axis.
- (v) semicubical parabola $ay^2 = x^3$ when s is measured from the cusp.
- (vi) $p = r \sin \alpha$. (vii) $p^2 = r^2 - a^2$.
- (viii) cycloid $x = a(t + \sin t)$, $y = a(1 + \cos t)$ when s is measured from $t = 0$.

41. Find the intrinsic equation of evolute of the following curves :

- (i) cardioid $s = 8a \sin^2 \frac{\psi}{6}$. (ii) $s = c(e^{a\psi} - 1)$.
- (iii) $s = a \tan \psi + a \log (\tan \psi + \sec \psi)$.

42. Find the intrinsic equation of involute of the following curves :

- (i) $s = a \psi$. (ii) $9s = 4a (\sec^2 \psi - 1)$.

43. Find the length of the parabola $y^2 = 8x$ intercepted by the line $4x - y - 4 = 0$.

44. Find the length of the parabola $y^2 = 2x$ intercepted by the line $y = 4x - 1$.

45. A chain is hanging with ends fixed on a horizontal line 20 feet apart. The chain has a sag of 5 feet. If it is in the form of (i) circle and (ii) parabola, find its length.

[Hints : (i) $x^2 + y^2 - 25y = 0$, integrate from $y = 0$ to $y = 5$ and take the double (iii) $x^2 = 20y$.]

46. The chain of suspension bridge is in the form $x^2 = \left(\frac{b^2}{h} \right) y$. The lowest point of the chain is the origin of the curve, the y -axis is vertical, b is half the span of the chain and h is the sag. Express the length of the chain. When b is small compared with h , expand the integrand binomially, and show that the length of the chain is $(2b + 4h^2/3b)$ units.

Answers

1. $\left[\sqrt{b^2 + pb} + \frac{p}{2} \log \frac{2b + p + 2\sqrt{b^2 + pb}}{p} \right] \text{ units.}$
2. $a \{ \sqrt{2} + \log(1 + \sqrt{2}) \} \text{ units.}$
3. $2\pi a \text{ units.}$
4. $\sqrt{e^2 + 1} - \sqrt{2} + \log(1 + \sqrt{e^2 - 1}) / \{e(1 + \sqrt{2})\} \text{ units.}$
5. $\frac{1}{27} (80\sqrt{10} - 13\sqrt{13}) \text{ units.}$
6. $\log(2 + \sqrt{3}) \text{ units.}$
7. $\log(2 + \sqrt{3}) \text{ units.}$
8. $(\log(e^{2b} - 1) / (e^{2a} - 1) + a - b) \text{ units.}$
9. $a(\beta - \alpha) \text{ units.}$
10. $4a/\sqrt{3} \text{ units.}$
11. $4a/\sqrt{3} \text{ units.}$
12. $4a/\sqrt{3} \text{ units.}$
13. 137.7 units.
14. $\frac{8}{9} \left\{ 64 - \left(\frac{7}{4} \right)^{3/2} \right\} \text{ units.}$
15. $\frac{8}{243} (82\sqrt{82} - 1) \text{ uni.}$
16. $\sqrt{2} (e^x - 1) \text{ units.}$
17. $8a \text{ units.}$
18. $a [\log(1 + \sqrt{2}) + \sqrt{2}] \text{ units.}$
19. $4\sqrt{3} \text{ units.}$
20. $6a \text{ units.}$
21. $\frac{a^2 + ab + b^2}{a + b} \text{ units.}$
22. $6a \text{ units.}$
23. $\left[\frac{1}{2} a [\pi \sqrt{\pi^2 + 1} + \log(\sqrt{\pi^2 + 1} + \pi)] \right] \text{ units.}$
24. $8a \text{ units.}$
25. $8a \text{ units.}$
26. $2\pi a \text{ units.}$
27. $a \left[\frac{\sqrt{1 + 3 \cos^2 \theta}}{\cos \theta} - \frac{\sqrt{3}}{2} \log \frac{\sqrt{1 + 3 \cos^2 \theta} + \sqrt{3} \cos \theta}{\sqrt{1 + 3 \cos^2 \theta} - \sqrt{3} \cos \theta} \right]_0^{\theta} \text{ units.}$
28. $\left(\frac{r}{a} \sqrt{a^2 + 1} \right) \text{ units.}$
29. $\frac{16a}{3} \text{ units.}$
30. 4 units.
31. $a [\sqrt{2} + \log(\sqrt{2} + 1)] \text{ units.}$
32. $(r_2 - r_1) \sec \alpha \text{ units.}$
33. (i) $s = a [\tan \psi \sec \psi + \log(\tan \psi + \sec \psi)]$
34. (ii) $s = a [\cot \psi \operatorname{cosec} \psi + \log(\cot \psi + \operatorname{cosec} \psi)]$
35. (iii) (i); Hints : the tangent is inclined at an angle of $\left(\frac{\pi}{2} - \psi \right)$ with x -axis.
36. (iv) $s = \frac{3}{2} a \sin^2 \psi$ and $s = \frac{3}{2} a \cos^2 \psi$.
37. (v) $8s = 3a(2\psi - \sin 2\psi) + c\psi + \text{const.}$

$$(vi) s = c e^{\psi \cot \alpha}$$

$$(vii) 2s = a \psi^2$$

$$(viii) s = 4a \sin \psi$$

$$41. (i) s = \frac{4}{3} a \sin \frac{\psi}{3}$$

$$(ii) s = c (e^{a\psi} - 1) \sqrt{1 + a^2}$$

$$(iii) s = 2a (\sec^3 \psi - 1).$$

$$42. (i) s = \frac{a}{2} \psi^2 + c \psi + \text{const.}$$

$$(ii) 9s = 2a \{ \sec \psi \tan \psi + \log (\sec \psi + \tan \psi) \} + c \psi + \text{const.}$$

$$43. \frac{1}{2} \left\{ 4\sqrt{2} + \sqrt{5} + 4 \sinh^{-1} 1 + 4 \sinh^{-1} \frac{1}{2} \right\} = 5.18 \text{ units.}$$

$$44. \frac{1}{8} (4\sqrt{2} + \sqrt{5}) + \frac{1}{2} \log \frac{1}{2} (1 + \sqrt{2}) (1 + \sqrt{5}) \text{ units.}$$

$$45. (i) 23.18 \text{ feet, (ii) } 22.95 \text{ feet.}$$

$$46. 2 \int_0^b \sqrt{1 + 4h^2 x^2 / b^4} dx.$$

VOLUME AND SURFACE AREA OF SOLIDS OF REVOLUTION

13.1 SOLIDS OF REVOLUTION

A solid is formed by the revolution of an area about a line. The line is called the axis of revolution. For example :

(i) the revolution of the area of a right-triangle formed by $y = mx$, $x = 0$, $x = a$ and the x -axis about the x -axis produces a right circular cone (fig. 66) with axis as x -axis and vertex at the origin and radius of the base is ' ma ';

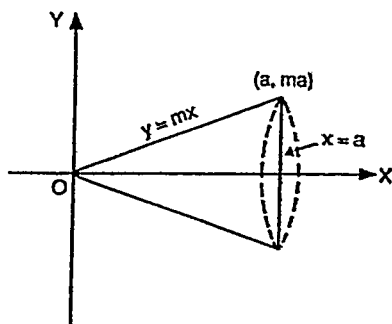


Fig. 66

(ii) the revolution of the same triangular area in (i) about $x = a$ also produces a right circular cone (fig. 67) with axis as $x = a$ and vertex at (a, ma) , and with radius of the base as ' a ';

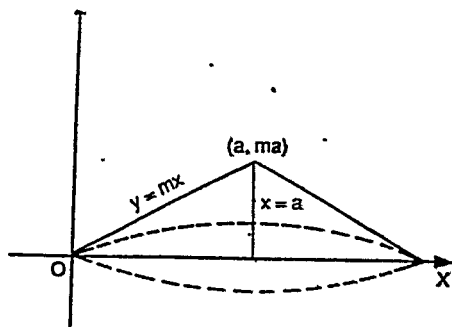


Fig. 67

(iii) the revolution of the same triangular area in (i) about the hypotenuse $y = mx$ produces a double right circular cone with same axis $y = mx$, opposite vertices $(0, 0)$ and (a, ma) and same radius of the base $(ma/\sqrt{m^2 + 1})$ (fig. 68);

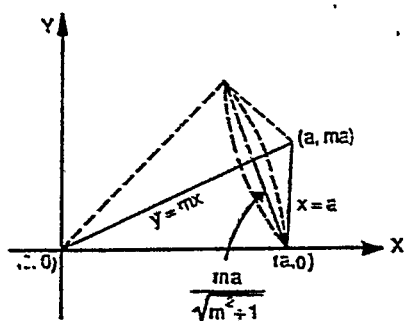


Fig. 68

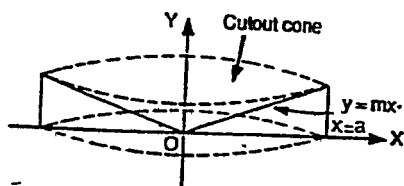


Fig. 69

(iv) the revolution of the same triangular area in (i) about y -axis or the origin produces a solid cylinder with conical groove formed by cutting out a cone with vertex at the origin and base radius ' a ' equal to that of the cylinder. Like the adjoining figure (fig. 69);

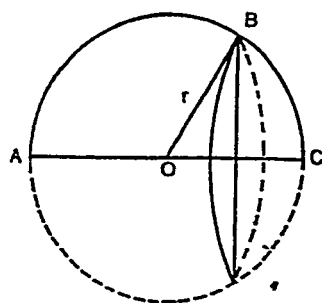


Fig. 70

(v) the revolution of the semi-circle ABC about its diameter AC produces a sphere with centre O and radius OB (fig. 70);

(vi) the revolution of the area of rectangle formed by $y = b, x = 0, x = a$ and the x -axis about any one of those 4 lines produces a right circular cylinder (fig. 71 shows the revolution about x -axis);

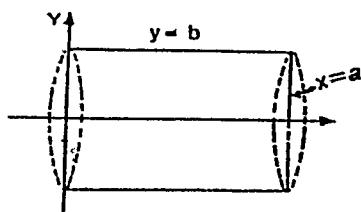


Fig. 71

(vii) in this way the revolution of a parabola, ellipse, hyperbola etc. about co-ordinate axes produces solids called paraboloid, ellipsoid, hyperboloid etc.

It should be noted that the axis of revolution may be any line, not necessarily the coordinate lines or the lines of the boundary of the enclosed area.

13.2 VOLUME OF SOLID OF REVOLUTION

Fundamental principle : The volume of any solid is determined by the product of the area of the base and its height

i.e. volume = (area of base) $\times$ height.

Let us determine the volume generated by the revolution of the area enclosed by $y = f(x)$, x -axis, the ordinates $x = a$ and $x = b$, about the x -axis (fig. 72).

The interval $[a, b]$ is divided into subintervals

$[a, x_1], [x_1, x_2], \dots, [x_r, x_{r+1}], \dots$

$\dots [x_r, b]$.

Let the length of the subinterval $[x_r, x_{r+1}]$ be Δx and x_r^* be a point in it. Let the rectangle with Δx as breadth and $f(x_r^*)$ as length be constructed.

If $y = f(x)$ is revolved about x -axis, the ordinate $f(x_r^*)$ will produce a circle with radius $f(x_r^*)$ and area $= \pi \{f(x_r^*)\}^2$. Due to this revolution the constructed rectangle will produce a cylinder of base area $\pi \{f(x_r^*)\}^2$ and height Δx . So, the volume of this cylinder $= \pi \{f(x_r^*)\}^2 \Delta x$. This volume is called an elementary volume. Let the length of the largest interval $\rightarrow 0$, then all Δx will $\rightarrow 0$. Under such condition limiting value of the sum of all the elementary volumes gives the volume of the solid generated. If V be this volume, then

$$V = \lim_{\Delta x \rightarrow 0} \sum \pi \{f(x_r^*)\}^2 \Delta x = \pi \lim_{\Delta x \rightarrow 0} \sum \{f(x_r^*)\}^2 \Delta x \quad (1)$$

$$= \pi \int_a^b \{f(x)\}^2 dx = \pi \int_a^b y^2 dx$$

$\therefore$ the volume of the solid generated by the revolution of the area enclosed by $y = f(x)$, $x = a$, $x = b$ and x -axis about x -axis is

$$V = \pi \int_a^b y^2 dx \quad (2)$$

Similarly, the volume of the solid generated by the revolution of the area enclosed by $x = f(y)$, $y = a$, $y = b$ and y -axis is

$$V = \pi \int_a^b x^2 dy. \quad (3)$$

N.B. : The curve should *not* cross the axes of co-ordinates, the axes of revolution.

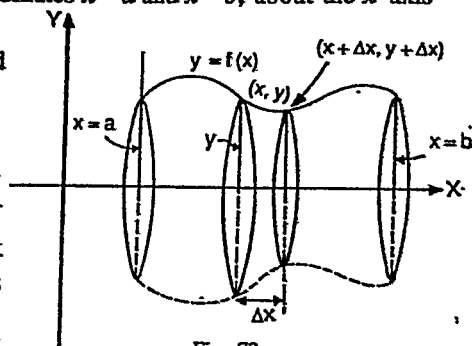


Fig. 72

13.2.1 OTHER AXIS OF REVOLUTION

The axis of revolution may be other than axes of co-ordinates. The following possibilities may occur :

(i) *The axis of revolution is parallel to y-axis as well as a boundary of the area.* This may happen in simple case when the area is enclosed by $y = f(x)$, $x = c$, $y = 0$ (fig. 73 (a) or fig. 73 (b)) showing only area which revolves about $x = c$, a line parallel to y-axis. For fig. 73 (a) radius of revolution $= (x - c)$ and for fig. 73 (b) that $= (c - x)$. So,

$$\text{Vol.} = \pi \int_0^a (x - c)^2 dy \text{ or } \pi \int_0^a (c - x)^2 dy.$$

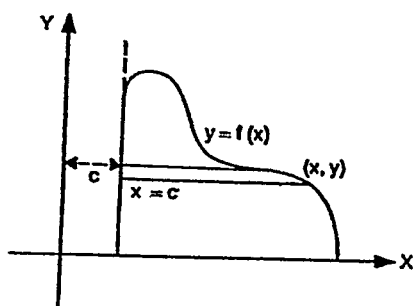


Fig. 73 (a)

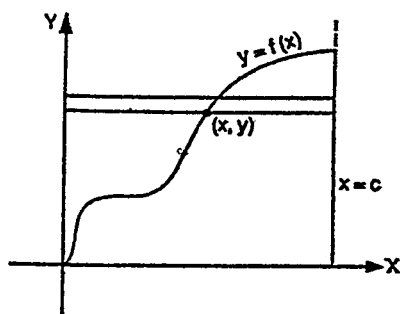
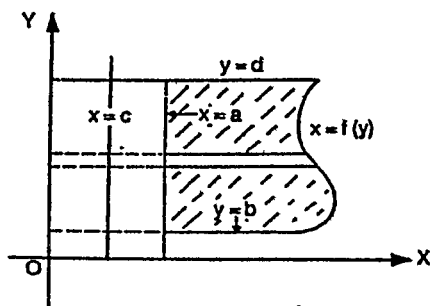


Fig. 73 (b)

(ii) The axis of revolution is parallel to y-axis but not a boundary of the area. This may happen when the area is enclosed by $x = f(y)$, $x = a$, $y = b$ and $y = d$, $b < d$ and the enclosed area is revolved about $x = c < a$ (fig. 74 shows only the area). The (radius)² of revolution $= (x - c)^2 - (a - c)^2$ and the



$$\text{Vol.} = \pi \int_b^d \{(x - c)^2 - (a - c)^2\} dy.$$

Fig. 74

(iii) Similar conclusions can be drawn for axis of revolution parallel to x-axis.

(iv) In general if any line $y = mx + c$ be the axis of revolution as well as one of the boundaries of the area then volume generated is

$$\text{Vol.} = \pi \int_B^C (PM)^2 d(AM)$$

where PM is the radius of revolution and A is a fixed point of line AC (fig. 75). If the line $y = mx + c$ does not form a part of the bounded area then radius of revolution is to be calculated as in (ii).

13.3 VOLUME OF HOLLOW SOLID OF REVOLUTION : CYLINDER METHOD

A hollow solid is formed by the revolution of a plane area about an axis beyond the area. Let $ABCD$ be a closed area not intersecting the x -axis. Let it be revolved about the x -axis and the revolution results into a hollow solid, two portions are shown in fig. 76, the hollow part may be seen as due to $A'B'C'B$ and solid parts are shown by dashes.

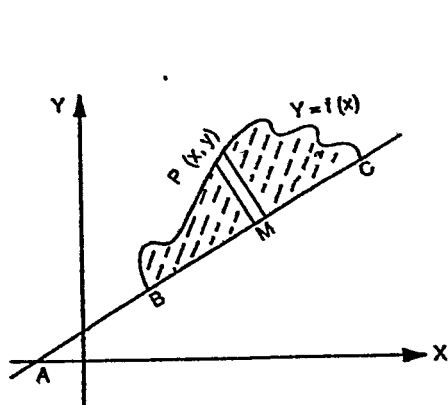


Fig. 75

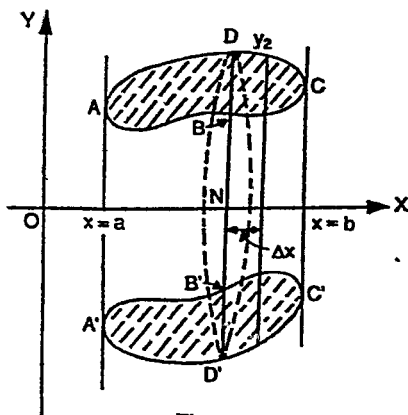


Fig. 76

Let equidistant planes Δx apart be pass through the solid perpendicular to the x -axis. As a result a set of hollow circular cylinders is created. Let the inner radius be $BN = y_1$ and outer radius be $DN = y_2$. Hence its volume $= \pi (y_2^2 - y_1^2) \Delta x$. Now summing up and taking limits when $\Delta x \rightarrow 0$, the total volume is

$$V_x = \pi \int_a^b (y_2^2 - y_1^2) dx \quad (1)$$

where a and b are the extreme x -co-ordinates of the plane area $ABCD$ and the subscript x denotes revolution about x -axis. The radii y_1 and y_2 are functions obtainable from the equation of the curve.

13.3.1 VOLUME OF HOLLOW SOLID OF REVOLUTION : SHELL METHOD

Revolution of a rectangle about one side produces a right circular cylinder with axis as the axis of revolution and radius equal to the other side (13.1 (vi)). If the axis of revolution be a line beyond the rectangle, the

revolution produces a cylindrical shell with a common axis. If R and r are the outer and inner radii of the shell with height h , then its volume is given by

$$V = \pi R^2 h - \pi r^2 h = \pi h (R + r) (R - r) \quad (2)$$

Let the closed curve in 13.3 be revolved about y -axis. The ordinates at the extreme points $x = a$ and $x = b$ of the curve forms rectangle with y -axis and due to revolution cylindrical shells are formed with common axis as y -axis (fig. 77).

Let $y = f(x)$ be the equation of the curve. Let $P(x, y)$ and $Q(x + \Delta x, y + \Delta y)$ be two neighbouring points on the curve. Due to revolution about y -axis the cylindrical shells are formed by them with radii $(x + \Delta x)$ and x respectively. Therefore,

$$\begin{aligned} \Delta V &= \pi y (x + \Delta x + x) (x + \Delta x - x) \text{ by (2)} \\ &= \pi y (2x + \Delta x) \Delta x \end{aligned}$$

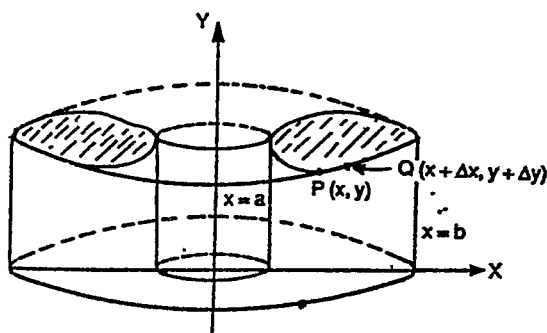


Fig. 77

Now taking limits of the sum of all such small volumes, we get,

$$V = 2\pi \int_a^b xy \, dx \quad (3)$$

$$[\because 2x + \Delta x \rightarrow 2x \text{ with } \Delta x \rightarrow 0.]$$

13.4 AREA OF SURFACE OF SOLID OF REVOLUTION

Describing as in 13.2 and instead of considering the area and the height of elementary cylinders, let us consider perimeter $2\pi f(x_r^*)$ of the base & arc length Δs . So, surface area approximates the area of a rectangle with dimensions $2\pi f(x_r^*)$ and Δs . Hence elementary surface area $\Delta S_x = 2\pi f(x_r^*) \Delta s$. Now forming sum of all such surface areas and taking limit when $\Delta s \rightarrow 0$, the total surface area of solids of revolution about x -axis is

$$S_x = 2\pi \int_{x=a}^b f(x) \, ds. \quad (4)$$

Similarly, surface area of revolution about y -axis is

$$S_y = 2\pi \int_{y=c}^d f(y) ds. \quad (5)$$

But $ds^2 = dx^2 + dy^2$. So, replacing ds by $\sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$ in (4) and by $\sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$ in (5) we have the formulae for surface area of revolution as

$$S_x = 2\pi \int_a^b f(x) \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \quad (6)$$

and
$$S_y = 2\pi \int_c^d f(y) \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy. \quad (7)$$

If the axis of revolution is other than axes of co-ordinates then we have to seek for the radius of elementary cylinders which is the **perpendicular distance** from any point (x, y) to the axis of revolution.

13.5 VOLUME AND SURFACE OF SOLID OF REVOLUTION IN POLAR CO-ORDINATES

The volume and the surface of solid of revolution in cartesian co-ordinates can easily be transferred to polar co-ordinates using the relation

$$x = r \cos \theta \text{ and } y = r \sin \theta \quad (8)$$

between them and identifying the axis of revolution the x -axis to initial line, y -axis to $\theta = \pi/2$ and any axis by changing to polar co-ordinates using (8). Thus note the following transformations.

(i) Let the area enclosed by $r = f(\theta)$, $\theta = \alpha$, $\theta = \beta$ be revolved about the initial line. Let this revolution be identical to that of cartesian $y = f(x)$, $x = a$ and $x = b$.

$\therefore$ **Volume** of solids of revolution about the initial line is given by

$$\begin{aligned} V_i &= \pi \int_a^b y^2 dx = \pi \int_\alpha^\beta (r \sin \theta)^2 \frac{d}{d\theta} (r \cos \theta) d\theta, \because dx = \frac{dx}{d\theta} d\theta \\ &= \pi \int_\alpha^\beta r^2 \sin^2 \theta \frac{d}{d\theta} (r \cos \theta) d\theta \\ &= \pi \int_\alpha^\beta [f(\theta)]^2 \sin^2 \theta \frac{d}{d\theta} \{f(\theta) \cos \theta\} d\theta \\ &= \pi \int_\alpha^\beta [f(\theta)]^2 (\sin^2 \theta) [f'(\theta) \cos \theta - f(\theta) \sin \theta] d\theta \end{aligned} \quad (9)$$

Also, **surface** area of solid of revolution about the initial line is given by

$$\begin{aligned}
 S_i &= 2\pi \int_{x=a}^b y \, ds = 2\pi \int_{\theta=\alpha}^{\beta} r \sin \theta \left(\frac{ds}{d\theta} \right) d\theta \\
 &= 2\pi \int_{\alpha}^{\beta} r \sin \theta \sqrt{r^2 + \left(\frac{dr}{d\theta} \right)^2} d\theta \\
 &= 2\pi \int_{\alpha}^{\beta} f(\theta) \sin \theta \sqrt{\{f(\theta)\}^2 + \{f'(\theta)\}^2} d\theta \quad (10)
 \end{aligned}$$

(ii) Similarly, the volume of solid of revolution about the line $\theta = \frac{\pi}{2}$

which corresponds to y -axis is given by

$$\begin{aligned}
 V_{\pi/2} &= \pi \int_c^d x^2 \, dy = \pi \int_{\alpha}^{\beta} (r \cos \theta)^2 \frac{d}{d\theta} (r \sin \theta) d\theta \\
 &= \pi \int_{\alpha}^{\beta} r^2 \cos^2 \theta \frac{d}{d\theta} (r \sin \theta) d\theta \\
 &= \pi \int_{\alpha}^{\beta} \{f(\theta)\}^2 (\cos^2 \theta) [f'(\theta) \sin \theta + f(\theta) \cos \theta] d\theta \quad (11)
 \end{aligned}$$

Also, surface area of solid of revolution about the line $\theta = \pi/2$ is given

by

$$\begin{aligned}
 S_{\pi/2} &= 2\pi \int_{y=c}^d x \, ds = 2\pi \int_{\theta=\alpha}^{\beta} r \cos \theta \frac{ds}{d\theta} d\theta \\
 &= 2\pi \int_{\alpha}^{\beta} r \cos \theta \sqrt{r^2 + \left(\frac{dr}{d\theta} \right)^2} d\theta \\
 &= 2\pi \int_{\alpha}^{\beta} f(\theta) \cos \theta \sqrt{\{f(\theta)\}^2 + \{f'(\theta)\}^2} d\theta \quad (12)
 \end{aligned}$$

(iii) If the axis of revolution be a line other than the standard axes i.e. initial line and $\theta = \pi/2$ in case of polar, then the volume and surface area are first set in cartesian and hence transfer to polar as above.

Examples

1. Find volume and surface area of the solid formed by the revolution of the area enclosed by x -axis, $y = 2x - x^2$ about the x -axis.

Let the solid formed by the revolution of the given area about x -axis be as in fig. 78. The curve intersects the x -axis at $(0, 0)$ and $(2, 0)$. So, volume is given by

$$\begin{aligned}
 V &= \pi \int_0^2 y^2 \, dx = \pi \int_0^2 (2x - x^2)^2 \, dx \\
 &= \frac{16}{15} \pi \text{ c. units} = 3.35 \text{ c. units}
 \end{aligned}$$

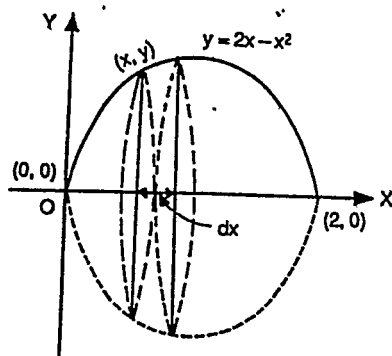


Fig. 78

Also, surface area of solid of revolution is

$$\begin{aligned}
 S &= 2\pi \int_0^2 y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\
 &= 2\pi \int_0^2 (2x - x^2) \sqrt{1 + (2 - 2x)^2} dx, \quad \because y = 2x - x^2 \\
 &= 2\pi \int_0^2 x(2 - x) \sqrt{1 + 4(1 - x)^2} dx \\
 &= 2\pi \int_{\tan^{-1} 2}^{-\tan^{-1} 2} \left(1 - \frac{1}{2} \tan \theta\right) \left(1 + \frac{1}{2} \tan \theta\right) \cdot \sec \theta \cdot \left(-\frac{1}{2}\right) \sec^2 \theta d\theta
 \end{aligned}$$

$$\text{when } 2(1 - x) = \tan \theta$$

$$dx = -\frac{1}{2} \sec^2 \theta d\theta$$

$$= \pi \int_{-\tan^{-1} 2}^{\tan^{-1} 2} \left(1 - \frac{1}{4} \tan^2 \theta\right) \sec^3 \theta d\theta$$

$$= 2 \cdot \frac{1}{4} \int_0^{\tan^{-1} 2} (4 - \tan^2 \theta) \sec^3 \theta d\theta$$

$$= \frac{1}{2} \int_0^{\tan^{-1} 2} (5 \sec^3 \theta - \sec^5 \theta) d\theta$$

$$= \frac{1}{2} \left[\frac{17}{8} \log (\sec \theta + \tan \theta) + \frac{17}{8} \frac{\sin \theta}{\cos^2 \theta} - \frac{\sin \theta}{4 \cos^3 \theta} \right]_0^{\tan^{-1} 2}$$

x	0	2
θ	$\tan^{-1} 2$	$-\tan^{-1} 2$

$$\begin{aligned}
 &= \frac{1}{2} \left[\frac{17}{8} \log(2 + \sqrt{5}) + \frac{17\sqrt{5}}{4} - \frac{5}{2} \right] \\
 &= \frac{1}{16} [17 \log(2 + \sqrt{5}) + 34\sqrt{5} - 20] = 4.17 \text{ s. units.}
 \end{aligned}$$

2. Find volume and surface area of the solid formed by the revolution of the cissoid

$x = 2a \sin^2 \theta$, $y = 2a \sin^3 \theta / \cos \theta$
about its asymptote.

The given cissoid is symmetrical about the x -axis. The equation of its asymptote is $x = 2a$ which is parallel to y -axis. The solid formed by the revolution is shown in fig. 79. Since the cissoid revolves about its asymptote, so the perpendicular distance of any point (x, y) on the cissoid to its asymptote

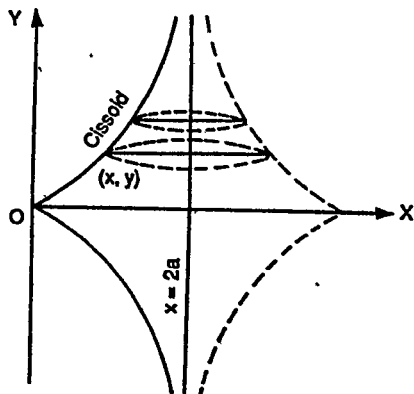


Fig. 79

$$= (2a - x).$$

Therefore, volume of solid of revolution is

$$\begin{aligned}
 V &= 2\pi \int_0^\infty (2a - x)^2 dy \\
 &= 2\pi \int_0^{\pi/2} 4a^2 \cos^4 \theta \cdot 2a \cdot \frac{3 \sin^2 \theta \cos^2 \theta + \sin^4 \theta}{\cos^2 \theta} d\theta \\
 &= 16\pi a^3 \int_0^{\pi/2} (3 \cos^4 \theta - 3 \cos^6 \theta + \sin^4 \theta - \sin^6 \theta) d\theta \\
 &= 2\pi^2 a^3 \text{ c. units.}
 \end{aligned}$$

Now, surface area of solid of revolution is

$$\begin{aligned}
 S &= 2 \cdot 2\pi \int_0^\infty (2a - x) \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy \\
 &= 4\pi \int_0^{\pi/2} 2a \cos^2 \theta \sqrt{1 + \frac{4 \cos^6 \theta}{\sin^2 \theta (2 \cos^2 \theta + 1)^2}} \\
 &\quad \cdot 2a \cdot \frac{3 \sin^2 \theta \cos^2 \theta + \sin^4 \theta}{\cos^2 \theta} d\theta \\
 &= 16\pi a^2 \int_0^{\pi/2} \frac{\sqrt{3 \cos^2 \theta + 1}}{\sin \theta (2 \cos^2 \theta + 1)} \cdot \sin^2 \theta (2 \cos^2 \theta + 1) d\theta
 \end{aligned}$$

$$\begin{aligned}
 &= 16\pi a^2 \int_0^{\pi/2} \sqrt{3 \cos^2 \theta + 1} \cdot \sin \theta \, d\theta \\
 &= \frac{16\pi a^2}{\sqrt{3}} \int_0^{\sqrt{3}} \sqrt{u^2 + 1} \, du \quad \text{when } u = \sqrt{3} \cos \theta \\
 &= \frac{16\pi a^2}{\sqrt{3}} \left(\sqrt{3} + \frac{1}{2} \log(2 + \sqrt{3}) \right) = 29.02 a^2 \text{ s. units.}
 \end{aligned}$$

3. Find volume and surface area of the solid formed by the revolution of the area enclosed by the parabola $y^2 = 4ax$ and a chord joining vertex and one end of latus rectum of the parabola, about the chord.

The solid of revolution is shown in fig. 80. The co-ordinate of L of the latus rectum LL' is $(a, 2a)$. Let A be the vertex of the parabola. So, equation of AL is $y = 2x$. If $P(x, y)$ is a point on the parabola, then let $PN = \text{perp. from } (x, y) \text{ to } AL = \frac{y-2x}{\sqrt{5}} = \frac{2\sqrt{ax}-2x}{\sqrt{5}}$.

Let $AN = h$. So, $h^2 = AP^2 - PN^2$

$$\begin{aligned}
 &= x^2 + y^2 - \left(\frac{y-2x}{\sqrt{5}} \right)^2 \\
 &= (x+2y)^2/5 \\
 \therefore h &= (x+2y)/\sqrt{5} \\
 &= \frac{x}{\sqrt{5}} + \frac{2}{\sqrt{5}} \cdot 2\sqrt{ax} \\
 \therefore dh &= \left(\frac{1}{\sqrt{5}} + \frac{2\sqrt{a}}{\sqrt{5}\sqrt{x}} \right) dx
 \end{aligned}$$

$\therefore$ the volume of the solid of revolution is

$$\begin{aligned}
 V &= \pi \int_0^{AL} PN^2 dh = \pi \int_0^a \frac{4}{5} (\sqrt{ax} - x)^2 \cdot \frac{1}{\sqrt{5}} \left(1 + \frac{2\sqrt{a}}{\sqrt{x}} \right) dx \\
 &= \frac{4\pi}{5\sqrt{5}} \int_0^a (ax + x^2 - 2\sqrt{a} x^{3/2}) \left(1 + \frac{2\sqrt{a}}{\sqrt{x}} \right) dx \\
 &= \frac{2\pi a^3}{15\sqrt{5}} = .19 a^3 \text{ c. units.}
 \end{aligned}$$

Also the surface area of the solid of revolution is

$$\begin{aligned}
 S &= 2\pi \int_0^{AL} PN \, ds = 2\pi \int_0^a \frac{2}{\sqrt{5}} (\sqrt{ax} - x) \sqrt{1 + \left(\frac{dy}{dx} \right)^2} dx \\
 &= \frac{4\pi}{\sqrt{5}} \int_0^a (\sqrt{ax} - x) \sqrt{\frac{x+a}{x}} dx = \frac{4\pi}{\sqrt{5}} \int_0^a (\sqrt{a} - \sqrt{x}) \sqrt{x+a} dx
 \end{aligned}$$

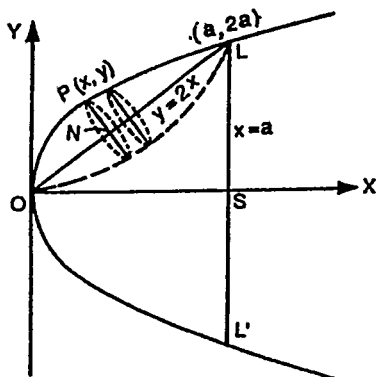


Fig. 80

$$\begin{aligned}
 &= \frac{4\pi}{\sqrt{5}} \cdot \left\{ \left[\sqrt{a} \cdot \frac{2}{3} (x+a)^{3/2} - \frac{2x+a}{4} \sqrt{x^2+ax} \right. \right. \\
 &\quad \left. \left. + \frac{a^2}{8} \log \left(x + \frac{a}{2} + \sqrt{x^2+ax} \right) \right]_0^a \right\} \\
 &= \frac{4\pi}{\sqrt{5}} a^2 \left[\frac{7\sqrt{2}}{12} - \frac{2}{3} + \frac{1}{8} \log(3+2\sqrt{2}) \right] = 1.43 a^2 \text{ s. units.}
 \end{aligned}$$

4. Find volume and surface area of the solid of revolution of the lemniscate $r^2 = a^2 \cos 2\theta$ about (i) the initial line; (ii) the line $\theta = \pi/2$ and (iii) tangent at the pole.

(i) The equation is in polar co-ordinates.

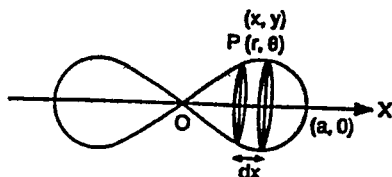


Fig. 81

So, volume in cartesian is

$$\begin{aligned}
 V &= 2\pi \int_0^a y^2 dx, [\because \text{lemniscate (fig. 81) is symmetrical} \\
 &\quad \text{about the initial line.}] \\
 &= 2\pi \int_{\pi/4}^0 r^2 \sin^2 \theta \frac{d}{d\theta} (r \cos \theta) d\theta \\
 &= 2\pi \int_{\pi/4}^0 a^2 \cos 2\theta \sin^2 \theta \frac{d}{d\theta} (a \sqrt{\cos 2\theta} \cos \theta) d\theta \\
 &= 2\pi a^3 \int_0^{\pi/4} \sin^2 \theta \sin 3\theta \sqrt{\cos 2\theta} d\theta \\
 &= -2\pi a^3 \int_1^{1/\sqrt{2}} \frac{-8t^6 + 14t^4 - 7t^2 + 1}{\sqrt{2t^2 - 1}} dt \text{ when } \cos \theta = t \\
 &= \frac{\pi a^3}{2} \left[\frac{1}{\sqrt{2}} \log(\sqrt{2} + 1) - \frac{1}{3} \right] \text{ c. units, [see Exercise 4. Prob. 47, Type 4.7]}
 \end{aligned}$$

Also, surface area of the solid formed is

$$\begin{aligned}
 S &= 2 \cdot 2\pi \int_0^{\pi/4} y \frac{ds}{d\theta} d\theta = 4\pi \int_0^{\pi/4} r \sin \theta \sqrt{r^2 + \left(\frac{dr}{d\theta} \right)^2} d\theta \\
 &= 4\pi a^2 (2 - \sqrt{2}) \text{ s. units.}
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii) } V &= 2\pi \int_0^{\pi/4} x^2 \frac{dy}{d\theta} d\theta = 2\pi \int_0^{\pi/4} r^2 \cos^2 \theta \cdot \frac{a \cos 3\theta}{\sqrt{\cos 2\theta}} d\theta \\
 &= 2\pi a^3 \int_0^{\pi/4} \sqrt{\cos 2\theta} \cos^2 \theta \cos 3\theta d\theta = \frac{\pi^2 a^3}{4\sqrt{2}} \text{ c. units} \\
 S &= 2 \cdot 2\pi \int_0^{\pi/4} x \frac{ds}{d\theta} d\theta = 4\pi a^2 \int_0^{\pi/4} \cos \theta d\theta = 2\sqrt{2} \pi a^2 \text{ s. units.}
 \end{aligned}$$

(iii) The equation of the tangent at the origin is $\theta = \pi/4$ i.e. $y = x$. Let $P(x, y)$ be a point (fig. 82) on the lemniscate. Let the foot of the perpendicular on the tangent be N . Let $ON = h$. So, $PN = \frac{x-y}{\sqrt{2}}$.

$$\therefore h = \sqrt{OP^2 - PN^2} = \frac{x+y}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}} a \sqrt{\cos 2\theta} (\cos \theta + \sin \theta)$$

$$\therefore dh = \frac{a}{\sqrt{2}} \left[\frac{\cos 3\theta - \sin 3\theta}{\sqrt{\cos 2\theta}} \right] d\theta.$$

$\therefore$ the required volume,

$$\begin{aligned}
 V &= 2\pi \int_{-\pi/4}^{\pi/4} PN^2 dh \\
 &= \frac{a\pi}{\sqrt{2}} \int_{-\pi/4}^{\pi/4} (r \cos \theta - r \sin \theta)^2 \left[\frac{\cos 3\theta - \sin 3\theta}{\sqrt{\cos 2\theta}} \right] d\theta \\
 &= \frac{a^3\pi}{\sqrt{2}} \int_{-\pi/4}^{\pi/4} \frac{\cos 2\theta (1 - \sin 2\theta) (\cos 3\theta - \sin 3\theta)}{\sqrt{\cos 2\theta}} d\theta \\
 &= \frac{1}{4} \pi^2 a^3 \text{ c. units}
 \end{aligned}$$

$$\begin{aligned}
 \text{and } S &= 2 \cdot 2\pi \int_{\theta=-\pi/4}^{\pi/4} PN ds = \frac{2 \cdot 4\pi a^2}{\sqrt{2}} \int_0^{\pi/4} \cos \theta d\theta \\
 &= 4\pi a^2 \text{ s. units.}
 \end{aligned}$$

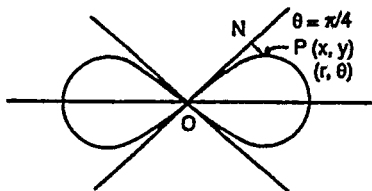


Fig. 82

Exercises 13

1. Find volume and surface area of the cylinder of revolution of the rectangle contained by $x = a$, $y = h$ about the x -axis.
2. Find volume and surface area of the cone formed by the revolution of the area enclosed by $y = mx$ and $x = a$ about the x -axis.
3. Find volume and surface area of the sphere formed by the revolution of the circle $x^2 + y^2 = r^2$ about its diameter.
4. Find volume and surface area of the solid formed by the revolution of the triangle contained by $y = 2 - x$ and the co-ordinate axes about the x -axis.

5. Find the volume of the truncated cone formed due to the revolution of the area enclosed by $y = 6 - x$, $y = 0$ and $x = 4$ about x -axis,
6. Find the volume of the solid formed by revolving the closed area. $y = x^2$, $y = 0$ and $x = 2$ about x -axis.
7. Find the volume of paraboloid of revolution whose surface is generated by revolving the arc of the parabola $y^2 = 2px$ between the origin and the point (x_1, y_1) about (i) x -axis, (ii) y -axis.
8. Find the volume generated by the revolution of the area enclosed by $y = 4 - x^2$, $y = 0$ about x -axis.
9. Find the volume generated by revolving about OX the areas bounded by the following loci :
 - (i) $ay^2 = x^2$, $y = 0$, $y = 2$;
 - (ii) the parabola $\sqrt{x} + \sqrt{y} = \sqrt{a}$, $x = 0$, $y = 0$;
 - (iii) $ay^2 = x^3$, $y = 0$, $x = a$;
 - (iv) $x^{2/3} + y^{2/3} = a^{2/3}$;
 - (v) $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$;
 - (vi) $a^2 x^2 + b^2 y^2 = a^2 b^2$;
 - (vii) $y = \frac{1}{(x-1)^3}$, $x = -1$, $x = 0$, $y = 0$;
 - (viii) an arch of $y = \sin x$.
10. Find the volume generated by revolving about OY the areas enclosed by the following curves :
 - (i) $y = x^3$, $y = 0$, $x = 2$;
 - (ii) $y = e^x$, $x = 0$, $y = 0$;
 - (iii) $9x^2 + 16y^2 = 144$;
 - (iv) $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$;
 - (v) $x = 2y - y^2$, $x = 0$.
11. The equation of the curve OA in the fig. 83 is $y^2 = x^3$. Find the volume generated when the area :
 - (i) OAB is revolved about OX .
 - (ii) OAB is revolved about AB .
 - (iii) OAB is revolved about CA .
 - (iv) OAB is revolved about OY .
 - (v) OAC is revolved about OY .
 - (vi) OAC is revolved about CA .
 - (vii) OAC is revolved about AB .
 - (viii) OAC is revolved about OX .

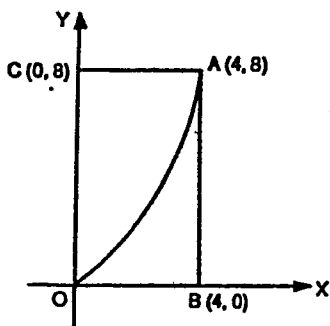


Fig. 83

12. Calculate the volume generated by revolving about each of the following lines the area which it cuts from the corresponding curve :
- (i) $y = x$, $y = x^2$. (ii) $y = x$, $y = 3x - x^2$.
13. Determine the volume generated when the area $y^2 = 4ax$ and $x = a$ is revolved about each of the following lines :
- (i) $x = a$, (ii) y -axis,
(iii) $x = 2a$, (iv) $x = -2a$.
14. Find the volume of the solids generated by an arch of the cycloid $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$ about OX and OY respectively.
15. Find the volume generated by revolving one loop of $y^2(a+x) = x^2(3a-x)$ about x -axis.
16. Determine the volume of the solids generated by revolving the following areas about OY . Use shell method :
- (i) $y = \sqrt{x}$, $x = 4$, $y = 0$. (ii) $y^3 = x$, $x = 8$, $y = 0$.
(iii) $x^2 - y^2 = 1$, $x = 3$.
17. Find the surface area of the solid generated by revolving the cardioid $x = a(2 \cos \theta - \cos 2\theta)$, $y = a(2 \sin \theta - \sin 2\theta)$ about its axis.
18. Find volume and surface area of the solids generated by revolving the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about its (i) major axis and (ii) minor axis.
19. Find the surface area of the solids generated by revolving the portion of the catenary $y = \frac{a}{2}(e^{x/a} + e^{-x/a})$ between $x = 0$ to $x = a$ about (i) OX and (ii) OY .
20. Find volume and surface area of the solid generated by revolving the cardioid $r = a(1 + \cos \theta)$ about the initial line.
21. Find the surface area of the solids obtained by revolving (i) the cardioid $r = a(1 - \cos \theta)$ and (ii) the curve $r = 2a \cos \theta$ about the initial line.
22. Show that the volume generated by the revolution of the loop of the curve $r \cos \theta = a \cos 2\theta$ about the initial line is $2\pi a^3 \left(\log 2 - \frac{2}{3} \right)$ c. units.
23. Show that if the area lying within the cardioid $r = 2a(1 + \cos \theta)$ and without the parabola $r(1 + \cos \theta) = 2a$ revolves about the initial line, the volume generated is $18\pi a^3$ c. units.

Answers

1. πah^2 c. units, $2\pi ah$ sq. units.
 2. $\frac{1}{3}\pi m^2 a^3$ c. units, $ma^2 \pi \sqrt{1+m^2}$ s. units.
 3. $\frac{4}{3}\pi r^3$ c. units, $4\pi r^2$ s. units. 4. $\frac{8}{3}\pi$ c. units, $4\sqrt{28}$ s. units.

5. $\frac{208\pi}{3}$ c. units. 6. $\frac{32\pi}{3}$ c. units.
7. (i) $\frac{1}{2}\pi y_1^2 x_1^2$ c. units; (ii) $\frac{1}{5}\pi x_1^2 y_1$ c. units.
8. $\frac{512\pi}{15}$ c. units.
9. (i) $\frac{\pi a^3}{4}$ c. units. (ii) $\frac{1}{15}\pi a^3$ c. units.
 (iii) $\frac{1}{4}\pi a^3$ c. units. (iv) $\frac{32}{105}\pi a^3$ c. units.
 (v) $\frac{32}{105}\pi ab^2$ c. units. (vi) $\frac{4}{3}\pi a^2 b$ c. units.
 (vii) $\frac{31}{160}\pi$ c. units. (viii) $\frac{1}{2}\pi^2$ c. units.
10. (i) $\frac{64}{5}\pi$ c. units. (ii) 2π c. units.
 (iii) 64π c. units. (iv) $\frac{4}{5}\pi a^2 b$ c. units. (v) $\frac{56}{15}\pi$ c. units.
11. (i) 64π c. unit; (ii) $\frac{1024}{35}\pi$ c. unit; (iii) $\frac{704}{5}\pi$ c. unit;
 (iv) $\frac{512}{7}\pi$ c. unit; (v) $\frac{384}{7}\pi$ c. unit; (vi) $\frac{576}{5}\pi$ c. unit;
 (vii) $\frac{3456}{35}\pi$; (viii) 192π c. units
12. (i) $\frac{1}{60}\sqrt{2}\pi$ c. units, (ii) $\frac{8\sqrt{2}}{15}\pi$ c. units.
13. (i) $\frac{32}{15}\pi a^3$ c. units. (ii) $\frac{16}{15}\pi a^3$ c. units.
 (iii) $\frac{112}{15}\pi a^3$ c. units. (iv) $\frac{32}{3}\pi a^3$ c. units.
14. $5\pi^2 a^3$ and $64\pi^3 a^3$ c. units. 15. $\pi(8 \log 2 - 3)a^3$ indicates c. units.
16. (i) $\frac{128}{5}\pi$ c. units. (ii) $\frac{768}{7}\pi$ c. units.
 (iii) $\frac{64\sqrt{2}}{3}\pi$ c. units.
17. $\frac{128}{5}\pi a^2$ s. units.
18. (i) $\frac{4}{3}\pi ab^2$ c. units, $2\pi ab(\sqrt{1-e^2} + e^{-1} \sin^{-1} e)$ s. units.
 (ii) $\frac{4}{3}\pi a^2 b$ c. units, $2\pi \left(a^2 + \frac{b^2}{e} \log \sqrt{\frac{1+e}{1-e}} \right)$ s. units.
19. (i) $\frac{\pi a^2}{4}(e^2 + 4 - e^{-2})$ s. units, (ii) $2\pi a^2(1 - e^{-1})$ s. units.
20. $\frac{8}{3}\pi a^3$ c. units, $\frac{32}{5}\pi a^2$ s. units.
21. (i) $32\pi a^2/5$ s. units, (ii) $4\pi a^2$ s. units.

OTHER APPLICATIONS OF INTEGRATION

14.1 INTRODUCTION

This is a proper time to *interpret* the definite integral after learning a good number of its applications in Chs. 11, 12 and 13. It is now clear that in area $f(x)$ and dx in the definite integral $\int_a^b f(x) dx$ are the two dimensions, say, length and breadth of an elementary rectangle. In finding length $f(x)$ is replaced by the differential of arc length. In volume $f(x)$ plays the role of area of a circle and dx is the height of an elementary cylinder. In this way, with different interpretations of $f(x)$, provided $f(x)$ is a continuous function, definite integral is being applied in Business, Economics, Sociology, Medicine, Physics and Engineering. All these things cannot be discussed in a single cover in details due to limitation of space and time. Here an attempt is made to give just an idea only. Integration has an wide application in solving differential equations discussed in Chs. 18 to 22. So it has been rightly said that integral calculus is *richer* than differential calculus.

14.2 BUSINESS, ECONOMICS AND SOCIOLOGY

(i) **Accumulation/Investment**: If $I(t)$ denotes the capital investment for a small time dt , then accumulation of capital is $I(t) dt$. Therefore the total accumulation or investment for the time interval t_1 to t_2 is given by the defined integral $\int_{t_1}^{t_2} I(t) dt$.

(ii) **Revenue**: If $p = f(x)$ is the price or demand function per commodity then for a sale of dx items, the revenue or sale proceed is $f(x) dx$. Therefore, the total revenue for sale between a to b is defined as

$$R(a, b) = \int_a^b f(x) dx.$$

(iii) **Consumer surplus, C**: It is the difference between the revenue possible $R(0, b)$ and the actual revenue pb for sales of b units at fixed price p . This is expressed by

$$C = \int_0^b f(x) dx - pb.$$

If C is positive, the seller is in advantageous position otherwise in disadvantageous position.

(iv) **Population density** : Let $f(x)$ be the population density of a city at a distance x from the centre of city. Then the total population between two distances a and b from the centre of city is defined by sociologists as

$$p(a, b) = 2\pi \int_a^b x f(x) dx.$$

14.3 MOMENTS, CENTROIDS AND MOMENT OF INERTIA

Moment measures the turning effect of a force about a point or line plane. Archimedes first expressed it mathematically. Moment is the prod. of the weight or mass or force and its distance from a point or line or plane. Thus moment M of a mass m at a distance x from O is $M = mx$. Similarly moment M of a system of masses $m_1, m_2, \dots, m_n$ situated at $x_1, x_2, \dots, x_n$ from O is the algebraic sum of moments i.e.

$$M = \sum_{i=1}^n m_i x_i.$$

If $M = 0$, there will be no turning effect and the system will be in equilibrium. Such a point for which moment is zero or the system is in equilibrium is called **centroid** or **centre of gravity (C. G.)** of the system. At the centroid, mass of a rigid body is supposed to be concentrated.

Let $G(\bar{x}, \bar{y})$ be the centroid of a continuous body. Let it be divided into smaller parts. Let (x, y) be the C. G. of Δm . Now the moment of Δm about x -axis is $y \cdot \Delta m$ and hence moment of the body in question about x -axis is $\sum y \Delta m$. Also the moment of the body is $\bar{y} \sum \Delta m$ about x -axis when mass of the body is supposed to be concentrated at C. G., $G(\bar{x}, \bar{y})$.

$\therefore$ These two moments of the same body must balance each other i.e.

$$\bar{y} \sum \Delta m = y \sum \Delta m.$$

Now taking limits when $\Delta m \rightarrow 0$, we have by def. of integral,

$$\bar{y} \int dm = \int y dm \quad \because \bar{y} \text{ is constant.}$$

$$\therefore \bar{y} = \frac{\int y dm}{\int dm} = \frac{M_y}{M} \quad (1)$$

$$\text{Similarly, } \bar{x} = \frac{\int x dm}{\int dm} = \frac{M_x}{M} \quad (2)$$

where M_x and M_y denotes moment about x and y -axes respectively and M is the total mass.

Let the body in question be (i) a thin uniform wire or filament, (ii) a thin uniform plate or shell and (iii) a solid. The first two cases will be taken here and the last two will be considered in multiple integrals Ch. 16.

Case (i). Thin uniform wire or filament : In this case $dm = \delta ds$ where δ is the density defined as the mass per unit length. So, (1) and (2) becomes

$$\bar{x} = \frac{\int x \delta ds}{\int \delta ds} \quad \text{and} \quad \bar{y} = \frac{\int y \delta ds}{\int \delta ds} \quad (3)$$

when $\delta = \text{const.}$, (3) simplifies to

$$\bar{x} = \frac{\int x ds}{\int ds} \quad \text{and} \quad \bar{y} = \frac{\int y ds}{\int ds} \quad (4)$$

where, both for (3) and (4), either $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$

or $ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$.

Limits are to be so chosen to cover the whole of the wire or fillament in question. The numerators of (3) and (4) are the moments M_x and M_y .

Case (ii). Thin uniform plate or shell : In this case $dm = \delta dA$ where δ is the density defined as the mass per unit area. Also the centroid lies in the centre of $dA = y dx$ and it is $\left(x + \frac{dx}{2}, \frac{y}{2}\right)$. But as $dx \rightarrow 0$ the centroid $\rightarrow (x, y/2)$. Therefore, equation (1) and (2) becomes,

$$\left. \begin{aligned} \bar{x} &= \frac{\int x dm}{\int dm} = \frac{\int x \delta y dx}{\int \delta y dx} = \frac{\int \delta xy dx}{\int \delta y dx} \\ \text{and } \bar{y} &= \frac{\int \frac{1}{2} y dm}{\int dm} = \frac{\frac{1}{2} \int y \cdot \delta y dx}{\int \delta y dx} = \frac{\frac{1}{2} \int \delta y^2 dx}{\int \delta y dx} \end{aligned} \right\} \quad (5)$$

$$\text{and the moments } M_y = \int \delta xy dx \quad \text{and} \quad M_x = \frac{1}{2} \int \delta y^2 dx \quad (6)$$

For constant density (5) simplifies cancelling δ from both numerator and denominator.

For symmetrical cases, moment will be zero about the axis of symmetry and hence centroid lies on the axis of symmetry.

N.B. : If density = 1, then in case (i) arc length = mass and in case (ii) area = mass.

Moment of inertia : It is the product of the mass and the square of its distance from an axis of rotation. Thus moment of inertia I due to an elementary mass dm at the point (x, y) from the x -axis is $I_x = \int y^2 dm$ and that from y -axis

is $I_y = \int x^2 dm$. Therefore, moment of inertia of a **thin wire or filament** is as in (3) and (4) $I_x = \int y^2 \delta ds$ about x -axis and that about y -axis is $I_y = \int x^2 \delta ds$ where $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$ or its equivalent. Similarly moment of inertias of thin uniform plate of shell above are

$$I_x = \frac{1}{4} \int y^2 dm = \frac{1}{4} \int y^3 \delta dx \text{ and } I_y = \int x^2 dm = \int x^2 y \delta dx \text{ respectively.}$$

Radius of gyration : If $I = MR^2$ for a mass M then R is called the radius of gyration. See Ch. 16 for further discussion. Equivalent formulae for a thin wire or plate can be easily written.

14.3.1 CENTROID OF A SOLID OF REVOLUTION

The centroid of a homogeneous solid is identical with its geometrical centroid. Let an area be revolved about OX of (fig. 73) to generate a volume. Then its centroid lies on OX and let it be $(\bar{x}, 0)$. Let dV be an elementary volume of the generated cylinder with height, Δx and radius of the base, y . So, $dV = \pi y^2 \Delta x$. Now the **moment of volume** of this cylinder w.r.t. the plane through OY and perpendicular to OX is

$$dM_y = x dV = \pi xy^2 \Delta x$$

$$\therefore M_y = \int \pi xy^2 dx$$

$$\text{Also, } M_y = V \bar{x} = \int \pi xy^2 dx.$$

$$\therefore \bar{x} = \frac{\int \pi xy^2 dx}{V} = \frac{\pi \int xy^2 dx}{\pi \int y^2 dx} = \frac{\int xy^2 dx}{\int y^2 dx}$$

Theorem of Pappus :

It is an important theorem. It consists of two parts. Its one part gives volume and the other, surface area if C. G. is known and vice-versa.

Part one : *If an arc of a plane curve, lying on one side of a line L , is revolved about L , then the surface area generated by the arc is equal to the product of the length of the arc and the distance travelled by its C. G.*

Proof : Let the arc be revolved about x -axis. Then by (4) of 14.3,

$$\bar{y} = \frac{\int y ds}{s}$$

$$\therefore 2\pi \bar{y} s = \int 2\pi y ds, \text{ multiplying by } 2\pi.$$

Here R.H.S. = Surface area and L.H.S. = the distance travelled by C. G.. Hence proved.

Part two : *If a plane area, lying on one side of a line L in its plane, is revolved about L , then the volume generated is equal to the product of the area and the distance travelled by its C. G.*

Proof : Let the area be revolved about x -axis. Let the area be divided into smaller areas by drawing lines parallel to x -axis. Then y -coordinate of the C. G. is

$$\bar{y} = \frac{\int y \, dA}{A}, \quad A \text{ is the given area.}$$

$$\therefore 2\pi \bar{y} A = \int 2\pi y \cdot dA = \int 2\pi y x \, dy = 2\pi \int xy \, dy$$

$$= V, \text{ by shell method}$$

$\therefore$ L.H.S. = product of the area, A and the distance travelled by C. G. = volume in the right. Hence proved.

14.4 FORCE ON DAM, FLUID PRESSURE AND CENTRE OF PRESSURE

Pressure or force due to a body of water in a reservoir or behind a dam are of the same nature. Force of liquid against a vertical dam or on a plane region submerged vertically in a liquid is to be determined here. Before that the following facts regarding hydrostatic pressure should be remembered :

- (i) Pressure is force per unit area;
- (ii) If the liquid is at rest, the pressure is same in all directions;
- (iii) If ω = weight per unit, d = depth and p = pressure then $p = \omega d$ i.e. pressure is proportional to depth;
- (iv) As a result of (i) and (iii), if a region of area A is such that all points of the region are at the same depth, the force on the region is $F = pA = \omega dA$;
- (v) Force does not depend on the shape of the sides of the vessel, the force on the flat bottom is same.

For a flat plate submerged horizontally can be treated by (iv) but it is not valid for vertical submerge or for a vertical dam. To overcome this difficulty, the case of a vertical dam is taken for the determination of force and this will be prototype for the force exerted by a body of water in a container or on a vertically submerged plane region.

For the sake of convenience, let the bed of the river be horizontal i.e. not wavy. This is demonstrated by a co-ordinate system whose x -axis is horizontal on the surface of the liquid and +ve y -axis is downward through the middle of the dam (fig. 84) and its inversion coincide with river bed and actual dam. Let the width of the dam at a depth y be $W(y)$ and let it be a continuous function of y . Let the depth d be subdivided by $y_0, y_1, \dots, y_n$. By drawing horizontal lines at those y -values, the dam is decomposed into n rectangles.

$\therefore$ force on the i th rectangle is $\rho y_i W(y_i) \Delta y_i$

where ρ is the density of the liquid. Now forming the sum and taking limit as

$$\Delta y \rightarrow 0, \text{ the total force } F = \rho \int_0^d y W(y) dy \quad (1)$$

for constant density. Note that this is precisely the moment of a lamina in the form of a dam about the x -axis.

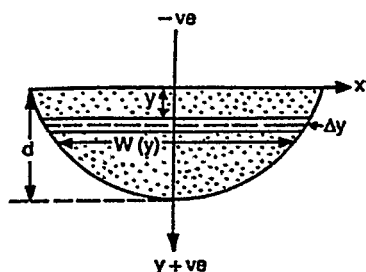


Fig. 84

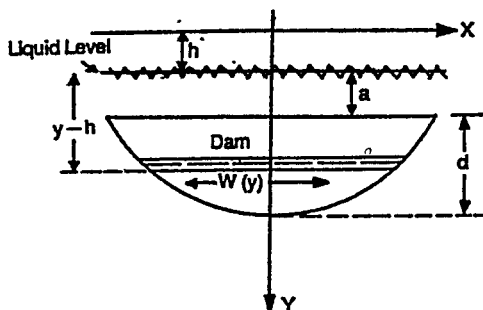


Fig. 85

Cor. 1 : If the x -axis is h units above the liquid surface and the dam be considered a units below the liquid-surface as in fig. 85. Then force

$$F = \rho \int_a^{a+d} (y-h) W(y) dy \quad (2)$$

This force will remain unaffected if the dam be unsymmetrical about the assumed y -axis.

Centre of pressure : The point at which the resultant of the liquid forces acts is called the centre of pressure. Since force does not depend on the position of assumed y -axis, so, sum of the moments of the horizontal elementary rectangles about axes is equal to the moment of resultant force about the axes.

Therefore, taking $(0, \bar{y})$ as the centre of pressure and taking moments about x -axis, we have, from equation (2)

$$\begin{aligned} \bar{y} \rho \int_a^{a+d} (y-h) W(y) dy &= \lim_{\Delta y \rightarrow 0} \Sigma (y-h) \rho W(y) y \Delta y \\ &= \rho \int_a^{a+d} (y-h) y W(y) dy \\ \bar{y} &= \frac{\int_a^{a+d} (y-h) y W(y) dy}{\int_a^{a+d} (y-h) W(y) dy} \end{aligned}$$

or,

14.5 WORK

Work, by definition of physics, is the product of force and distance *i.e.*

Work, $W = Fx$ ft. lbs if F is in lbs and distance x is in feet.

This holds good so long as force F is of constant magnitude. But for variable force, integral come into play.

Let there be a variable force F . So, work for small displacement Δx is $F \cdot \Delta x$. Taking the limit of all sums as $\Delta x \rightarrow 0$ we get,

$$W = \lim_{\Delta x \rightarrow 0} \sum F \cdot \Delta x = \int F dx.$$

Let the point of application undergoes a displacement from $x = a$ to $x = b$. Therefore, the total work done in moving from a to b is

$$W = \int_a^b F dx. \quad (1)$$

Let the force of mass m acting on the body be F and velocity attained be v . Therefore,

$$\begin{aligned} W &= \int_a^b \left(m \frac{dv}{dt} \right) dx, \text{ by Newton's second law } F = m f = m \frac{dv}{dt}. \\ &= \int_a^b m \cdot \frac{dv}{dx} \cdot \frac{dx}{dt} dx = \int_a^b m v \frac{dv}{dx} dx, \quad \because v = \frac{dx}{dt} \\ &= \int_a^b m v dv = \left[\frac{1}{2} m v^2 \right]_a^b = \left[\frac{1}{2} m v^2 \right]_{v=a}^{v=b} \\ &= (K.E. \text{ at } b) - (K.E. \text{ at } a) = \text{change in } K.E. \end{aligned} \quad (2)$$

$\therefore$ total work done = change in $K.E.$

Potential : It is the work done in moving the particle to infinity.

Examples

1. The rate of investment is defined by the function $I = 12t^{1/3}$. Find the amount of capital accumulation during the time interval 1 to 3.

$$\begin{aligned} \text{By def., capital accumulation} &= \int_1^3 I(t) dt \\ &= \int_1^3 12t^{1/3} dt = 12 \cdot \frac{3}{4} t^{4/3} \Big|_1^3 = 9 [3^{4/3} - 1^{4/3}] = 29.94 \text{ units.} \end{aligned}$$

2. If the demand function is $p = 11 - x^2/100$, find the consumer surplus when the selling price is $p = 2$.

Using $p = 2$ in the given demand function, we have,

$$2 = 11 - \frac{x^2}{100} \text{ or } x = 30 = \text{number of units sold.}$$

$$\therefore \text{consumer surplus, } C = \int_0^{30} \left(11 - \frac{x^2}{100} \right) dx - 2(30)$$

$$= \left[11x - \frac{x^3}{300} \right]_0^{30} - 60 = 180.$$

3. Find the population within 4 miles of the centre of a city whose population density is given by $f(x) = 10,000/(x^2 + 1)^4$ where x is the number of miles from the centre of the city.

$$\begin{aligned} \text{We have } P(0, 4) &= 2\pi \int_0^4 \frac{10,000x}{(x^2 + 1)^4} dx \\ &= 10,000\pi \left(\frac{1}{3} - \frac{1}{3(17)^3} \right) \approx 10,470. \end{aligned}$$

4. A thin uniform wire is bent into a quadrant of a circle of radius 'a'. Find the moments and the co-ordinates of the centroid.

Let the equation of the circle into which the wire is bent be (fig. 86) $x^2 + y^2 = a^2$. Therefore, $\frac{dy}{dx} = -\frac{x}{y}$.

$$\begin{aligned} \therefore ds &= \sqrt{1 + \left(\frac{dy}{dx} \right)^2} dx \\ &= \sqrt{1 + \frac{x^2}{y^2}} dx = \frac{a}{y} dx \end{aligned}$$

Now, moment about x-axis is

$$M_x = \int y ds = \int_0^a y \cdot \frac{a}{y} dx = a^2$$

and moment about y-axis is

$$\begin{aligned} M_y &= \int x ds = \int_0^a x \frac{a}{y} dx = \int_0^a \frac{ax}{\sqrt{a^2 - x^2}} dx \\ &= -\frac{1}{2}a \cdot \left[2\sqrt{a^2 - x^2} \right]_0^a = a^2 \end{aligned}$$

$$\text{Also length of the wire, } L = \int_0^a ds = \int_0^a \frac{a}{y} dx = \int_0^a \frac{a}{\sqrt{a^2 - x^2}} dx$$

$$= a \cdot \left[\sin^{-1} \frac{x}{a} \right]_0^a = \frac{a\pi}{2} \text{ units.}$$

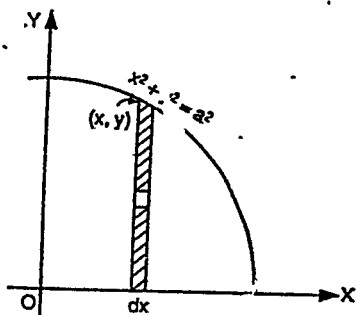


Fig. 86

$$\therefore \text{the } x\text{-co-ordinate of the centroid, } \bar{x} = \frac{\int x ds}{\int ds} = \frac{M_y}{L} = \frac{a^2}{\frac{a\pi}{2}} = \frac{2a}{\pi}$$

$$\text{and the } y\text{-co-ordinate of the centroid, } \bar{y} = \frac{\int y ds}{\int ds} = \frac{M_x}{L} = \frac{2a}{\pi}.$$

$$\therefore \text{the reqd. centroid is } (\bar{x}, \bar{y}) = \left(\frac{2a}{\pi}, \frac{2a}{\pi} \right)$$

5. Find moments and centroid of the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the first quadrant.

The required part of the ellipse is OAB shown in (fig. 87). Now the area A of

$$OAB = \int_0^a y dx = \frac{\pi}{4} ab.$$

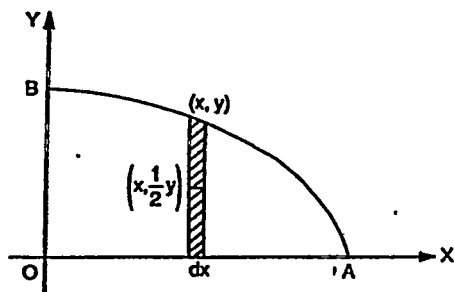


Fig. 87

Now the moment of OAB about x -axis is

$$\begin{aligned} M_x &= \frac{1}{2} \int_0^a y^2 dx = \int_0^a b^2 \left(1 - \frac{x^2}{a^2} \right) dx \\ &= \frac{1}{2} b^2 \cdot \frac{2}{3} a = \frac{2}{3} ab^2 \end{aligned}$$

and the moment of OAB about y -axis is $M_y = \int_0^a xy dx$

$$= \int_0^a bx \sqrt{1 - x^2/a^2} dx = \frac{1}{3} a^2 b$$

$$\therefore x \text{ co-ordinate of the centroid is } \bar{x} = \frac{M_y}{A} = \frac{\frac{1}{3} a^2 b}{\frac{\pi ab}{4}} = \frac{4a}{3\pi}$$

$$\text{and } y\text{-co-ordinate of the centroid is } \bar{y} = \frac{M_x}{A} = \frac{(2/3) ab^2}{\frac{\pi ab}{4}} = \frac{8b}{3\pi}$$

$$\therefore \text{ the centroid is } (\bar{x}, \bar{y}) = \left(\frac{4a}{3\pi}, \frac{8b}{3\pi} \right)$$

6. Find the moment of the area of the semi-circle $y = \sqrt{r^2 - x^2}$, $-r \leq x \leq r$, about the line $y = x - r$. Hence find the y -co-ordinate of the centroid of this area, the volume and the surface area of the solid of revolution about the given line.

The semi-circle is shown in fig. 88. Let $Q\left(x, \frac{1}{2}y\right)$ be centroid of the ordinate of $P(x, y)$. Let QN be drawn perpendicular upon the given line $y = x - r$. So,

$$QN = \frac{\frac{1}{2}y - x + r}{\sqrt{1^2 + 1^2}} = \frac{\frac{1}{2}y - x + r}{\sqrt{2}}$$

$\therefore$ the moment of the area of the semi-circle about $y = x - r$ is

$$\begin{aligned} M_x &= \int_{-r}^r QN \cdot y \, dx = \frac{1}{\sqrt{2}} \int_{-r}^r \left(\frac{1}{2}y - x + r \right) y \, dx \\ &= \frac{1}{\sqrt{2}} \int_{-r}^r \left[\frac{1}{2}(r^2 - x^2) - x\sqrt{r^2 - x^2} + r\sqrt{r^2 - x^2} \right] dx \\ &= \frac{r^3(3\pi + 4)}{6\sqrt{2}} \end{aligned}$$

Since area of the semi-circle, $A = \frac{1}{2} \pi r^2$. Therefore, y -co-ordinate of the centroid,

$$\bar{y} = \frac{M_x}{A} = \frac{r(3\pi + 4)}{3\pi\sqrt{2}}$$

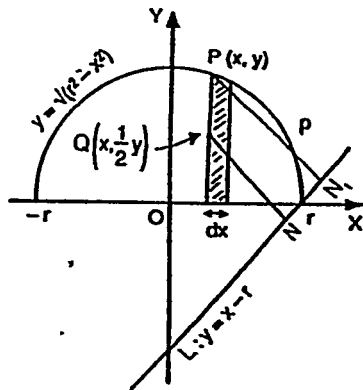


Fig. 88

Hence using Pappus theorem, vol., $V = 2\pi \bar{y} A = 2\pi \cdot \frac{r(3\pi+4)}{3\pi\sqrt{2}} \cdot \pi r^2$
 $= \frac{(3\pi+4)}{3} \cdot \frac{\pi r^3}{\sqrt{2}} \text{ c. units.}$

Surface Area : From Pappus theorem,

Surface area, $S = 2\pi \bar{y} s = 2\pi \int_{-r}^r p \, ds$ where p is the perpendicular distance of $P(x, y)$ from L . Therefore

$$\begin{aligned} S &= 2\pi \int_{-r}^r \frac{y-x+r}{\sqrt{2}} \cdot \sqrt{1+\left(\frac{dy}{dx}\right)^2} \, dx \\ &= \frac{2\pi}{\sqrt{2}} \int_{-r}^r (\sqrt{r^2-x^2}-x+r) \sqrt{1+\left(\frac{-2x}{2\sqrt{r^2-x^2}}\right)^2} \, dx \\ &= \sqrt{2}\pi \int_{-r}^r (\sqrt{r^2-x^2}-x+r) \cdot \frac{r}{\sqrt{r^2-x^2}} \, dx \\ &= \sqrt{2}\pi \int_{-r}^r \left(r - \frac{xr}{\sqrt{r^2-x^2}} + r^2 \cdot \frac{1}{\sqrt{r^2-x^2}} \right) \, dx \\ &= \sqrt{2}\pi \left[rx + r \frac{1}{2} \cdot 2\sqrt{r^2-x^2} + r^2 \sin^{-1} \frac{x}{r} \right]_{x=-r}^r \\ &= \sqrt{2}\pi r^2 (\pi+2) \text{ sq. units.} \end{aligned}$$

7. Find the force of water against an elliptical dam if the major axis of length 50 ft. is at the surface of water and if the semi-minor axis has length 20 ft. Assume water weighs 62.5 lbs per cubic foot.

The equation of the ellipse as per description is $\frac{x^2}{25^2} + \frac{y^2}{20^2} = 1$.

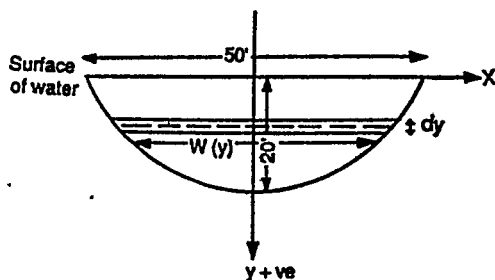


Fig.. 89

The shape of the dam is like fig. 89.

Let the width of the dam at a depth y be $W(y)$. Therefore, force is

$$\begin{aligned}
 F &= \rho \int_0^{20} y W(y) dy \\
 &= \rho \int_0^{20} 25.2 y \sqrt{1 - \frac{y^2}{20^2}} dy = 50 \rho \int_0^{20} y \sqrt{1 - \frac{y^2}{20^2}} dy \\
 &= -25 \rho \cdot 400 \cdot \frac{2}{3} \left(1 - \frac{y^2}{20^2} \right)^{3/2} \Bigg|_0^{20} \\
 &= 25 \rho \cdot 400 \cdot \frac{2}{3} = 25 (62.5) 400 \left(\frac{2}{3} \right) = 4,16,666.6667 \\
 &= 4,16,667 \text{ lbs. (approx.)}
 \end{aligned}$$

8. A tank in the shape of a cone is placed with its vertex at the bottom. The radius at the top is 4 ft. and the height is 12 ft. The tank contains liquid to a depth of 9 ft. If the liquid weighs ω lb per cu. ft., find the measure of work done in pumping the liquid out over the top.

The dimension of the cone is shown in fig. 90. The force necessary to overcome the force of gravity is the vertical force. Let the liquid in the tank be considered as being made up of disk-like slabs of thickness Δy . So, work done in lifting the slab at a height y to the top of the tank is, approximately,

$$(12 - y) \pi \omega x^2 \Delta y$$

where x is the radius of the slab and
weight of the slab = force due to the
slab = volume, $(\pi x^2 \Delta y) \times$ unit
weight, ω .

$\therefore$ total work done,

$$W = \int_0^9 \pi \omega x^2 (12 - y) dy$$

To express x in terms of y , we
have the properties of similar triangles,
as

$$\frac{x}{4} = \frac{y}{12} \text{ or } x = \frac{1}{3}y$$

$$\therefore W = \int_0^9 \pi \omega \frac{1}{9} y^2 (12 - y) dy = \frac{567}{4} \pi \omega \text{ ft. lbs.}$$

If the liquid is water, $\omega = 62.5$ lbs, & $W \approx 27,818.5$ ft. lbs.

9. A force of 10 lbs is required to compress a spring of natural length of 20 inches to a length of 19 inches. What is the work done in stretching the spring from a length of 24 inches to a length of 30 inches?

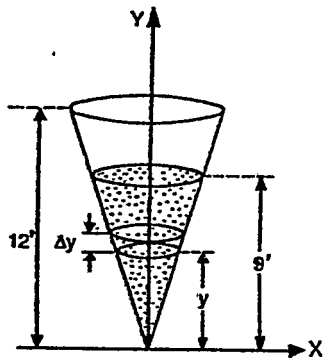


Fig. 90

Since work is measured in feet lbs. So, let us convert inches to feet.

$F = 10$ lbs. Length compressed $20'' - 19'' = 1'' = \frac{1}{12}$ ft. So, by Hooke's law

$$F = 10 = K \cdot \frac{1}{12}, \text{ const. of proportionality.}$$

Spring const. $K = 120$ lbs/foot

$$\text{Now work done, } W = \int_{1/3}^{5/6} 120x \, dx = 35 \text{ ft. lbs.}$$

10. Find the moment of inertia and radius of gyration of circular area about its diameter.

Let the equation of the circle be $x^2 + y^2 = a^2$ (fig. 91). Let the diameter be along x -axis. So taking moment of inertia about x -axis,

$$\begin{aligned} I_x &= \int_{-a}^a y^2 \cdot \rho \cdot 2x \, dy \\ &= 2\rho \int_{-a}^a \sqrt{a^2 - y^2} \, y^2 \, dy, \text{ taking } \rho = \text{const.} \end{aligned}$$

$$= \rho \pi a^2 \cdot \frac{a^2}{4} = M \frac{a^2}{4} \text{ if } M = \text{mass}$$

$$\therefore R_x^2 = \frac{I_x}{M} = \frac{a^2}{4}$$

$$\therefore R_x = \frac{a}{2}$$

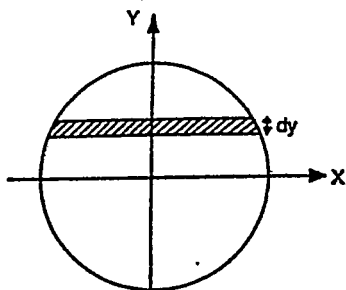


Fig. 91

Exercises 14

1. (i) The rate of investment is defined by $I(t) = 3t^{1/2}$ (thousands of dollars per year). What will be the capital formation during the time interval 1 to 4?
- (ii) For each demand function $p = f(x)$, find the consumer surplus for the given actual price p per unit sold or number x of units sold.
 - (a) $p = 200 - (x + 2)^2$, $x = 8$, (b) $p = 50 - \sqrt{x}$, $p = 47$,
 - (c) $p = 100 - x\sqrt{x^2 + 9}$, $x = 4$, (d) $p = 20 - \frac{x}{50}$, $p = 100$.
- (iii) Find the population between the given numbers of miles from the centre of a city for the indicated population density $f(x)$ at a distance x miles from the centre:
 - (a) $f(x) = 10,000/(x^2 + 2)^3$, within 34 miles.
 - (b) $f(x) = 10,000/(x^2 + 2)^3$ from 3 miles to 8 miles.

2. (i) Find the moment about the line $y = -r$ of the area in the semi-circle $y = \sqrt{r^2 - x^2}$, $-r \leq x \leq r$. Find also the volume using Pappus theorem.
- (ii) Find the moment of the area of the semi-circle $y = \sqrt{r^2 - x^2}$, $-r \leq x \leq r$, about x -axis.
- (iii) Find the centre of mass of a thin homogeneous plate covering the area of the first quadrant of the circle $x^2 + y^2 = a^2$.
- (iv) Find the centroid of the area enclosed by the following curves :
- (a) $y^2 = 4x$, $x = 4$; (b) $y = x^3$, $x = 2$, $y = 0$;
 (c) $\sqrt{x} + \sqrt{y} = a$, $x = 0$, $y = 0$;
 (d) $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$.
- (v) Find the centroid of the arc of the following curves :
- (a) $x^{2/3} + y^{2/3} = a^{2/3}$, above the x -axis;
 (b) $r = a(1 + \sin \theta)$.
- (vi) Find the centroid of the paraboloid generated by revolving the area enclosed by $y^2 = 4ax$, $x = 0$, $x = h$ about y -axis.
- (vii) The first quadrant of the ellipse is revolved about y -axis. Find the centroid of the solid generated.
- (viii) Find the centroid of a thin homogeneous plate in the form of the cardioid $r = a(1 + \cos \theta)$.
- [Hint : $dA = \frac{1}{2} r^2 d\theta$, and elementary area approximates a triangle. So its centroid lies $\frac{2}{3}$ along the median from the origin and the co-ordinates of the centroid are $x = \frac{2}{3} r \cos \theta$, $y = \frac{2}{3} r \sin \theta$. Due to symmetry $\bar{y} = 0$.
- $$M_y = \int x dA = \int_0^{2\pi} \frac{2}{3} r \cos \theta \cdot \frac{1}{2} r^2 d\theta \text{ and } A = \int_0^{2\pi} \frac{1}{2} r^2 d\theta]$$
3. (i) A dam has a parabolic shape, with the axis of the parabola vertical. If the dam is 50 ft. across at the water level and is 600 ft. deep, find the force of water against the dam.
- (ii) A plate in the shape of the region bounded by the parabola $y = 2x^2$ and the line $y = 3$ is submerged vertically in a fluid with its upper edge 3 feet below the surface of the fluid. Find the force exerted on one side of the plate if the density of the fluid is 50 pounds per cubic foot.
- (iii) A triangle is vertically submerged in a fluid. The vertices are at depths a , b , c respectively. Find the force of the fluid on it.

4. (i) A conical tank is 15 ft. high and has a radius at the top of 10 ft. It is filled to a depth of 10 ft. with a liquid weighing 63 lbs per cubic ft. Find the time taken by an engine of H.P. $\frac{1}{3}$ in pumping out all the contents from the top.
- (ii) A tank filled with water (weighing 62.5 lbs/cu. ft.) is in the shape of a circular cone with a 20 feet vertical axis and a radius of 4 feet at the top. Find the amount of work done needed to pump all the water out of the top of the tank.
- (iii) A piston slowly compresses a monoatomic gas. If the gas is maintained at constant temperature, the pressure p in pounds/sq. foot is $p = 20,000V^{-3/2}$, where V is the volume of the gas in cubic feet. Find the work done in compressing the gas from 9 cubic feet to 4 cubic feet.

[Hint : $W = \int_4^9 20,000V^{-3/2} dV$]

- (iv) A horizontal, cylindrical tank is 4 feet in diameter and 10 feet long. It is half full of oil weighing 50 pounds per cubic foot. Find the work needed to pump the oil out of a pump 6 feet above the top of the tank.
5. (i) Find the volume and the lateral surface of right circular cone.
[Hints : Take the vertex on the y -axis. Let r be the radius of the base, h be the height of the cone and l be the slant height. Equation of the slant height is $\frac{x}{r} + \frac{y}{h} = 1$. Take $V = 2\pi \bar{x} A = 2\pi M_y = 2\pi \frac{h r^2}{6} = \frac{1}{3} \pi r^2 h c$. units and $S = 2\pi \bar{x} s = 2\pi \cdot \frac{r}{2} \cdot \sqrt{h^2 + r^2} = \pi r l$ sq. units.]
- (ii) The semi-circular arc $y = \sqrt{r^2 - x^2}$, $-r \leq x \leq r$ is rotated about the line $y = r$. Find the surface area generated.
[Hints : $S = 2\pi \bar{y} s$]

Answers

1. (i) \$ 14 Thousands. (ii) (a) $469\frac{1}{3}$; (b) 9; (c) $47\frac{1}{3}$; (d) 100
(iii) (a) 3797; (b) 126
2. (i) Moment = $(4 + 3\pi) r^3/6$, $\bar{y}$ of centroid = $\frac{r(3\pi + 4)}{3\pi}$, $V = \pi r^3 (3\pi + 4)/3$ c. units.
(ii) $2r^3/3$ (iii) $\left[\left(\frac{4a}{3\pi}, \frac{4a}{3\pi} \right) \right]$
(iv) (a) (2.4, 0) (b) (8/5, 16/7) (c) (a/5, a/5) (d) (0, 2a/5)

$$- \text{(v)} \left(\frac{3\sqrt{3}}{(2\pi)^2}, \frac{3}{(2\pi)^2} \right) \text{(b)} \left(0, \frac{5a}{4} \right) \text{(vi)} 0, \frac{2h}{3}$$

$$\text{(vii)} \left(0, \frac{3b}{8} \right) \text{(viii)} \left(\frac{5a}{6}, 0 \right)$$

$$3. \text{(i)} 3 \times 10^6 \text{ lbs.} \text{(ii)} 1029 \text{ lbs.} \text{(iii)} \frac{1}{2} (3a^2 + 2ab + b^2)/(a + 2b)$$

$$4. \text{(i)} W = 22 \times 10^4 \text{ feet lbs., time} = 20 \text{ minutes, } \therefore 1 \text{ H.P.} = 33000 \text{ ft. lbs.}$$

$$\text{(ii)} 1,04,720 \text{ ft. lbs.} \text{(iii)} 5,666 \frac{2}{3} \text{ ft. lbs.}$$

$$\text{(iv)} \left(8000\pi + \frac{8000}{3} \right) \text{ ft.}$$

$$5. \text{(i)} V = \frac{1}{3} \pi r^2 h \text{ c. units } S = \pi r l \text{ sq. units} \text{(ii)} 2 \pi r^2 (\pi - 2) \text{ sq. units.}$$

PART II

Jacobians and Multiple Integrals
with
Applications

CH. 15—16

JACOBIANS

15.1 DEFINITION AND INTRODUCTION

Jacobian is a determinant comprising of the continuous first partial derivatives of n functions each of n independent variables such that each row/column contains all the first partial derivatives of one function.

Let $u_r = F_r(x_1, x_2, \dots, x_n)$, $r = 1, 2, \dots, n$ be n continuous functions each of n independent variables $x_1, x_2, \dots, x_n$ and let each of the first partial derivatives $\frac{\partial u_r}{\partial x_s}$, $r, s = 1, 2, \dots, n$, be a continuous function of x_s . Then the jacobian of u_r w.r.t. x_s is the determinant

$$J = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} & \dots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} & \dots & \frac{\partial u_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \frac{\partial u_n}{\partial x_3} & \dots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

Common symbols for the jacobian are $\frac{\partial(u_1, u_2, \dots, u_n)}{\partial(x_1, x_2, \dots, x_n)}$;

$$\frac{d(u_1, u_2, \dots, u_n)}{d(x_1, x_2, \dots, x_n)}, \quad J\left(\frac{u_1, u_2, \dots, u_n}{x_1, x_2, \dots, x_n}\right), \quad J\left(\frac{u_1, u_2, \dots, u_n}{x_1, x_2, \dots, x_n}\right) \quad \text{or simply}$$

$J(u_1, u_2, \dots, u_n)$. Of these first & fourth symbols will be interchangeably used.

When $n = 1$, the determinant reduces to $J = du_1/dx_1$ i.e. the derivative of u_1 w.r.t. x_1 .

The name jacobian originates from its first user C. G. Jacob Jacobi in his research paper in 1841 and he called it a functional determinant. The historian E.T. Bell called this year 1841 'a landmark' in the history of determinant.

The jacobian for a single function with single independent variable is the usual first derivative, $J = \frac{du_1}{dx_1}$.

The jacobian is mainly used in transformation like change of variables, co-ordinate transformations, functional dependence, partial derivatives of implicit function, scalar triple product of gradients etc.

15.2 FUNCTIONAL DEPENDENCE

The necessary and sufficient condition for the existence of a functional relation of continuous and differentiable functions defined in a region is the vanishing of their jacobian.

Let $u = u(x, y)$ and $v = v(x, y)$ be two continuous and differentiable functions of x and y . The necessary and sufficient condition for the existence of a functional relation $\phi(u, v) = 0$ is the vanishing of their jacobian i.e. $J\left(\frac{u, v}{x, y}\right) = 0$.

Proof. Necessary Condition : Given a functional relationship $\phi(u, v) = 0$, it is to prove that $J\left(\frac{u, v}{x, y}\right) = 0$.

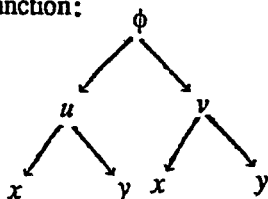
Since $\phi(u, v) = 0$, differentiating this partially w.r.t. x and y using chain rule of partial of differentiation, which can be easily remembered from the following tree structure of the composition of function:

$$\frac{\partial \phi}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v} \cdot \frac{\partial v}{\partial x} = 0 \quad (i)$$

$$\frac{\partial \phi}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v} \cdot \frac{\partial v}{\partial y} = 0 \quad (ii)$$

Eliminating partial derivatives of ϕ from (i) and (ii), we get

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0 \text{ i.e. } J\left(\frac{u, v}{x, y}\right) = 0.$$



[Rule : Follow a branch, say x , use $(\cdot)$ at the junction and $(+)$ for a different branch.]

Sufficient condition : Given $J\left(\frac{u, v}{x, y}\right) = 0$, it is to prove that $\phi(u, v) = 0$.

Since $J = 0$, so let $\frac{\partial u}{\partial x} = 0$ and $\frac{\partial u}{\partial y} = 0$ but $\frac{\partial v}{\partial x} \neq 0$, $\frac{\partial v}{\partial y} \neq 0$.

$\therefore u = \text{const.}$ and v is some function of x and y . Then, obviously u, v are functionally related. Similarly, for the converse, this is true.

Again J is zero when $\frac{\partial u}{\partial x} \neq 0$ and $\frac{\partial u}{\partial y} \neq 0$ simultaneously. As $u = u(x, y)$ and $\frac{\partial u}{\partial x} \neq 0$, so x can be solved for u and y i.e. Let $x = F(u, y)$.
Hence

$$dx = \frac{\partial F}{\partial u} du + \frac{\partial F}{\partial y} dy \quad (1)$$

Also, $du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \quad \because u = u(x, y)$

$$= \frac{\partial u}{\partial x} \left[\frac{\partial F}{\partial u} du + \frac{\partial F}{\partial y} dy \right] + \frac{\partial u}{\partial y} dy \text{ by (1)}$$

$$= \frac{\partial u}{\partial x} \frac{\partial F}{\partial u} du + \left(\frac{\partial u}{\partial x} \frac{\partial F}{\partial y} + \frac{\partial u}{\partial y} \right) dy$$

Equating coefficients of the differentials du and dy ,

$$\frac{\partial u}{\partial x} \frac{\partial F}{\partial u} = 1 \quad \text{and} \quad \frac{\partial u}{\partial x} \frac{\partial F}{\partial y} + \frac{\partial u}{\partial y} = 0$$

$$\therefore \frac{\partial F}{\partial y} = - \frac{\partial u / \partial y}{\partial u / \partial x} \quad \left[\because \frac{\partial u}{\partial x} \neq 0 \right] \quad (2)$$

Again, $dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy \quad (3)$

$$= \frac{\partial v}{\partial x} \left[\frac{\partial F}{\partial u} du + \frac{\partial F}{\partial y} dy \right] + \frac{\partial v}{\partial y} dy$$

$$= \frac{\partial v}{\partial x} \frac{\partial F}{\partial u} du + \left[\frac{\partial v}{\partial x} \frac{\partial F}{\partial y} + \frac{\partial v}{\partial y} \right] dy \quad \text{by (1)}$$

$$= \frac{\partial v}{\partial x} \frac{\partial F}{\partial u} du + \left[\frac{\partial v}{\partial x} \cdot \frac{-\partial u / \partial y}{\partial u / \partial x} + \frac{\partial v}{\partial y} \right] dy \quad \text{by (2)}$$

$$= \frac{\partial v}{\partial x} \frac{\partial F}{\partial u} du + \frac{\frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial v}{\partial x} \frac{\partial u}{\partial y}}{\frac{\partial u}{\partial x}} dy$$

$$= \frac{\partial v}{\partial x} \frac{\partial F}{\partial u} du \quad [\because J=0] \quad (4)$$

which shows that dv depends on du i.e. v depends on u . Therefore, there is a functional relationship between u and v i.e. $\phi(u, v) = 0$.

Cor. 1. If $J \neq 0$, then there exists no functional relationship, i.e. the functions are linearly independent. This condition permits a transformation from one system to another defined by the functions.

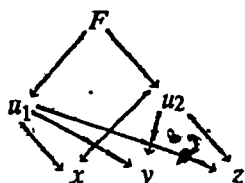
Cor. 2. Conditions for functional dependence of fewer functions than the number of independent variables. Let $u_1 = u_1(x, y, z)$ and $u_2 = u_2(x, y, z)$ be two functions with three independent variables x, y and z . Let there exists a functional relation $F(u_1, u_2) = 0$ between u_1 and u_2 .

Now differentiating $F(u_1, u_2) = 0$ partially w.r.t. x, y and z , we have,

$$\frac{\partial F}{\partial u_1} \frac{\partial u_1}{\partial x} + \frac{\partial F}{\partial u_2} \frac{\partial u_2}{\partial x} = 0 \quad (1)$$

$$\frac{\partial F}{\partial u_1} \frac{\partial u_1}{\partial y} + \frac{\partial F}{\partial u_2} \frac{\partial u_2}{\partial y} = 0 \quad (2)$$

$$\frac{\partial F}{\partial u_1} \frac{\partial u_1}{\partial z} + \frac{\partial F}{\partial u_2} \frac{\partial u_2}{\partial z} = 0 \quad (3)$$



Functional relations for chain rule of differentiation.

Now eliminating derivatives of F w.r.t. u_1 and u_2 from the above 3 equations taking 2 equations at a time, we get, the necessary conditions for functional dependence of u_1 and u_2 as

$$\frac{\partial (u_1, u_2)}{\partial (x, y)} = 0, \frac{\partial (u_1, u_2)}{\partial (z, x)} = 0 \text{ and } \frac{\partial (u_1, u_2)}{\partial (y, z)} = 0.$$

Proceeding in this way, it can be said that if there are n independent variables in each of the r functions such that $r < n$, then for a functional relationship among the r functions n_c , Jacobians of r functions w.r.t. r of the n independent variables must vanish.

15.3 PROPERTIES OF JACOBIAN

The two important properties of jacobian are

(i) If $x = f(u, v)$, $y = g(u, v)$ and $u = \phi(r, s)$, $v = \psi(r, s)$

$$\text{then } \frac{\partial (x, y)}{\partial (r, s)} = \frac{\partial (x, y)}{\partial (u, v)} \cdot \frac{\partial (u, v)}{\partial (r, s)}.$$

(ii) If $J = \frac{\partial (x, y)}{\partial (u, v)} \neq 0$ and $J' = \frac{\partial (u, v)}{\partial (x, y)}$ then $JJ' = 1$.

Proof (i) : The jacobian $\frac{\partial (x, y)}{\partial (r, s)}$ by can be written as

$$\frac{\partial (x, y)}{\partial (r, s)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial s} \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} \frac{\partial u}{\partial r} + \frac{\partial x}{\partial v} \frac{\partial v}{\partial r} & \frac{\partial x}{\partial u} \frac{\partial u}{\partial s} + \frac{\partial x}{\partial v} \frac{\partial v}{\partial s} \\ \frac{\partial y}{\partial u} \frac{\partial u}{\partial r} + \frac{\partial y}{\partial v} \frac{\partial v}{\partial r} & \frac{\partial y}{\partial u} \frac{\partial u}{\partial s} + \frac{\partial y}{\partial v} \frac{\partial v}{\partial s} \end{vmatrix}$$

using chain rule of partial differentiation

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \times \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} \end{vmatrix}, \text{ multiplication of determinants.}$$

$$= \frac{\partial (x, y)}{\partial (u, v)} \cdot \frac{\partial (u, v)}{\partial (r, s)}.$$

This chain can be extended to any number of variables.

(ii) Since $J \neq 0$, so x and y are not functionally related i.e.

$\frac{\partial x}{\partial y} = 0$ and $\frac{\partial y}{\partial x} = 0$. Therefore, using $r = x$ and $s = y$ in (i) we have,

$$\begin{aligned} \frac{\partial(x, y)}{\partial(u, v)} \cdot \frac{\partial(u, v)}{\partial(x, y)} &= \frac{\partial(x, y)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} \end{vmatrix} \\ &= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1. \end{aligned}$$

$\therefore JJ' = 1$ and $J' = \frac{1}{J}$ called the inverse of J .

Cor. The property (i) above is analogous to the chain rule of differentiation. If $x = f(u)$ and $u = g(r)$ only, then property (i) reduces to $\frac{dx}{dr} = \frac{dx}{du} \cdot \frac{du}{dr}$ and in (ii) if $r = x$ then $\frac{dx}{du} \cdot \frac{du}{dx} = \frac{dx}{dx} = 1$ i.e. $u = f(x)$ and $x = g(u)$ are inverse to one another.

15.4 APPLICATIONS

Jacobian is widely used in all kinds of transformation or mapping. For the existence of transformation, non-vanishing of jacobian is required. In this sense jacobian is used in ch. 16 for substitution or change of variables in multiple integral. It is indirectly used in 1-dimensional transformations also as mentioned in 16.3.5. One of the main uses of jacobian is testing of functional dependence, stated the condition & proved in 15.2. It is used in implicit function study for computing derivatives and inverse functions and their derivatives not included here. Interested readers are referred to Advanced Calculus by Hildebrand and also by Kaplan. Jacobian is used in determining the direction of the normal to a surface in parametric form. Thus if $x = f(u, v)$, $y = g(u, v)$ and $z = h(u, v)$ represents the parametric equation of a surface, then the direction of its normal are given by $J_1 : J_2 : J_3$ where $J_1 = \frac{\partial(x, y, z)}{\partial(u, v)}$, $J_2 = \frac{\partial(z, x)}{\partial(u, v)}$, $J_3 = \frac{\partial(x, y)}{\partial(u, v)}$. With these J 's in mind the surface area in 16.3.8 may be written as $S = \iint_R \sqrt{J_1^2 + J_2^2 + J_3^2} du dv$. In the same way scalar triple product of gradients in Vector Calculus may be expressed in terms of jacobian i.e. $[\Delta u \Delta v \Delta w] = \frac{\partial(u, v, w)}{\partial(x, y, z)}$ and its vanishing gives the co-planarity condition of the normals to the surfaces $u(x, y, z) = c_1$, $v(x, y, z) = c_2$ and $w(x, y, z) = c_3$.

Uses of jacobian are available in general curvilinear co-ordinates in Vector Analysis with Applicatons of the authors, A.K. Hazra and Ali Ashraf.

Examples

1. Find the jacobian of $u = \frac{x+y}{1-xy}$ and $v = \tan^{-1} x + \tan^{-1} y$ w.r.t. x and y . Are u and v functionally related? If so, find the relation.

$$\text{Here } \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{x+y^2} \end{vmatrix} = 0$$

Since the jacobian vanishes, so there is a functional relation between u and v . Now,

$$v = \tan^{-1} \frac{x+y}{1-xy} = \tan^{-1} u \text{ or } u = \tan v.$$

This functional relation may be obtained by other way also i.e. somehow or other eliminate x and y to get a relation between u and v .

2. Given the transformation $x = f(u, v)$, $y = g(u, v)$ with $J = \frac{\partial(x, y)}{\partial(u, v)}$, show that for the inverse functions one has

$$\frac{\partial u}{\partial x} = \frac{1}{J} \frac{\partial y}{\partial v}, \frac{\partial u}{\partial y} = -\frac{1}{J} \frac{\partial x}{\partial v}, \frac{\partial v}{\partial x} = -\frac{1}{J} \frac{\partial y}{\partial u} \text{ and } \frac{\partial v}{\partial y} = \frac{1}{J} \frac{\partial x}{\partial u}.$$

Hence show that $JJ' = 1$.

Here, $f(u, v) - x = 0$ and $g(u, v) - y = 0$.

Diff. w.r.t. x and y , we get the following 4 relations :

$$f_u u_x + f_v v_x - 1 = 0 \quad (1)$$

$$f_u u_y + f_v v_y = 0 \quad (2)$$

$$g_u u_x + g_v v_x = 0 \quad (3)$$

$$g_u u_y + g_v v_y - 1 = 0 \quad (4)$$

Form (1) and (3),

$$u_x = \frac{\begin{vmatrix} 1 & f_v \\ 0 & g_v \end{vmatrix}}{\begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix}} = \frac{g_v}{\begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix}} = \frac{y_v}{\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}} = \frac{1}{J} \frac{\partial y}{\partial v},$$

$$\text{Similarly, from (1) \& (3), } v_x = \frac{-g_u}{J} = -\frac{1}{J} y_u = -\frac{1}{J} \frac{\partial y}{\partial u}$$

$$\text{and from (2), (4), } u_y = -\frac{f_v}{J} = -\frac{1}{J} \frac{\partial x}{\partial v}$$

$$v_y = \frac{f_u}{J} = \frac{1}{J} \frac{\partial x}{\partial u}$$

2nd part :

$$\begin{aligned} \text{Now } J' &= \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \\ &= \begin{vmatrix} \frac{1}{J} \frac{\partial y}{\partial v} & -\frac{1}{J} \frac{\partial x}{\partial v} \\ -\frac{1}{J} \frac{\partial y}{\partial u} & \frac{1}{J} \frac{\partial x}{\partial u} \end{vmatrix} = -\frac{1}{J^2} \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial v} & -\frac{\partial x}{\partial v} \end{vmatrix} \\ &= \frac{1}{J^2} \begin{vmatrix} \frac{\partial x}{\partial u} & -\frac{\partial y}{\partial u} \\ -\frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{1}{J^2} \cdot J = \frac{1}{J} \end{aligned}$$

$$\therefore JJ' = 1.$$

3. If u, v are two polynomials in x, y that are homogeneous of the n th degree, prove that

$$u \, dv - v \, du = \frac{1}{n} \frac{\partial(u, v)}{\partial(x, y)} (x \, dy - y \, dx).$$

Since u and v are homogeneous functions of degree n , then by Euler's theorem on homogeneous functions,

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu \tag{1}$$

$$x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = nv \tag{2}$$

$$\text{Also, } dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$\text{and } du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$\begin{aligned} \therefore u \, dv - v \, du &= \left(u \frac{\partial v}{\partial x} dx + u \frac{\partial v}{\partial y} dy \right) - \left(v \frac{\partial u}{\partial x} dx + v \frac{\partial u}{\partial y} dy \right) \\ &= \frac{1}{n} \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right) \left(\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy \right) - \frac{1}{n} \left(x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} \right) \left(\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \right) \\ &\qquad\qquad\qquad \text{by (1) \& (2)} \end{aligned}$$

$$= \frac{1}{n} \frac{\partial(u, v)}{\partial(x, y)} (x dy - y dx).$$

Exercises 15

1. If $x = u - v + w$, $y = u^2 - v^2 - w^2$ and $z = u^3 + v$, find the Jacobian $\frac{\partial(x, y, z)}{\partial(u, v, w)}$.
2. If $F(u, v) = 3u^2 - uv$ and $G(u, v) = 2uv^2 + v^3$ then find $\frac{\partial(F, G)}{\partial(u, v)}$.
3. If $F = x + 3y^2 - z^3$, $G = 2x^2 yz$ and $H = 2z^2 - xy$ then find $\frac{\partial(F, G, H)}{\partial(x, y, z)}$ at $(1, -1, 0)$.
4. Is there any functional relationship between $u = \sin^{-1} x + \sin^{-1} y$ and $v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$. If so, find the relation.
5. Is there any functional relationship among $F = xy + yz + zx$, $G = x^2 + y^2 + z^2$ and $H = x + y + z$? If so, find the relation.
6. Prove that if $x = r \cos \theta$ and $y = r \sin \theta$, then $J = \frac{\partial(x, y)}{\partial(r, \theta)} = r$ and if $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$, $z = r \cos \theta$, then $J = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2 \sin \theta$.
7. If $x + y + z = u$, $y + z = uv$ and $z = uvw$, then show that $\frac{\partial(x, y, z)}{\partial(u, v, w)} = u^2 v$ and if

$$x_1 + x_2 + x_3 + x_4 + \dots + x_n = y_1$$

$$x_2 + x_3 + x_4 + \dots + x_n = y_1 y_2$$

$$x_3 + x_4 + \dots + x_n = y_1 y_2 y_3$$

$$\dots$$

$$x_r + x_{r+1} + \dots + x_n = y_1 y_2 y_3 \dots y_r$$

$$\dots$$

$$x_n = y_1 y_2 y_3 \dots y_n$$
 then show that $\frac{\partial(x_1, x_2, \dots, x_n)}{\partial(y_1, y_2, \dots, y_n)} = y_1^{n-1} y_2^{n-2} \dots y_{n-2}^2 y_{n-1}$.
8. If $u = f(x) + f(y) + f(z)$, $v = f(y)f(z) + f(z)f(x) + f(x)f(y)$ and $w = f(x)f(y)f(z)$ are functionally related then determine f .
9. Under what condition will the linear functions $ax + by$, $cx + dy$ be functionally dependent?
10. If $t = r \cos \phi$, $x = r \sin \phi \cos \psi$, $y = r \sin \phi \sin \psi \cos \theta$ and

$z = r \sin \phi \sin \psi \sin \theta$ show that $r^2 + x^2 + y^2 + z^2 = r^2$ and find $\frac{\partial (t, x, y, z)}{\partial (r, \phi, \psi, \theta)}$.

11. Given the transformation $x = u - 2v$, $y = 2u + v$.

(i) Write the equations of inverse transformation;

(ii) Evaluate the jacobian of the transformation and of the inverse transformation.

(iii) Verify that $JJ' = 1$.

12. If $u = x/(1 - r^2)^{1/2}$, $v = y/(1 - r^2)^{1/2}$, $w = z/(1 - r^2)^{1/2}$ where $r^2 = x^2 + y^2 + z^2 \neq 1$, show that $J = \frac{\partial (u, v, w)}{\partial (x, y, z)} = \frac{1}{(1 - r^2)^{5/2}}$.

Extend to the case of n functions $y_1, y_2, \dots, y_n$ where $y_s = x_s/(1 - r^2)^{1/2}$, $r^2 = x_1^2 + x_2^2 + \dots + x_n^2 \neq 1$, $s = 1, 2, \dots, n$.

13. Prove that the functions $(3x + 2y - z)$, $(x - 2y + z)$ and $(x + 2y - z)$ are not independent and find the equation that connects them.

14. Given the transformation $x = f(u, v, w)$, $y = g(u, v, w)$, $z = h(u, v, w)$ with Jacobian, $J = \frac{\partial (x, y, z)}{\partial (u, v, w)}$, show that for the inverse functions one has

$$\frac{\partial u}{\partial x} = \frac{1}{J} \frac{\partial (y, z)}{\partial (v, w)}, \quad \frac{\partial u}{\partial y} = \frac{1}{J} \frac{\partial (z, x)}{\partial (v, w)}, \quad \frac{\partial u}{\partial z} = \frac{1}{J} \frac{\partial (x, y)}{\partial (v, w)};$$

$$\frac{\partial v}{\partial x} = \frac{1}{J} \frac{\partial (y, z)}{\partial (w, u)}, \quad \frac{\partial v}{\partial y} = \frac{1}{J} \frac{\partial (z, x)}{\partial (w, u)}, \quad \frac{\partial v}{\partial z} = \frac{1}{J} \frac{\partial (x, y)}{\partial (w, u)};$$

$$\frac{\partial w}{\partial x} = \frac{1}{J} \frac{\partial (y, z)}{\partial (u, v)}, \quad \frac{\partial w}{\partial y} = \frac{1}{J} \frac{\partial (z, x)}{\partial (u, v)}, \quad \frac{\partial w}{\partial z} = \frac{1}{J} \frac{\partial (x, y)}{\partial (u, v)}.$$

15. Determine whether the functions are functionally dependent or not? If dependent, find a relation among them.

(i) $u_1 = \frac{x-y}{x+z}$, $u_2 = \frac{x+z}{y+z}$.

(ii) $u_1 = \frac{yz-x}{x}$, $u_2 = \frac{xyz-y^2z^2+x^2}{xyz+y^2z^2}$;

[Hint: Show $\frac{\partial (u_1, u_2)}{\partial (x, y)}$, $\frac{\partial (u_1, u_2)}{\partial (y, z)}$, $\frac{\partial (u_1, u_2)}{\partial (z, x)}$ are all zero.]

(iii) $f = u + v - x$, $g = x - y + u$, $h = u - 2v + 5x - 3y$.

(iv) $f = x + y^3 + u^3 + v^3$, $g = x^3 + y - u^4 + v^4$.

(v) $f = u^2 + v^2 - x - y$, $g = u^2 - v^2 + 3x + y$.

(vi) $u_1 = y + z$, $u_2 = x + 2z^2$.

16. Verify that $JJ' = 1$ for the following transformations :

(i) $x = r \cos \theta, y = r \sin \theta$

(ii) $x = r \sin \theta \cos \phi, y = r \sin \theta \sin \phi, z = r \cos \theta$

(iii) $x = r \cos \theta, y = r \sin \theta, z = z$

(iv) $u = y/x, v = xy$

(v) $r = \sqrt{x^2 + y^2}, \theta = \tan^{-1} \frac{y}{x}$

(vi) $x = \sqrt{(v/u)}, y = \sqrt{uv}$

17. (i) If $x = a \cosh u \cos v, y = a \sinh u \sin v$, show that

$$J \left(\frac{x, y}{u, v} \right) = \frac{1}{2} a^2 (\cosh 2u - \cos 2v)$$

(ii) If $y_1 = x_2 x_3 / x_1, y_2 = x_2 x_1 / x_3, y_3 = x_1 x_2 / x_3$

show that $J \left(\frac{y_1, y_2, y_3}{x_1, x_2, x_3} \right) = 4$.

Answers

1. $6wu^2 + 2u + 6u^2v + 2w$.

4. Yes, $\sin u = v$.

9. $ad = bc$.

(ii) 5 and $\frac{1}{5}$.

15. (i) Yes, $u_1 u_2 + 1 = u_2$.

(iii) Yes, $h + 2f - 3g = 0$.

(vi) No.

2. $24u^2v + 16uv^2 - 3v^3$.

5. Yes, $H^2 - G - 2F = 0$.

10. $r^3 \sin^2 \phi \sin \psi$.

(iii) Verified.

(ii) Yes, $(u_1 + 1)(u_1 u_2 + 2u_2 + u_1) = 1$.

(iv) No

3. 10

8. $f = \text{const.}$

11. (i) $u = \frac{x+2y}{5}, v = \frac{y-2x}{5}$;

13. $u = v + 2w$.

(v) No.

MULTIPLE INTEGRATION

16.1 INTRODUCTION

Integration of a single variable was considered from ch.1 through 14 both as, "The inverse process of differentiation" and "The process of summation". But in differential calculus ordinary differentiation of single variable extends upto partial differentiation of multi-variables. So, integral calculus has a natural demand of its extension from single to multi-variables. Integration of multi-variables is called **multiple integration**. This book limits its discussion to double and triple integration only. In these cases the fundamental definitions of integration for a single variable is extended. Thus partial derivatives of several variables are integrated several times to obtain the original function and correspondingly the integration is called **iterated** or **repeated integration** in contrast to partial differentiation. The integrals of multi-variables defined as a process of summation is normally called **multiple integrals** and its process of evaluation or computation using fundamental theorem of integral calculus is **iterated** or **repeated integrals**.

16.2 ITERATED INTEGRALS

Repeated definite and indefinite integrals of the form $\int_a^b \left[\int_{y_1}^{y_2} f(x, y) dy \right] dx$ or $\int_a^b \int_{y_1}^{y_2} f(x, y) dy dx$ and $\iint f(x, y) dy dx$ are

called **iterated integrals**. This is very important from computational point of view of the multiple integrals. *Sometimes* multiple and iterated are not distinguished and used interchangeably.

Computational procedure : Keeping x constant both the definite integral $\int_{y_1}^{y_2} f(x, y) dy$, indefinite integral $\int f(x, y) dy$ are first integrated w.r.t.

y . Let they respectively be $\left[F(x, y) \right]_{y=y_1}^{y_2} = F(x, y_2) - F(x, y_1) = F_1(x)$ and

$F(x, y) + g_1(x)$ where $g_1(x)$ is the constant of integration and it is constant w.r.t. y . Note that here constant of integration should be some arbitrary function instead of an arbitrary constant in case of single indefinite integrals. Secondly, the resulting integrands are to be integrated with respect to x in the same manner as

$$\int_a^b F_1(x) dx = \left[G_1(x) \right]_a^b = G_1(b) - G_1(a)$$

and $\int F(x, y) dx + g_1(x) = G_1(x, y) + g_1(x) + g_2(y),$

y is kept constant in this time, respectively. This procedure may be continued upto any number of variables.

Examples

1. Compute the integral $\iint (x^2 + 2xy + y^2) dx dy.$

Integrating first w.r.t. x keeping y constant,

$$\begin{aligned} & \iint (x^2 + 2xy + y^2) dx dy \\ &= \left[\int x^2 dx + 2y \int x dx + y^2 \int dx \right] dy + f_1(y) \\ &= \int \left(\frac{x^3}{3} + x^2 y + xy^2 \right) dy + f_1(y), \text{ where } f_1(y) \text{ is the first const. of} \\ & \hspace{15em} \text{integration.} \\ &= \frac{x^3}{3} \int dy + x^2 \int y dy + x \int y^2 dy + f_1(y) + f_2(x), \text{ where } f_2(x) \text{ is} \\ & \hspace{15em} \text{second constant of integration and } x \text{ is kept} \\ & \hspace{15em} \text{const. for integration w.r.t. } y. \\ &= \frac{x^3 y}{3} + \frac{1}{2} x^2 y^2 + \frac{1}{3} xy^3 + f_1(y) + f_2(x) \text{ which is the original function} \end{aligned}$$

whose partial derivative $\frac{\partial^2 (...)}{\partial y \partial x}$ is the given integrand. Verify the statement.

2. Evaluate $\int_0^1 f(y) dy$ where $f(y) = \int_0^y y^2 \sin xy dx.$

$$\begin{aligned} \text{Here } \int_0^1 f(y) dy &= \int_0^1 \int_0^y y^2 \sin xy dx dy \\ &= \int_0^1 y^2 \left(\int_0^y \sin xy dx \right) dy \\ &= \int_0^1 y^2 \left[-\frac{1}{y} \cos xy \right]_{x=0}^y dy = \int_0^1 y (1 - \cos y^2) dy \\ &= \left[\frac{1}{2} y^2 - \frac{1}{2} \sin y^2 \right]_0^1 = \frac{1}{2} (1 - \sin 1). \end{aligned}$$

3. Evaluate $\int_0^1 \int_x^x \int_{xy}^{\frac{x^2}{y^3}} xy \, dz \, dy \, dx$

$$\begin{aligned} \int_0^1 \int_x^x xy \left(\int_{xy}^{\frac{x^2}{y^3}} dz \right) dy \, dx &= \int_0^1 \int_x^x xy \left[z \right]_{z=xy}^{\frac{x^2}{y^3}} dy \, dx \\ &= \int_0^1 \int_x^x xy (x^2 y^3 - xy) dy \, dx = \int_0^1 x^2 \left(\int_x^x (xy^4 - y^2) dy \right) dx \\ &= \int_0^1 x^2 \left[\frac{x}{5} y^5 - \frac{y^3}{3} \right]_{y=x}^x dx = \int_0^1 x^2 \left[\frac{x}{5} (x^{10} - x^5) - \frac{1}{3} (x^6 - x^3) \right] dx \\ &= \frac{2}{189}. \end{aligned}$$

Exercises 16 (1)

Compute the following iterated integrals :

1. $\iint (xy - y^2) (dy)^2$. [Hints : Integrate twice with respect y]

2. $\iiint 2 \, dy \, dx \, dz$.

3. $\int_0^3 \int_0^2 x^3 \, dy \, dx$.

4. $\int_0^1 \int_0^2 (x+2) \, dy \, dx$.

5. $\int_b^a \int_\beta^\alpha r^2 \sin \theta \, d\theta \, dr$.

6. $\int_0^6 \int_{\frac{1}{2}y}^3 y \, dx \, dy$.

7. $\int_0^4 \int_0^x y \, dy \, dx$.

8. $\int_2^4 \int_0^{\frac{4-x}{2}} x^2 \, dy \, dx$.

9. $\int_0^a \int_0^{\sqrt{x}} dy \, dx$.

10. $\int_0^2 \int_x^{2x} xy \, dy \, dx$.

11. $\int_2^6 \int_0^y \frac{dx \, dy}{x^2 + y^2}$.

12. $\int_0^3 \int_x^{\sqrt{25-x^2}} x \, dy \, dx$.

13. $\int_0^\pi \int_0^{a \cos \theta} r \sin \theta \, dr \, d\theta$.

14. $\int_{\pi/6}^{\pi/4} \int_0^{4 \sin 2\theta} r \sin \theta \cos \theta \, dr \, d\theta$.

15. $\int_1^2 \int_1^{\tan y} (1+x^2)^{-1} dx \, dy$.

16. $\int_0^4 \int_0^{\sqrt{2 \sin x^2}} \int_{y-x}^{y+x} y \, dz \, dy \, dx$.

17. $\int_0^a \int_0^x \int_0^y x^3 y^2 z \, dz \, dy \, dx$.

18. $\int_1^2 \int_0^z \int_0^{x\sqrt{3}} \frac{x}{x^2 + y^2} dy \, dx \, dz$.

19. $\int_0^{\sqrt{5}} \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{(9-x^2-y^2)^2}} (\sqrt[3]{9-x^2-y^2}) x \, dz \, dy \, dx.$
20. $\int_1^2 \int_x^{4/x} \int_0^{xy} xy \, dz \, dy \, dx.$
21. $\int_0^1 \int_y^{2y} \int_0^x ye^z \, dz \, dx \, dy.$
22. $\int_0^{\pi/2} \int_0^{\cos \theta} \int_0^{6-r \sin \theta} r \, dz \, dr \, d\theta.$
23. $\int_0^{\pi/2} \int_0^{\pi/3} \int_0^{4 \cos \theta} r^3 \sin \theta \sin \phi \, dr \, d\theta \, d\phi.$
24. $\int_0^{\pi} \int_1^{\sin z} \int_0^{\log y} y e^x \, dx \, dy \, dz.$
25. $\int_0^{\pi/2} \int_0^{\pi/2} \int_{\pi/2}^{z+y} \cos(x+y) \, dx \, dz \, dy.$

Answers

1. $\frac{1}{6}xy^3 - \frac{1}{12}y^4 + \phi(x) + \psi(x)$ 2. $2xyz + f_1(x, z) + f_2(z) + c$ 3. $\frac{81}{2}$, 4. 5; 5. $\frac{1}{3}(a^3 - b^3)(\cos \beta - \cos \alpha)$; 6. 18; 7. $\frac{32}{3}$; 8. $\frac{22}{3}$; 9. $\frac{2}{3}a^{3/2}$; 10. $\frac{8}{3}$; 11. $\frac{\pi}{4} \log 3$; 12. $\frac{34}{3}$; 13. $\frac{a^3}{3}$; 14. $\frac{11}{12}$; 15. $\frac{1}{4}(6 - \pi)$; 16. .04; 17. $\frac{a^2}{90}$; 18. $\frac{\pi}{2}$; 19. $\frac{422}{15}$; 20. $\frac{1}{6}(128 \log 2 - 21)$; 21. $\frac{1}{12}(3e^2 - 13)$; 22. $\frac{1}{12}(9\pi - 1)$; 23. $\frac{62}{5}$; 24. $\frac{4}{9} - \frac{1}{12}\pi$; 25. $\frac{1}{2}(2 - \pi).$

16.3 DOUBLE INTEGRATION AS A SUMMATION

Let $f(x, y)$ be a single-valued continuous function of two variables x and y within and on a closed region R bounded by C (fig. 92). Let the region R be sub-divided into n sub-regions $\Delta R_1, \Delta R_2, \dots, \Delta R_n$. Let ΔA_i denote the area of a sub-region ΔR_i , $i = 1, 2, \dots, n$ and $P_i(\xi_i, \eta_i)$ be any point within ΔR_i .

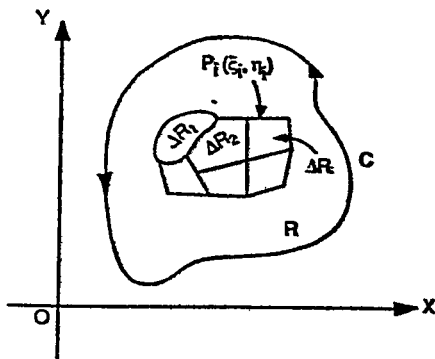


Fig. 92

Now evaluating the functional values $f(\xi_i, \eta_i)$ at each point P_i form the sum of the products with ΔA_i as $\sum_{i=1}^n f(\xi_i, \eta_i) \Delta A_i$. As $n \rightarrow \infty$ as each

$\Delta R_i \rightarrow 0$, if the limiting value of the sum of the products exists then that value is called the **double integral** of $f(x, y)$ over the region R i.e.

$$\lim_{\substack{\Delta A_i \rightarrow 0 \\ n \rightarrow \infty}} \sum_{i=1}^n f(\xi_i, \eta_i) \Delta A_i = \iint_R f(x, y) dA.$$

This analytic definition of double integral does not provide any practical means of evaluating the integral to get a numerical result. So we look for following iterated integral of the same double integral.

16.3.1 ITERATED INTEGRAL (Double)

To evaluate the double integral $\iint_R f(x, y) dA$, let us consider two cases— (1) a regular shaped rectangular region R , (2) an irregular shaped closed region.

Case 1 : Let R be a rectangular region bounded by the lines $x=a, x=b, y=c$ and $y=d$ (fig. 93). Let the region R be subdivided into $m \times n$ smaller rectangles by drawing lines $x=a, x=x_1, \dots, x=x_{i-1}, x=x_i, \dots, x_n=b$ and $y=c, y=y_1, \dots, y=y_{j-1}, y=y_j, \dots, y_m=d$. Let $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_j = y_j - y_{j-1}$. Let the area of R_{ij} rectangle be $\Delta A_{ij} = \Delta x_i \cdot \Delta y_j$. Let an arbitrary point (ξ_i, η_j) be in ΔR_{ij} . Then the sum

$$\sum_{i=1, j=1}^{i=n, j=m} f(\xi_i, \eta_j) \Delta A_{ij} = \sum_{i=1, j=1}^{i=n, j=m} f(\xi_i, \eta_j) \Delta x_i \Delta y_j \quad (1)$$

Now the sum (1) can be performed in two ways — (a) first adding all the smaller rectangles in a column i.e. $R_{i1}, R_{i2}, R_{i3}, \dots, R_{im}$, $i=1, 2, \dots, n$ and then adding the columns together; (b) secondly adding all the smaller rectangles in a row i.e. $R_{1j}, R_{2j}, \dots, R_{nj}$, $j=1, 2, \dots, m$, and then adding the rows together. Therefore, using procedure (a) equation (1) can be written as

$$\sum_{i=1, j=1}^{i=n, j=m} f(\xi_i, \eta_j) \Delta A_{ij} = \sum_{i=1}^n \left[\sum_{j=1}^m f(\xi_i, \eta_j) \Delta y_j \right] \Delta x_i \quad (2)$$

$$\text{But } \lim_{\substack{\Delta y_j \rightarrow 0 \\ m \rightarrow \infty}} \sum_{j=1}^m f(\xi_i, \eta_j) \Delta y_j = \int_c^d f(x_i, y) dy = \phi(x_i) \text{ (say)} \quad (3)$$

$\therefore$ from (2)

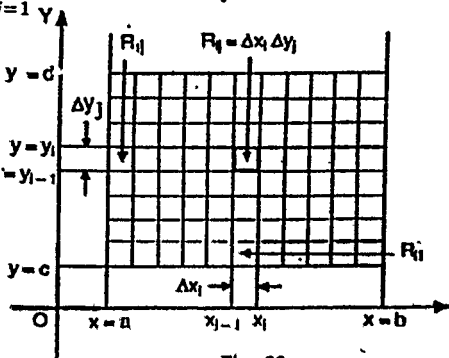


Fig. 93

$$\lim_{\substack{\Delta y_j \rightarrow 0 \\ m \rightarrow \infty}} \sum_{i=1, j=1}^{i=n, j=m} f(\xi_{ij}, \eta_j) \Delta A_{ij} = \sum_{i=1}^n \phi(x_i) \Delta x_i$$

Now, taking limit $n \rightarrow \infty$ and $\Delta x_i \rightarrow 0$, we have,

$$\lim_{\substack{\Delta y_j \rightarrow 0 \\ \Delta x_i \rightarrow 0 \\ m \rightarrow \infty \\ n \rightarrow \infty}} \sum_{i=1, j=1}^{i=n, j=m} f(\xi_{ij}, \eta_j) \Delta A_{ij} = \int_a^b \phi(x) dx$$

$$\therefore \iint_R f(x, y) dA = \int_a^b \int_c^d f(x, y) dy dx \quad (4)$$

[Replacing $\phi(x)$ by (3)]

Similarly using procedure (b), we can have,

$$\iint_R f(x, y) dA = \int_c^d \int_a^b f(x, y) dx dy. \quad (5)$$

Equation (4) and (5) shows the procedure for evaluating a double integral on the left by evaluating the inner integral w.r.t. to either y or x , as the case may be, keeping the other fixed and then integrate the remaining one w.r.t. to the remaining variable x or y .

Case 2 : The region R is not rectangular but bounded by a closed curve C such that any line parallel to co-ordinate axes does not intersect C in not more than two points (fig. 94). Let us draw tangents to C at its extreme points and parallel to axes of co-ordinates. Let the tangents at B_1 and B_2 to C be parallel to x -axis and those at A_1 and A_2 to C be parallel to y -axis respectively.

Now the curve C is divided into two parts $B_1 A_2 B_2$ and $B_2 A_1 B_1$. Let their equation be $x = \phi_2(y)$ and $x = \phi_1(y)$ respectively. As in (a) if the smaller regions along row are summed first then in equation (5) the limits of inner integrals namely $x = a$ to $x = b$ will be replaced by $x = \phi_1(y)$ to $x = \phi_2(y)$ and those of outer integrals namely $y = c$ to $y = d$ will be replaced by the y -coordinates of B_1 , say β_1 and of B_2 , say β_2 .

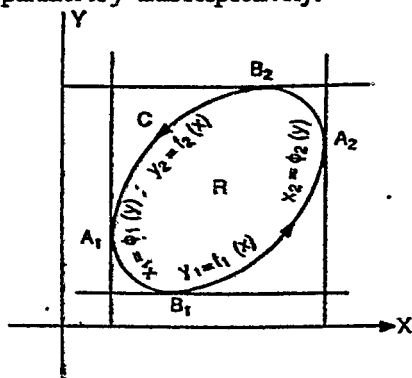


Fig. 94

$\therefore$ equation (5) stands as

$$\iint_R f(x, y) dA = \int_{\beta_1}^{\beta_2} \int_{\phi_1(y)}^{\phi_2(y)} f(x, y) dx dy \quad (6)$$

Similarly, if the equations of $A_1 B_1 A_2$ and $A_2 B_2 A_1$ are $y_1 = f(x)$ and $y_2 = f_2(x)$ and the x -coordinates of A_1 and A_2 are α_1 and α_2 respectively. Then equation (4) stands as

$$\iint_R f(x, y) dA = \int_{\alpha_1}^{\alpha_2} \int_{f_1(x)}^{f_2(x)} f(x, y) dy dx \quad (7)$$

If the boundary curve C of R be such that the lines parallel to co-ordinate axes intersect in more than two points then the region R is to be divided in such a way that each portion admits the above condition and the above procedure is to be applied in each case separately and their algebraic sum is to be taken to get the double integral in R .

16.32 AREA (CARTESIAN SYSTEM) BY DOUBLE INTEGRALS

The area between curves has been discussed in 11.6 by single integral. Double integrals also determines the same area. Let the area $ABCD$ be enclosed by the curves $y = f(x)$ and $y = \phi(x)$ and the ordinates at $x = a$ and $x = b$ (fig. 95).

Let the length between $x = a$ and $x = b$ be divided into n parts of length Δx and ordinates be erected at those points of division. The lines parallel to x -axis are drawn at a distance of Δy each. As a result elementary areas $\Delta A = \Delta x \cdot \Delta y$ are created. When both $\Delta x, \Delta y \rightarrow 0$ simultaneously, the area $ABCD$ will be expressed by double integrals.

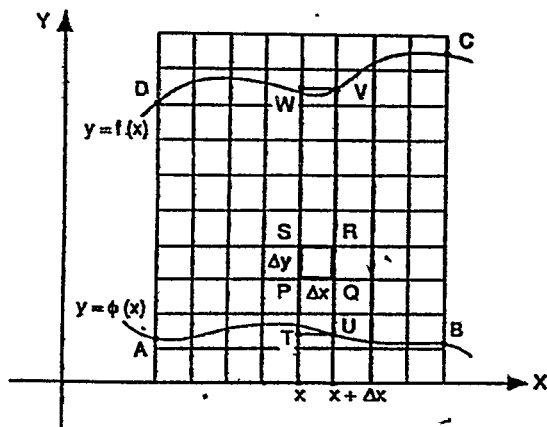


Fig. 95

Let the ordinates at x and $x + \Delta x$ intersect the curve $y = \phi(x)$ at T and U and the curve $y = f(x)$ at W and V respectively. When $\Delta y \rightarrow 0$, the area $TUVW$ may be considered as the sum of the elementary areas $PQRS$ i.e.

$$\text{Area } TUVW = \left[\lim_{\Delta y \rightarrow 0} \sum \Delta y \right] \Delta x = \left[\int_{\phi(x)}^{f(x)} dy \right] \Delta x$$

Now as $n \rightarrow \infty$ i.e. $\Delta x \rightarrow 0$ then the limiting value of the sum of the n areas like $TUVW$ will be equal to the area $ABCD$. Therefore.

$$\text{Area } ABCD = \int_a^b \left[\int_{\phi(x)}^{f(x)} dy \right] dx \quad (1)$$

Thus the plane area between two curves can be determined by double integrals. The right integral of (1) will be evaluated as an iterated integral.

Similarly if the area $BCDE$ is bounded by the curves $x = \phi(y)$, $y = \psi(y)$ and the abscissae $y = c$ and $y = d$, (fig. 96), then the elementary areas $TUVW$ parallel to x -axis are considered first and hence the sum of those areas give the total area $BCDE$. Thus

$$\text{Area } TUVW = \left[\int_{\psi(y)}^{\phi(y)} dx \right] \Delta y$$

$$\text{and area } BCDE = \int_c^d \left[\int_{\psi(y)}^{\phi(y)} dx \right] dy. \quad (2)$$

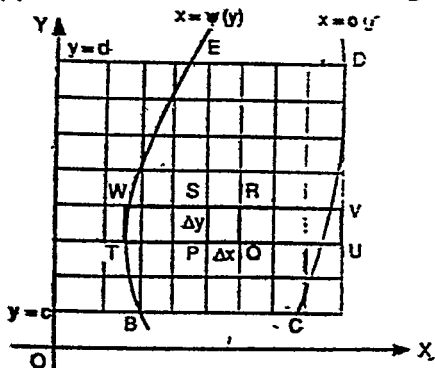


Fig. 96

Examples

1. Find the area enclosed by $xy=4$, $2x=5+4y-y^2$ and $y=4$ using double integration.

Let the curves $xy=4$ and $2x=5+4y-y^2$ intersect at the point $B(4, 1)$ (fig. 97). So, the area enclosed is BCD . Here horizontal elementary area is convenient. Therefore, the required area BCD

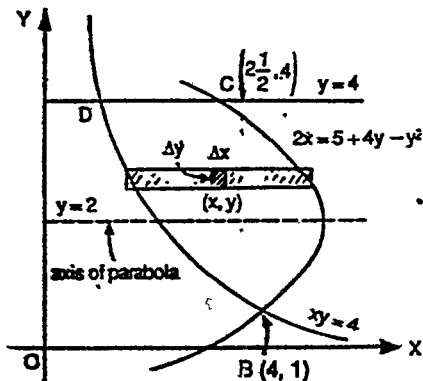


Fig. 97

$$= \int_1^4 \int_{4/y}^{(5+4y-y^2)/2} dx dy \quad [\because y \text{ has fixed limit and } x \text{ has}$$

variable limit which must be the inner integral.]

$$= \int_1^4 \left(\frac{5}{2} + 2y - \frac{y^2}{2} - \frac{4}{y} \right) dy = (12 - 8 \log 2) \text{ sq. units.}$$

2. Find the area bounded by $y^2 = x^3$ and $y = x$ using double integration.

The curves intersect at $(0, 0)$ and $(1, 1)$. Here elementary areas can be summed first in both ways. Because any one can be rolled to cover the whole area. First horizontal sum is taken (fig. 98 (a)).

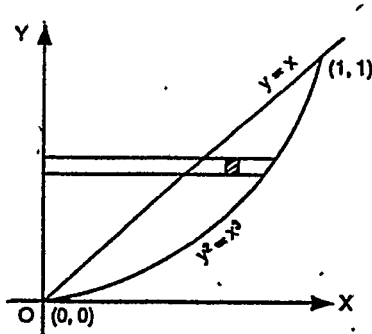


Fig. 98 (a)

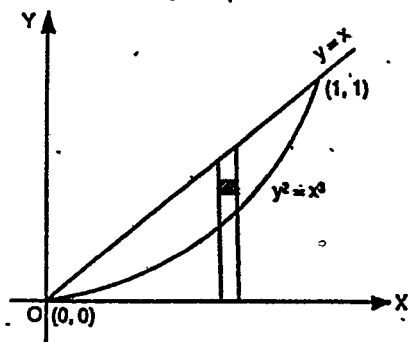


Fig. 98 (b)

$$A = \int_0^1 \int_y^{y^{2/3}} dx dy = \int_0^1 (y^{2/3} - y) dy$$

$$= \frac{1}{10} \text{ sq. units.}$$

In fig. 98 (b), vertical sum is taken first. So,

$$A = \int_0^1 \int_{x^{3/2}}^x dy dx = \int_0^1 (x - x^{3/2}) dx$$

$$= \frac{1}{10} \text{ sq. units.}$$

Exercises 16 (2)

Find the area enclosed by the following curves using double integration :

1. $x^2 + y^2 = 10$, $y^2 = 9x$ and x -axis, in the 1st. quadrant.
2. $y = 6x - x^2$, $y = x^2$.
3. $3y^2 = 25x$, $5x^2 = 9y$.
4. $y^2 = 4x + 16$, $x^2 + y^2 = 16$ right of the circle.
5. $y = 4x - x^2$, $y = x$.
6. $y^2 = 4x$, $2x - y = 4$.
7. $y = x^2$, $2x - y + 3 = 0$.
8. $y^2 = 2x$, $x^2 + y^2 = 4x$.
9. $\sqrt{x} + \sqrt{y} = \sqrt{a}$, $x + y = a$.
10. $y^2 = 9 + x$, $y^2 = 9 - 3x$.

11. $x^{2/3} + y^{2/3} = a^{2/3}$, $x + y = a$. 12. $(x^2 + 4a^2)y = 8a^3$, $2y = x$, $x = 0$.

13. $xy^2 = 4a^2(2a - x)$, $y = 2x$, $y = 0$.

14. $y = \frac{6}{x^2 + 3}$, $x^2 + y^2 - 4y = 0$ (lower part).

15. $xy = 2$, $y = 8 - x^2$, $x = 1$, $x = 2$. 16. $y = x^3 - 2x$, $y = 6x - x^3$.

[Hints : Q. 16. Both curves are symmetrical about origin]

Answers

1. 6.75 sq. units 2. 9 sq. units 3. 5 sq. units 4. $\frac{8}{3}(3\pi + 8)$ sq. units 5. $\frac{9}{2}$ sq. units 6. 9 sq. units

7. $\frac{32}{3}$ sq. units 8. $\pi - \frac{8}{3}$ sq. units 9. $\frac{1}{3}a^2$ sq. units 10. 48 sq. units 11. $\frac{1}{32}(16 - 3\pi)a^2$ sq. units

12. $a^2(\pi - 1)$ sq. units 13. $(\pi - 1)a^2$ sq. units 14. $\frac{1}{3}(3\sqrt{3}\pi - 9\sqrt{3} + 4\pi)$ sq. units

15. $\frac{1}{3}(17 - 6\log 2)$ sq. units 16. 16 sq. units.

16.3.3 MASS, MOMENTS, CENTROIDS AND MOMENT OF INERTIA BY DOUBLE INTEGRATION

Let a region R_1 of area A be bounded by $y = f(x)$, $y = g(x)$, $x = a$, and $x = b$. Let a plane sheet of uniform thickness and texture covers this area A . Then the mass of any portion of the sheet is proportional to the area i.e. $dm \propto dA$ and the constant of proportionality $\delta(x, y)$ is called the *areal density*.

i.e. $dm = \delta(x, y) dA$

$\therefore$ by double integration, mass is

(a) $M = \int dm = \iint_{R_1} \delta(x, y) dA = \int_a^b \int_{g(x)}^{f(x)} \delta(x, y) dy dx$ by (fig. 99).

If density $\delta(x, y) = 1$, then mass

$M = \iint_{R_1} dA = \int_a^b \int_{g(x)}^{f(x)} dy dx$ as

per fig. i.e. mass equals the area.

Let (x, y) be any vertex of the elementary rectangle dA . Then the **first moment** of the mass M about x -axis is

(b) $M_x = \iint_{R_1} y \delta(x, y) dA$
 $= \int_a^b \int_{g(x)}^{f(x)} y \delta(x, y) dy dx$

and that about y -axis is

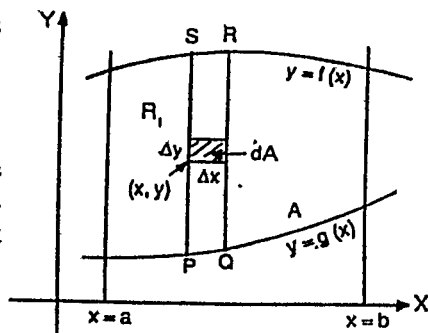


Fig. 99

$$(c) M_y = \iint_{R_1} x \delta(x, y) dA = \int_a^b \int_{g(x)}^{f(x)} x \delta(x, y) dy dx$$

[$\therefore$ moment of a mass is the product of the mass and the distance from the axis of rotation]

Let $(\bar{x}, \bar{y})$ be the **centroid** of the mass. Then moment of the mass is given

$$M\bar{x} = M_y \text{ and } M\bar{y} = M_x$$

$\therefore$ co-ordinates of the centroid $(\bar{x}, \bar{y})$ are

$$(d) \bar{x} = \frac{M_y}{M} = \frac{\text{moment about y-axis}}{\text{mass}} = \frac{\int_a^b \int_{g(x)}^{f(x)} x \delta(x, y) dy dx}{\int_a^b \int_{g(x)}^{f(x)} \delta(x, y) dy dx}$$

and
$$\bar{y} = \frac{M_x}{M} = \frac{\text{Moment about x-axis}}{\text{mass}}$$

$$= \frac{\int_a^b \int_{g(x)}^{f(x)} y \delta(x, y) dy dx}{\int_a^b \int_{g(x)}^{f(x)} \delta(x, y) dy dx}$$

Again, **moment of inertia**, also called **moment of a moment** or **second moment** is the product of the mass and the square of the distance or "arm of lever" and denoted by I .

So, **moment of inertia** about x and y axes are given by

$$(e) I_x = \iint_{R_1} y^2 \delta(x, y) dA = \int_a^b \int_{g(x)}^{f(x)} y^2 \delta(x, y) dy dx$$

and
$$I_y = \iint_{R_1} x^2 \delta(x, y) dA = \int_a^b \int_{g(x)}^{f(x)} x^2 \delta(x, y) dy dx.$$

Also the distance of (x, y) from the origin is $\sqrt{x^2 + y^2}$. So **polar moment of inertia** is

$$(f) I_o = \iint (x^2 + y^2) \delta(x, y) dA = I_x + I_y$$

$\therefore$ z -axis is perpendicular to xy -plane at the origin, so I_o may be called the moment of inertia about z -axis.

(g) For a lamina occupying a region R in the xy plane, the double integral $I_{xy} = \iint_R \delta(x, y) xy dA$ is called the **product of inertia** of the lamina w.r.t. the co-ordinate axes.

In the above integrals the limits are the limits of the area. But care should be taken for symmetrical cases. Limits of total area are to be considered for these cases instead of multiplying one symmetrical part as in area problems.

Uses of moment of Inertia :

(1) When a particle of mass m is rotating about an axis in a circle of radius r with angular velocity ω and linear velocity $v = \omega r$, its K.E. $= \frac{1}{2} m v^2 = \frac{1}{2} m \omega^2 r^2 = \frac{1}{2} \omega^2 I$, where I is the moment of inertia about that axis.

(2) In the theory of deflection of beams under transverse loading, the **stiffness factor** $= EI$, E is Young's modulus and I is the moment of inertia of a cross-section about an horizontal axis through C . G .

(3) In statistics first moment is used in computing mean or average of a set of data, second moment for computing variance (σ^2) or standard deviation (σ), third and fourth moments for computing skewness and kurtosis. The n th moment is defined as

$$M_n = \sum_{k=1}^n m_k r_k^n$$

where m_k is a number but not mass and r_k is some measure but not distance.

(4) Moment of inertia is used in determining **radius of gyration** $R_y = \sqrt{I_y/M}$ about y -axis. It tells how far from the y -axis the entire mass M might be concentrated and still give the same I_y .

Examples

1. Find mass, moments, centroid, moment of inertias, radius of gyrations and product of inertia of the area bounded by $x = y^2$, $x = 2y - y^2$ if the density at any point (x, y) is $\delta = y + 1$.

The given curves are parabolas and intersect at $(0, 0)$ and $(1, 1)$ respectively (fig. 100). Although both horizontal and vertical elementary strip covers the whole area as per figure, yet horizontal strip is convenient for obtaining simple limits. So, horizontal strip is considered. Now

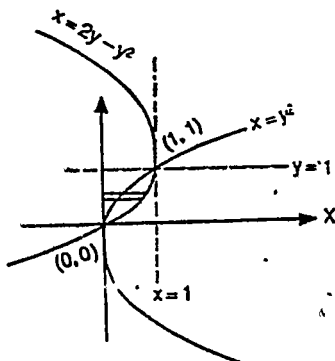


Fig. 100

$$\text{Mass, } M = \int_0^1 \int_{y^2}^{2y-y^2} \delta(x, y) dx dy$$

$$= \int_0^1 \int_{y^2}^{2y-y^2} (y+1) dx dy$$

$$= 2 \int_0^1 y(y+1)(1-y) dy = \frac{1}{2}$$

Moment about x -axis is

$$M_x = \int_0^1 \int_y^{2y-y^2} y(y+1) dx dy = 2 \int_0^1 y^2(1-y^2) dy = \frac{4}{15}$$

Moment about y-axis is

$$\begin{aligned} M_y &= \int_0^1 \int_y^{2y-y^2} x(y+1) dx dy = \frac{1}{2} \int_0^1 (y+1) [(2y-y^2)^2 - y^4] dy \\ &= 2 \int_0^1 y^2(1-y^2) dy = \frac{4}{15} \end{aligned}$$

Now centroid $(\bar{x}, \bar{y})$ is given by

$$\bar{x} = \frac{M_y}{M} = \frac{4}{15} \cdot 2 = \frac{8}{15}$$

and $\bar{y} = \frac{M_x}{M} = \frac{8}{15}$

$\therefore$ the centroid is $\left(\frac{8}{15}, \frac{8}{15}\right)$

Now moment of inertia about x and y-axes are

$$I_x = \int_0^1 \int_y^{2y-y^2} y^2(y+1) dx dy = 2 \int_0^1 y^3(1-y^2) dy = \frac{1}{6}$$

and
$$\begin{aligned} I_y &= \int_0^1 \int_y^{2y-y^2} x^2(y+1) dx dy = \frac{1}{3} \int_0^1 (y+1) [(2y-y^2)^3 - y^6] dy \\ &= \frac{1}{3} \int_0^1 (8y^3 - 4y^4 - 6y^5 + 4y^6 - 2y^7) dy = \frac{73}{420} \end{aligned}$$

Also, radius of gyration about x and y-axes are

$$R_x = \sqrt{I_x/M} = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}}$$

and $R = \sqrt{I_y/M} = \sqrt{(73/210)}$

Now product of inertia with respect to axes is given by

$$\begin{aligned} I_{xy} &= \iint_R \delta xy dA = \int_0^1 \int_y^{2y-y^2} (y+1) xy dx dy \\ &= \frac{1}{2} \int_0^1 (y+1) y [(2y-y^2)^2 - y^4] dy \\ &= \frac{1}{2} \int_0^1 4y^3(1-y^2) dy = \frac{1}{6} \end{aligned}$$

Exercises 16 (3)

Find the centroid of the following areas enclosed by the given curves :

1. $y = x^3$ and $y = 4x$ area in 1st quadrant.

2. $y = 4x - x^2$ and $y = x^2$.

3. $y = 6x - x^2$ and $y = x$.

4. $y = 3 - x^2$ and $y = x + 1$. 5. $y = x^2 - 2x - 3$, $y = 2x - 3$.
 6. $y^3 = x^2$ and $2y = x$. 7. $y^2 = 8x$ and $x + y = 6$.
 8. $y = 6x - x^2$ and $x + y = 6$. 9. $x = 4y - y^2$ and $y = x$.
 10. $x^{2/3} + y^{2/3} = a^{2/3}$ (1st quadrant). 11. $\sqrt{x} + \sqrt{y} = \sqrt{a}$, $x = 0$, $y = 0$.
 12. $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$ one arch.
 13. $y = 2x^2 - 8x + 5$, $y = 2x - 3$. 14. $y^2 = x^3$, $y = x$, (1st quadrant).
 15. Find the centre of gravity of the area bounded by $y = \sqrt{a^2 - x^2}$, $x = \pm a$, $y = -a$ if the density at (x, y) is $\delta = (y + a)$.
 16. Find the moment of inertia about the x -axis of the area bounded by $x = y - y^2$, $x + y = 0$, if the density at (x, y) is $\delta = x + y$.
 Find I_x, I_y, I_0 of the following enclosed area :
 (in Ans. A denotes enclosed area).
 17. $x^2 + y^2 = r^2$, right half of the y -axis.
 18. $(0, 0)$, (b, a) and $(b, 0)$ are the vertices of a triangle.
 19. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
 20. $y^2 = 4x$, $x = 4$, $y = 0$ (1st quadrant).
 21. $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and $x^2 + y^2 = 2y$ between them.
 22. $\frac{x^2}{16} + \frac{y^2}{9} = 1$ and $x^2 + \frac{y^2}{4} = 1$ between them.
 23. Between the circles $x^2 + y^2 = 36$, $x^2 + (y + 3)^2 = 4$.
 24. $x^{2/3} + y^{2/3} = a^{2/3}$.
 25. Find the product of inertia for the following lamina assuming constant density :
 (i) first quadrant of circular lamina $x^2 + y^2 = a^2$.
 (ii) square lamina bounded by $x = -a$, $x = 2a$, $y = -2a$, $y = a$.
 (iii) lamina bounded by $y^2 = 2a(x + a)$ and $x = a$.

Answers

1. $\left(\frac{16}{15}, \frac{64}{21}\right)$ 2. $(1, 2)$ 3. $\left(\frac{5}{2}, 5\right)$ 4. $\left(-\frac{1}{2}, \frac{7}{5}\right)$ 5. $\left(2, -\frac{3}{5}\right)$ 6. $\left(\frac{10}{3}, \frac{40}{21}\right)$ 7. $\left(\frac{34}{5}, -4\right)$
 8. $\left(\frac{7}{2}, 5\right)$ 9. $\left(\frac{12}{5}, \frac{3}{2}\right)$ 10. $\left(\frac{256a}{315\pi}, \frac{256a}{315\pi}\right)$ 11. $\left(\frac{a}{5}, \frac{a}{5}\right)$ 12. $\left(\pi a, \frac{5a}{6}\right)$ 13. $\left(\frac{15}{2}, \frac{1}{5}\right)$

14. $\left(\frac{10}{21}, \frac{5}{12}\right)$, 15. $\left(0, \frac{8+3\pi}{40+12\pi}\right)$, 16. $\frac{64}{105}$, 17. $I_x = I_y = \frac{Ar^2}{4}$, 18. $I_x = \frac{Aa^2}{6}$, $I_y = \frac{Ab^2}{2}$,
 19. $I_x = \frac{Ab^2}{4}$, $I_y = \frac{Aa^2}{4}$, 20. $I_x = \frac{16A}{5}$, $I_y = \frac{48A}{7}$, 21. $I_x = \frac{19A}{20}$, $I_y = \frac{53A}{20}$,
 22. $I_x = \frac{5A}{2}$, $I_y = \frac{19A}{4}$, 23. $I_x = \frac{71A}{8}$, $I_y = 10A$, 24. $I_x = I_y = \frac{7Aa^2}{64}$,
 25. (i) $6a^4/8$ (ii) $-9a^4/4$ (iii) 0.

N.B. A denotes given area.

16.3.4 CHANGE OF ORDER OF INTEGRATION

Continuous functions are integrable. In evaluating double integrals using iterated integrals it is found that the outer variable is a parameter for inner integral. So, continuity of this parametric function is essential for integration. The order of a proper iterated integral can be changed easily by changing the order of limits as is seen from equation (4) and (5) of case 1 of 16.3.1 for *constant* limits of rectangular region of integration and from equation (6) and (7) of case 2 of the same section 16.3.1 for *variable* limits of inner integral in closed region R requires changing actual values of the limits of integration and it is **not so easy** as with constant limits.

Procedure : (1) Draw diagram of the region enclosed by the limits. (2) Observe whether elementary area strip can be drawn in both horizontal and vertical ways and its rolling covers the whole area. (3) Next observe the easiness or uneasiness of integration and choose the easy one. (4) Over all continuity must be satisfied.

Illustration : In Ex. 1 of 16.3.1 order cannot be interchanged because of (1) procedure (2) fails to cover the whole area by the rolling of one vertical strip while Ex. 2 of 16.3.1 obeys all procedures and interchanges are also shown. Ex. 1 of 16.3.2 satisfies all conditions of procedure and so, change of order is possible but not possible for the first case.

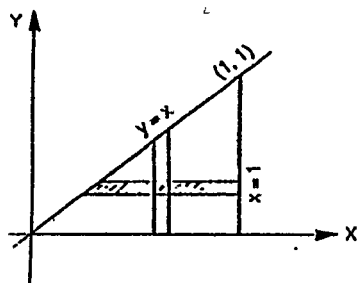


Fig. 101

Sometimes change of order is *essential* otherwise integration *cannot* be performed easily in closed form. For example $\int_0^1 \left(\int_y^1 e^{x^2} dx \right) dy$. Inner integral cannot be integrated in closed form. The area enclosed by the limits is shown in (fig. 101). All conditions of procedures are satisfied. So, new setting is

$$\int_0^1 \int_0^x e^{x^2} dy dx = \int_0^1 e^{x^2} [y]_0^x dx = \int_0^1 x e^{x^2} dx = \frac{1}{2} (e - 1).$$

Improper multiple integrals sometimes occur in practice as in 10.2.9 and 10.4 and integration order needs to interchange. In such cases the inner integral must be uniformly convergent as can be seen to satisfy in 10.2.9 and 10.4 using test of uniform convergence namely Weierstrass M -test.

16.3.5 CHANGE OF VARIABLES

Substitution is a powerful technique for single integrals. This substitution is a single variable relation like $x = \phi(u)$ and on differentiation $dx = \phi'(u) du$ gives the relation between old and new differentials dx and du provided $\phi'(u)$ exists and $\neq 0$. This $\phi(u)$ is the jacobian for single variable (Ref. Ch. 15) and non-vanishing of jacobian assures transformation or substitution whatever is said. This idea needs extension in multi-variable case when integrand or region is complicated. Here double variables will be considered.

Let $\iint_R F(x, y) dA$ be a

double integral in xy co-ordinate system. If substitutions or transformations $x = f(u, v)$ and $y = g(u, v)$

are made then "what will be the corresponding dA in uv system" is the present vital issue. So, using the above transformation corresponding

to $\iint_R F(x, y) dA$, a new integral in terms of u, v is to be determined.

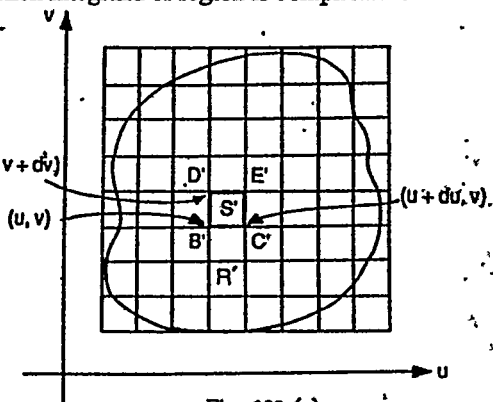
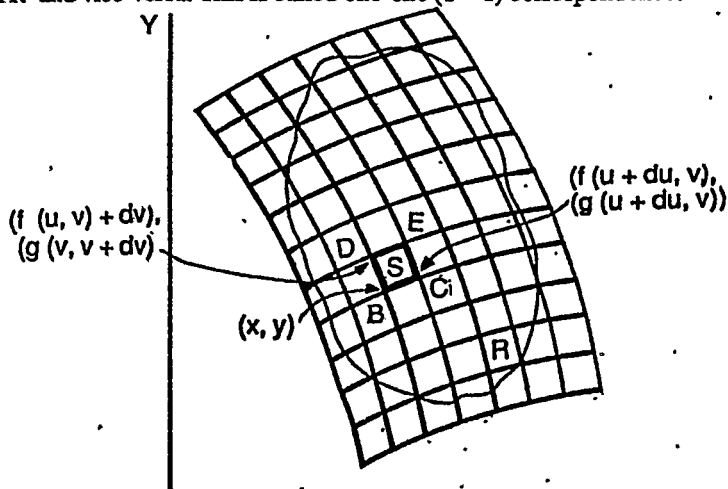


Fig. 102 (a)

In this transformation let each point of R corresponds a single point of R' and vice-versa. This is called one-one (1 - 1) correspondence.



Let the region R' be sub-divided into small rectangular sub-regions. The relations $x=f(u, v)$, $y=g(u, v)$ then give sub-regions of R into corresponding subregions in R' (fig. 102 a, b).

Let the respective sub-regions S and S' have respective areas dA and dA' . Let the dimensions of S' be du and dv so that the co-ordinates of the vertices of the rectangle $B'C'E'D'$ are $B'(u, v)$, $C'(u+du, v)$, $E'(u+du, v+dv)$ and $D'(u, v+dv)$; so that the corresponding vertices of the sub-region $BCED$ which is approximately a parallelogram are given by $B(x, y)$, $C(f(u+du, v), g(u+du, v))$, $E(f(u+du, v+dv), g(u+du, v+dv))$, $D(f(u, v+dv), g(u, v+dv))$. But by Taylor's series expansion $f(u+du, v) = f(u, v) + \frac{\partial f}{\partial u} du + \dots = f(u, v) + \frac{\partial f}{\partial u} du$ (approximately) neglecting higher powers of small quantities du etc. So the co-ordinates of C, D, E simplify to approximations

$$B(x, y), C\left(x + \frac{\partial f}{\partial u} du, y + \frac{\partial g}{\partial u} du\right), E\left(x + \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv, y + \frac{\partial g}{\partial u} du + \frac{\partial g}{\partial v} dv\right), \\ D\left(x + \frac{\partial f}{\partial v} dv, y + \frac{\partial g}{\partial v} dv\right). \text{ Now the area of this approximate parallelogram } BCED \text{ can be determined using (i) analytic co-ordinate geometry or (ii) cross product of vector algebra.}$$

Area, dA calculation :

(i) Area of a parallelogram $BCED = 2$ (area of the triangle BCD)

$$= \begin{vmatrix} x & y & 1 \\ \left(x + \frac{\partial f}{\partial u} du\right) & \left(y + \frac{\partial g}{\partial u} du\right) & 1 \\ \left(x + \frac{\partial f}{\partial v} dv\right) & \left(y + \frac{\partial g}{\partial v} dv\right) & 1 \end{vmatrix} = \begin{vmatrix} x & y & 1 \\ \frac{\partial f}{\partial u} du & \frac{\partial g}{\partial u} du & 0 \\ \frac{\partial f}{\partial v} dv & \frac{\partial g}{\partial v} dv & 0 \end{vmatrix} \\ = \begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial g}{\partial u} \\ \frac{\partial f}{\partial v} & \frac{\partial g}{\partial v} \end{vmatrix} du dv = J \begin{pmatrix} f, g \\ u, v \end{pmatrix} du dv$$

[$\because$ higher order infinitesimals du, dv are neglected.]

But $dA' = du dv$. Therefore $dA = J \begin{pmatrix} f, g \\ u, v \end{pmatrix} dA'$. It is better to take absolute value of jacobian, otherwise plotting of vertices is needed to note anticlockwise direction. $\therefore dA = |J| dA'$.

(ii) The approximate vectors i.e. tangents at B are $\vec{BC}$ and $\vec{BD}$. Their components are their co-ordinates as above.

$\therefore \vec{BC} \times \vec{BD}$ = Vector area of the parallelogram $BCED$ = dA

$$\approx \begin{vmatrix} i & j & k \\ \left(x + \frac{\partial f}{\partial u} du\right) & \left(y + \frac{\partial g}{\partial u} du\right) & 0 \\ \left(x + \frac{\partial f}{\partial v} dv\right) & \left(y + \frac{\partial g}{\partial v} dv\right) & 0 \end{vmatrix} = k \begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial g}{\partial u} \\ \frac{\partial f}{\partial v} & \frac{\partial g}{\partial v} \end{vmatrix} du dv$$

$$= k J \left(\frac{f, g}{u, v} \right) du dv$$

$$\therefore dA = |J| dA' \quad \therefore dA' = du dv$$

$$\therefore \iint_R F(x, y) dA = \iint_{R'} F(f(u, v), g(u, v)) |J| dA'$$

Therefore, this transformation or substitution or change of variables is only possible when $|J| \neq 0$ and for the existence of jacobian partial derivatives of f, g must be continuous with the continuity of f and g themselves.

16.3.5.1 PROCEDURE FOR CHANGE OF VARIABLES

Change of variables is essential in two cases – (i) the integrand f is complicated, (ii) region R of integration is complicated. In either case the following procedures may be adopted step by step.

Step 1. Define a suitable transformation T in R' of the uv plane if the transformation is not specified.

Step 2. Determine the image R' of R under the defined or given transformation and draw the figure in R' .

Step 3. Solve for x and y in terms of the new variables u and v to get the transformation from R' to R if the transformation is not given.

Step 4. Compute the jacobian $J = \frac{\partial(x, y)}{\partial(u, v)}$.

Step 5. Evaluate $\iint_R f(x, y) dA$ using $\iint_{R'} H(u, v) |J| dA'$.

Examples

1. Discuss the feasibility of changing the order of integration of Ex. 2 of 16.3.1.

$$\text{The integration is } \int_1^4 \int_{4/y}^{(5+4y-y^2)/2} dx dy.$$

The region of integration is depicted in fig. 97. To change the order, first we have to integrate w.r.t. y i.e. construction of a vertical strip is necessary. When this vertical strip rolls from left to right, it lies in (i) $xy=4$, $y=4$, $x=1$, $x=5/2$, then moves further to right and lies in (ii) $xy=4$, $2x=5+4y-y^2$, $x=5/2$, $x=4$ and finally within (iii) $2x=5+4y-y^2$, $x=4$, $x=4\frac{1}{2}$. In 2nd and last cases $y=2\pm\sqrt{9-2x}$. So, due to order change, the given integral becomes the sum of three integrals as.

$$\int_1^4 \int_{4/y}^{(5+4y-y^2)/2} dx dy = \int_1^{5/2} \int_{y=4/x}^4 dy dx + \int_{5/2}^4 \int_{4/x}^{2+\sqrt{9-2x}} dy dx + \int_4^{4\frac{1}{2}} \int_{2-\sqrt{9-2x}}^{2+\sqrt{9-2x}} dy dx.$$

From the academic point of view, it is O.K. But from the practical point view it is unwise and in such case order change is not advisable. This discussion is again important in setting iterated integrals for other problems too. Always choose a convenient strip of area so that minimum labour gives maximum benefit. The following problem shows that order change is essential.

2. Change the order of $\int_0^1 \int_y^1 \sin x^2 dx dy$ if essential and evaluate.

The given region of integration is obtained by setting limits to equation as $x=y$, $x=1$ from inner integral and $y=0$, $y=1$ from outer integral. Now, draw the region enclosed by those equations as (fig. 103). The original horizontal strip is shown clear and new vertical one is crossed. From figure the change of order is no problem. But from the given setting it is essential to change, because inner integral cannot be integrated easily in closed form. Therefore, in the new setting y varies from $y=0$ to $y=x$ and x varies from $x=0$ to $x=1$. Hence the required change of order is

$$\begin{aligned} I &= \int_0^1 \int_y^1 \sin x^2 dx dy \\ &= \int_{x=0}^1 \int_{y=0}^x \sin x^2 dy dx \end{aligned}$$

which is easily integrable. Therefore,

$$\begin{aligned} I &= \int_0^1 \sin x^2 \left[y \right]_0^x dx \\ &= \int_0^1 x \sin x^2 dx \\ &= \frac{1}{2} \left[-\cos x^2 \right]_0^1 = \frac{1}{2} (1 - \cos 1). \end{aligned}$$

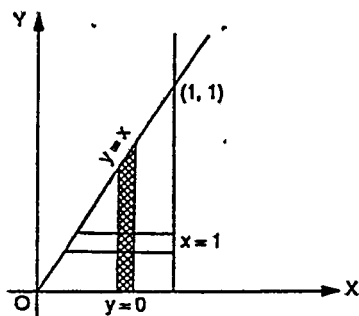


Fig. 103

3. Transform the integral $\int_0^1 \int_x^1 xy \, dy \, dx$ by using the relations $x = u + v$ and $y = u - v$. Hence evaluate it.

First construct the original region according to the limits of the integral. The boundaries of the given region are (i) $y = x$, lower curve of the vertical strip, dx ; (ii) $y = 1$, upper curve of that vertical strip, dx ; (iii) $x = 0$, the left boundary of R and (iv) $x = 1$, is the x -co-ordinate of the intersection of $y = x$ & $y = 1$. Therefore, region R is given in fig. 104 (a).

Therefore, the new boundaries, in uv plane, of region R' are correspondingly given by (i) $u + v = u - v$ or $v = 0$, (ii) $u - v = 1$, (iii) $u + v = 0$ or $v = -u$ (fig. 104 (b)). Here (ii) & (iii) intersect at $\left(\frac{1}{2}, -\frac{1}{2}\right)$.

$\therefore$ The jacobian,

$$J\left(\frac{x, y}{u, v}\right) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

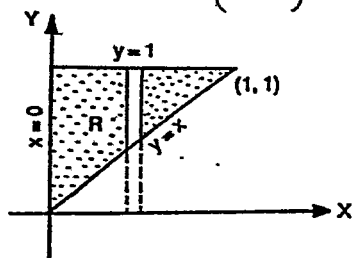


Fig. 104 (a)

$\therefore |J| = 2$. Therefore, relation between the two integrals in two system is

$$\begin{aligned} & \int_0^1 \int_x^1 xy \, dy \, dx \\ &= 2 \int_{-1/2}^0 \int_{-v}^{v+1} (u+v)(u-v) \, du \, dv \\ &= 2 \int_{-1/2}^0 \int_{-v}^{v+1} (u^2 - v^2) \, du \, dv \\ &= 2 \int_{-1/2}^0 \left[\frac{1}{3} u^3 - v^2 u \right]_{u=-v}^{u=v+1} dv \\ &= 2 \int_{-1/2}^0 \left[-\frac{4}{3} v^3 + v + \frac{1}{3} \right] dv = \frac{1}{8}. \end{aligned}$$

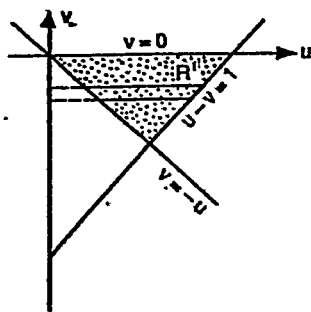


Fig. 104 (b)

4. Evaluate $\iint_R 49x^2y \, dA$, where R is the region bounded by the lines

$$2x - y = 1, 2x - y = -2, x + 3y = 0 \text{ and } x + 3y = 1,$$

by a suitable change of variables.

Here the integrand is simple for integration. But the region R bounded is a parallelogram (fig. 105 (a)) in which integration is complicated.

Step 1. Let $u = 2x - y$ and $v = x + 3y$ be the change of variables. Because $u = \text{const.}$, $v = \text{const.}$ will construct a rectangle (fig. 105 (b)) where integration is easy.

Step 2. The above step 1 shows that the images are $u = 1$, $u = -2$ and $v = 0$, $v = 1$ in R' of the uv plane and the figure is a rectangle.

Step 3. From step 1, let us solve for x and y . They are

$$x = \frac{1}{7}(3u + v) \quad \text{and} \quad y = \frac{1}{7}(-u + 2v).$$

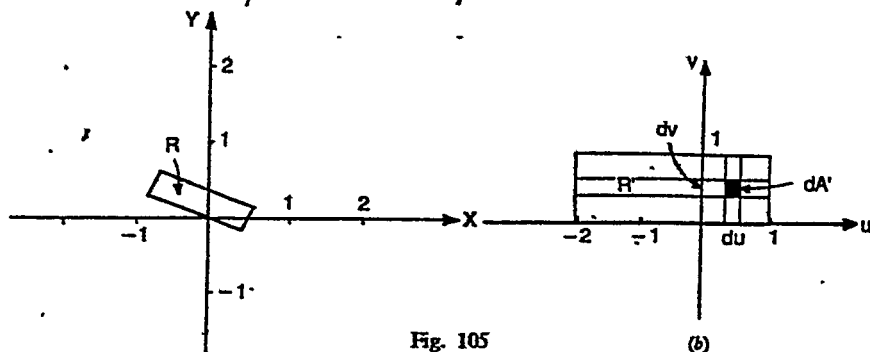


Fig. 105

(b)

(a)

$$\text{Step 4. Now } J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{3}{7} & \frac{1}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{vmatrix} = 1.$$

Step 5. Therefore,

$$\begin{aligned} \iint_R 49x^2y \, dA &= \iint_{R'} 49 \cdot \frac{1}{49} (3u + v)^2 \cdot \frac{1}{7} (-u + 2v) \, dA' \\ &= \frac{1}{7} \int_{-2}^1 \int_0^1 (-9u^3 + 12u^2v + 11uv^2 + 2v^3) \, dv \, du \\ &= \frac{191}{28}. \end{aligned}$$

Exercises 16 (4)

If change of order of integration is essential, then sketch the region of integration and change the order and hence or otherwise compute the integrals followed, if possible:

1. $\int_0^1 \int_v^1 e^{u^2} \, du \, dv.$

2. $\int_0^1 \int_y^1 \frac{\sin x}{x} \, dx \, dy.$

3. $\int_0^{\pi/2} \int_y^{\pi/2} \frac{\cos x}{x} \, dx \, dy.$

4. $\int_0^1 \int_{\sqrt{w}}^1 \frac{dv}{1+v^3} \, dw.$

5. (i) $\int_0^1 \int_y^1 x^2 \cos(x^2 - xy) dx dy$. (ii) $\int_0^a \int_0^y \frac{dx dy}{\sqrt{(a^2 - x^2)(y - x)}}$.

Change the order of integration of the following :

6. $\int_0^1 \int_x^{\sqrt{x}} f(x, y) dy dx$.

7. $\int_0^4 \int_{-\sqrt{4-y}}^{\frac{y-4}{2}} dx dy$.

8. $\int_0^a \int_0^{\sqrt{2ay-y^2}} f(x, y) dx dy$.

9. $\int_0^{\sqrt{2}} \int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} y dx dy$.

10. $\int_{-2}^1 \int_{\frac{x}{2}+4x}^{3x+2} dy dx$.

11. $\int_0^1 \int_{\frac{x}{2}}^x dy dx$.

What is the value of the following repeated integrals ? Write the equivalent repeated integrals with the order of integrals reversed and verify that each has the same value as the given integral :

12. $\int_0^1 \int_x^{2x} 6x^2 y dy dx$.

13. $\int_0^2 \int_1^x dy dx$.

14. $\int_0^1 \int_{\sqrt{y}}^1 dx dy$.

Use the given transformation to transform the integral to an iterated integral in u and v in each of the following cases :

15. $\iint_R (x+y) dA$ where R is the region bounded by $y=2x$, $x=1$ and $y=0$. Use $x=u-v$, $y=2v$.

16. $\int_{-2}^0 \int_{-x-2}^0 (xy+3) dy dx$ using $x=u-v$, $y=u+v$.

17. $\int_0^4 \int_{\sqrt{y}}^2 y dx dy$ using $x=2u$, $y=-v$.

18. By considering the co-ordinate transformation $x+y=u$ and $y=v$, show

$$\begin{aligned} \text{that } \int_0^a \int_0^{a-y} e^{-(x+y)} F(x) G(y) dx dy \\ = \int_0^a e^{-u} \int_0^u F(u-v) G(v) dv du. \end{aligned}$$

19. Given that $x+y=u$, $y=uv$, change the variables to u, v in the integral $\iint [xy(1-x-y)]^{1/2} dx dy$ taken over the area enclosed by the lines

$x=0, y=0, x+y=1$. Hence show that the value of the integral is $2\pi/105$.

20. Find the mass of a plate bounded by the four parabolas $y^2=4ax$, $y^2=4bx$, $x^2=4cy$, $x^2=4dy$, if the density $\rho=kxy$.

[Hints: Substitute $u=x^2/y$, $v=y^2/x$ then the parabolas in xy plane transform to straight lines $v=4a$, $v=4b$, $u=4c$, $u=4d$.]

21. Change the order of integration and then evaluate

$$\int_0^1 \int_y^{2y} f(x) dx dy \text{ where } f(x) = \begin{cases} x^{-1/2}, & 0 \leq x \leq 1 \\ (2-x)^{-1/2}, & 1 \leq x \leq 2 \end{cases}$$

[Hints: Enclosed region of area breaks by the line $x=1$ into two sub-regions when y is summed first to change the given order of integration.]

22. Sketch the region of integration and evaluate

$$\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2) \tan^{-1}\left(\frac{y}{x}\right) dx dy \text{ where 'a' is a constant.}$$

23. Evaluate the following integral using a suitable change of variables:

- (i) $\int_R x^3 y dA$ where R is in first quadrant bounded by the lines $y=x$, $y=2x$ and the hyperbolas $xy=1$ and $xy=3$.

[Hints: Region is complicated, $x=u/v$, $y=v$ may be chosen.]

- (ii) $\iint_R 7xy dA$ where R is bounded by $2x+3y=1$, $2x+3y=3$, $x-2y=2$ and $x-2y=-2$.

- (iii) $\iint_R \frac{\sin(x-y)}{\cos(x+y)} dA$ where R is bounded by $y=0$, $y=x$ and $x+y=\pi/4$.

[Hints: Integrand is complicated. $u=x-y$, $v=x+y$ may be taken.]

- (iv) $\iint_R \left(\frac{x-2y}{x+2y}\right)^3 dA$ where R is bounded by $x-2y=1$, $x-2y=2$, $x+2y=1$ and $x+2y=3$.

- (v) $\iint_R \exp[(2x-y)/(x+y)] dA$ where R is bounded by $x=2y$, $y=2x$, $x+y=1$ and $x+y=2$.

- (vi) $\iint_R \left(1 + \frac{x^2}{16} + \frac{y^2}{25}\right)^{3/2} dA$ where R is $\frac{x^2}{16} + \frac{y^2}{25} = 1$.

[Hints: Choose $x=4u$, $y=5v$. Then polar co-ordinates may be used to make evaluation easy.]

Answers

1. $\frac{1}{2}(e^2-1)$; 2. $1-\cos 1$; 3. 1; 4. $\frac{1}{3}\log 2$; 5. (i) $\frac{1}{2}(1-\cos 1)$, (ii) $4(\sqrt{2}-1)\sqrt{a}$;
 6. $\int_0^1 \int_2^y f(x,y) dx dy$; 7. $\int_{-2}^0 \int_{2x+4}^{4-x^2} dy dx = \frac{4}{3}$; 8. $\int_0^a \int_{a-\sqrt{a^2-x^2}}^a f(x,y) dy dx$;
 9. $\int_{-2}^2 \int_0^{\sqrt{(4-x^2)/2}} y dy dx = \frac{8}{3}$; 10. $\int_{-4}^5 \int_{(y-2)/3}^{-2+\sqrt{y+4}} dx dy = \frac{9}{2}$;
 11. $\int_0^1 \int_y^{\sqrt{y}} dx dy = \frac{1}{6}$; 12. $\int_0^1 \int_{y/2}^y 6x^2y dx dy + \int_1^2 \int_{y/2}^1 6x^2y dx dy = 9/5$;
 13. $\int_1^e \int_{\log y}^2 dx dy = e^2 - 3$; 14. $\int_0^1 \int_0^{x^2} dy dx = \frac{1}{3}$; 15. $2 \int_0^1 \int_{2v}^{v+1} (u+v) du dv$;
 16. $2 \int_{-1}^0 \int_u^{-u} (u^2 - v^2 + 3) dv du$; 17. $-\int_{-4}^0 \int_{\frac{1}{2}\sqrt{-v}}^1 v du dv$;
 20. $\frac{64}{3}k(b^2 - a^2)(d^2 - c^2)$; 21. $\int_0^1 \int_{x/2}^x x^{-1/2} dy dx + \int_1^2 \int_{x/2}^1 (2-x)^{-1/2} dy dx = \frac{2}{3}$;
 22. $\frac{\pi^2 a^4}{3^2}$; 23. 23. (i) $\frac{13}{16}$, (ii) $\frac{16}{147}$, (iii) $\frac{1}{2} \left(\log(\sqrt{2}+1) - \frac{\pi}{4} \right)$; (iv) $\frac{5}{12}$; (v) $\frac{1}{2}(2e-1)$; (vi) $256\sqrt{2}\pi$.

163.6 POLAR CO-ORDINATES IN DOUBLE INTEGRATION

The relationship of cartesian and polar co-ordinates is $x = r \cos \theta$ and $y = r \sin \theta$. Now the jacobian of x, y w.r.t. r, θ is

$$J \begin{pmatrix} x, y \\ r, \theta \end{pmatrix} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r \neq 0$$

$\therefore$ cartesian elementary area, $dA = 1 J | dr d\theta = r dr d\theta$.

$\therefore$ all two dimensional formulae upto now transform to:

(i) Area: The area bounded by $r = f(\theta)$, $r = g(\theta)$, $\theta = \alpha$ and $\theta = \beta$ is

$$\int_{\alpha}^{\beta} \int_{r=f(\theta)}^{r=g(\theta)} r dr d\theta$$

and the area bounded by $\theta = \phi(r)$, $\theta = \psi(r)$, $r = a$ and $r = b$ is

$$\int_a^b \int_{\theta=\phi(r)}^{\theta=\psi(r)} r d\theta dr.$$

(ii) Moments: The moment about x -axis is

$$M_x = \iint_R y dA = \iint_R r \sin \theta \cdot r dr d\theta = \iint_R r^2 \sin \theta dr d\theta.$$

The moment about y -axis is

$$M_y = \iint_R x dA = \iint_R r \cos \theta \cdot r dr d\theta = \iint_R r^2 \cos \theta dr d\theta.$$

- (iii) **Mass :** The mass of a solid of uniform thickness and texture with areal density $\delta(x, y)$ is

$$M = \iint_R \delta(x, y) dA = \iint_{R'} \delta(r \cos \theta, r \sin \theta) r dr d\theta.$$

- (iv) **Centroid $(\bar{x}, \bar{y})$:** The co-ordinates of centroid are

$$\bar{x} = \frac{M_y}{M} = \frac{\iint_{R'} r^2 \cos \theta dr d\theta}{\iint_{R'} r dr d\theta} \text{ if } \delta = 1,$$

$$\bar{y} = \frac{M_x}{M} = \frac{\iint_{R'} r^2 \sin \theta dr d\theta}{\iint_{R'} r dr d\theta} \text{ if } \delta = 1.$$

- (v) **Moments of inertia :** The moment of inertia about x -axis is

$$I_x = \iint_R y^2 dA = \iint_{R'} r^2 \sin^2 \theta \cdot r dr d\theta = \iint_{R'} r^3 \sin^2 \theta dr d\theta$$

and that about y -axis is

$$I_y = \iint_R x^2 dA = \iint_{R'} r^2 \cos^2 \theta \cdot r dr d\theta = \iint_{R'} r^3 \cos^2 \theta dr d\theta.$$

$$\text{The moment } I_o = I_x + I_y = \iint_{R'} r^3 dr d\theta.$$

- (vi) **Product of inertia :** The product of inertia w.r.t. x and y -axes is

$$\begin{aligned} I_{xy} &= \iint_R xy dA = \iint_{R'} r^2 \sin \theta \cos \theta \cdot r dr d\theta. \\ &= \frac{1}{2} \iint_{R'} r^3 \sin 2\theta dr d\theta. \end{aligned}$$

- (vii) **Radius of gyration :** The radius of gyration about y -axis is

$$R_y = \sqrt{I_y / M} = \sqrt{\left\{ \left(\iint_{R'} r^3 \cos^2 \theta dr d\theta \right) / \iint_{R'} r dr d\theta \right\}}.$$

Examples

1. Find area bounded by lemniscate $r^2 = 2a^2 \cos 2\theta$, circle $r = 2a \sin \theta$ and the line $\theta = \pi/4$ using double integration. Also find moments, moment of inertia, centroid, product of inertia and radius of gyration.

The lemniscate and the circle intersect at $\left(a, \frac{\pi}{6}\right)$. The area in question is $QABCO$ (fig. 106).

$$\text{Now, area, } A = \int_{\pi/6}^{\pi/4} \int_{a\sqrt{2}\cos 2\theta}^{2a\sin \theta} r dr d\theta$$

$$\begin{aligned}
 &= \int_{\pi/6}^{\pi/4} (2a^2 \sin^2 \theta - a^2 \cos 2\theta) d\theta \\
 &= \frac{a^2}{12} [\pi + 6\sqrt{3} - 12] \text{ sq. units.}
 \end{aligned}$$

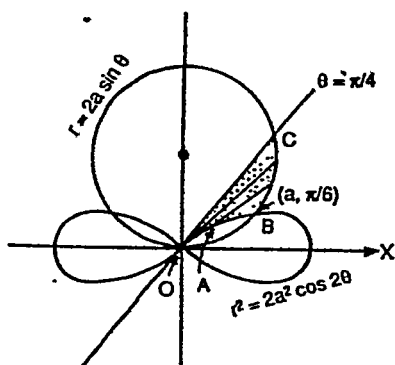


Fig. 106

Since nothing is said about the density of the area, δ . So, we take $\delta = 1$.

Now moment about x-axis is

$$\begin{aligned}
 M_x &= \int_{\pi/6}^{\pi/4} \int_{a\sqrt{2}\cos\theta}^{2a\sin\theta} (r^2 \sin \theta) dr d\theta \\
 &= \frac{1}{3} \int_{\pi/6}^{\pi/4} [8a^3 \sin^3 \theta - a^3 (2 \cos 2\theta)^{3/2}] \sin \theta d\theta \\
 &= \frac{a^3}{3} \int_{\pi/6}^{\pi/4} [2 (2 \sin^2 \theta)^2 - (4 \cos^2 \theta - 2)^{3/2} \sin \theta] d\theta \\
 &= \frac{a^3}{3} \left(\frac{\pi}{4} - 2 + \frac{7\sqrt{3}}{8} \right) - \frac{a^3}{3} \int_{\pi/6}^{\pi/4} (4 \cos^2 \theta - 2)^{3/2} \sin \theta d\theta \\
 &= \frac{a^3}{3} \left(\frac{\pi}{4} - 2 + \frac{7\sqrt{3}}{8} \right) - \frac{a^3}{3} \cdot \frac{1}{2} \int_{\sqrt{2}}^{\sqrt{3}} (u^2 - 2)^{3/2} du \text{ when } 2 \cos \theta = u \\
 &= \frac{a^3}{3} \left(\frac{\pi}{4} - 2 + \frac{7\sqrt{3}}{8} \right) - \frac{a^3}{6} \cdot 4 \int_0^{\phi} \tan^4 \phi \sec \phi d\phi, \text{ when } u = \sqrt{2} \sec \phi \\
 &\hspace{15em} \text{where } \phi = \sec^{-1} \sqrt{3/2} \\
 &= \frac{a^3}{3} \left(\frac{\pi}{4} - 2 + \frac{7\sqrt{3}}{8} \right) - \frac{2a^3}{3} \int_0^{\phi} (\sec^5 \phi - 2 \sec^3 \phi + \sec \phi) d\phi
 \end{aligned}$$

$$= \frac{a^3}{3} \left(\frac{\pi}{4} - 2 + \frac{7\sqrt{3}}{8} \right) - \frac{2a^3}{3} \left[\frac{1}{4} \sin \phi \sec^4 \phi - \frac{5}{8} \sin \phi \sec^2 \phi + \frac{3}{8} \log (\sec \phi + \tan \phi) \right]_0^{\phi} \quad \text{where } \phi = \sec^{-1} \sqrt{3/2}$$

Using reduction for $\int \sec^n x \, dx$.

$$= \frac{a^3}{3} \left(\frac{\pi}{4} - 2 + \frac{7\sqrt{3}}{8} \right) - \frac{2a^3}{3} \left[\frac{1}{4} \cdot \frac{1}{\sqrt{3}} \cdot \frac{9}{4} - \frac{5}{8} \cdot \frac{1}{\sqrt{3}} \cdot \frac{3}{2} + \frac{3}{8} \log \left(\frac{\sqrt{3}}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) \right]$$

$$= \frac{a^3}{24} \left[2\pi - 16 + 9\sqrt{3} - 3 \log \frac{\sqrt{3} + 1}{\sqrt{2}} \right]$$

and moment about y-axis is

$$M_y = \int_{\pi/6}^{\pi/4} \int_{a\sqrt{2}\cos 2\theta}^{2a\sin \theta} (r^2 \cos \theta) \, dr \, d\theta$$

$$= \frac{1}{3} \int_{\pi/6}^{\pi/4} [8a^3 \sin^3 \theta - a^3 (2 \cos 2\theta)^{3/2}] \cos \theta \, d\theta$$

$$= \frac{a^3}{3} \int_{\pi/6}^{\pi/4} 8 \sin^3 \theta \cos \theta \, d\theta - \frac{a^3}{3} \int_{\pi/6}^{\pi/4} (2 \cos 2\theta)^{3/2} \cos \theta \, d\theta$$

$$= \frac{a^3}{8} - \frac{a^3}{3} \int_{\pi/6}^{\pi/4} (2 - 4 \sin^2 \theta)^{3/2} \cos \theta \, d\theta$$

$$= \frac{a^3}{8} - \frac{a^3}{3} \cdot \frac{1}{2} \int_1^{\sqrt{2}} (2 - u^2)^{3/2} \, du \quad \text{when } 2 \sin \theta = u$$

$$= \frac{a^3}{8} - \frac{a^3}{6} \cdot 4 \int_{\pi/4}^{\pi/2} \cos^4 \phi \, d\phi \quad \text{when } u = \sqrt{2} \sin \phi$$

$$= \frac{a^3}{16} (10 - \pi)$$

Moment of inertia about x-axis is

$$I_x = \int_{\pi/6}^{\pi/4} \int_{a\sqrt{2}\cos 2\theta}^{2a\sin \theta} (r^3 \sin^2 \theta) \, dr \, d\theta$$

$$= \frac{1}{4} \int_{\pi/6}^{\pi/4} [16a^4 \sin^4 \theta - a^4 \cdot 2^2 \cos^2 2\theta] \sin^2 \theta \, d\theta$$

$$= a^4 \int_{\pi/6}^{\pi/4} [4 \sin^6 \theta - \cos^2 2\theta \sin^2 \theta] \, d\theta$$

$$= a^4 \int_{\pi/6}^{\pi/4} [4 \sin^6 \theta - \sin^2 \theta + 4 \sin^4 \theta - 4 \sin^6 \theta] \, d\theta$$

$$= a^4 \int_{\pi/6}^{\pi/4} [4 \sin^4 \theta - \sin^2 \theta] \, d\theta = \frac{a^4}{48} (4\pi - 36 + 15\sqrt{3})$$

$$\begin{aligned}
 \text{Also } I_y &= \int_{\pi/6}^{\pi/4} \int_{a\sqrt{2\cos 2\theta}}^{2a\sin \theta} r^3 \cos^2 \theta \, dr \, d\theta \\
 &= \frac{1}{4} \int_{\pi/6}^{\pi/4} [16a^4 \sin^4 \theta - a^4 \cdot 2^2 \cos^2 2\theta] \cos^2 \theta \, d\theta \\
 &= a^4 \int_{\pi/6}^{\pi/4} [4 \sin^4 \theta \cos^2 \theta - \cos^2 2\theta \cdot \cos^2 \theta] \, d\theta \\
 &= -\frac{a^4}{2} \int_{\pi/6}^{\pi/4} [\cos 2\theta + \cos 4\theta] \, d\theta = \frac{a^4}{16} (3\sqrt{3} - 4)
 \end{aligned}$$

$$\therefore I_o = I_x + I_y = \frac{a^4}{12} (\pi - 12 + 6\sqrt{3})$$

$$\begin{aligned}
 \text{Centroid. } \bar{x} &= \frac{M_y}{M} = \frac{a^3 (10 - \pi)}{16} \cdot \frac{12}{a^2 (\pi + 6\sqrt{3} - 12)} \\
 &= \frac{3a}{4} \cdot \frac{10 - \pi}{\pi + 6\sqrt{3} - 12} = 3.35a
 \end{aligned}$$

$$\begin{aligned}
 \bar{y} &= \frac{M_x}{M} = \frac{a^3}{24} \left(2\pi - 16 + 9\sqrt{3} - 3 \log \frac{\sqrt{3} + 1}{\sqrt{2}} \right) \cdot \frac{12}{a^2 (\pi + 6\sqrt{3} - 12)} \\
 &= \frac{a}{2} \cdot \left(2\pi - 16 + 9\sqrt{3} - 3 \log \frac{\sqrt{3} + 1}{\sqrt{2}} \right) / (\pi + 6\sqrt{3} - 12) = 2.43a
 \end{aligned}$$

$\therefore$ Centroid is (3.34a, 2.43a)

$$\begin{aligned}
 \text{Product of inertia, } I_{xy} &= \frac{1}{2} \int_{\pi/6}^{\pi/4} \int_{a\sqrt{2\cos 2\theta}}^{2a\sin \theta} r^3 \sin 2\theta \, dr \, d\theta \\
 &= \frac{1}{8} \int_{\pi/6}^{\pi/4} [16a^4 \sin^4 \theta - a^4 \cdot 2^2 \cos^2 2\theta] \sin 2\theta \, d\theta \\
 &= \frac{a^4}{2} \int_{\pi/6}^{\pi/4} [8 \sin^5 \theta \cos \theta - \cos^2 2\theta \sin 2\theta] \, d\theta = -\frac{7a^4}{64}
 \end{aligned}$$

Radius of gyration : Radius of gyration about y-axis is

$$R_y = \sqrt{I_y / M} = \sqrt{\left\{ \frac{a^4}{16} (3\sqrt{3} - 4) \right\} / \left\{ \frac{a^2 (\pi + 6\sqrt{3} - 12)}{1} \right\}} = .76a$$

and that about x-axis is $R_x = \sqrt{I_x / M} = .64a$.

Exercises 16 (5)

1. Change the following integrals into polar co-ordinates and hence evaluate :
(Note the advantage of transformation)

$$(i) \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy \, dx. \quad (ii) \int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2 + y^2) \, dx \, dy.$$

$$(iii) \int_0^{a\sqrt{2}} \int_0^{\sqrt{a^2-y^2}} x \, dx \, dy. \quad (iv) \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} \, dx \, dy.$$

$$(v) \int_0^2 \int_0^x y \, dy \, dx, \quad (vi) \int_0^{2a} \int_0^{\sqrt{2ax-x^2}} x^2 \, dy \, dx.$$

[Hints : Draw the curves in cartesian. Convert their equations from cartesian to polar and hence determine the limits in each case].

2. Find the following areas :

- (i) Bounded by straight line $\theta = \frac{\pi}{4}$, the lower part of a loop of $r = 4 \sin 2\theta$ in the first quadrant and the curve $r = 4 \cos 2\theta$.
- (ii) Common to both $r = 2\sqrt{3} \cos \theta$ and $r = 3$.
- (iii) Inside the circle $r = 3 \cos \theta$ and outside the circle $r = \frac{3}{2}$.
- (iv) Bounded by the circle $r = 4 \cos \theta$ and the straight line $\theta = \frac{\pi}{4}$ (smaller one).
- (v) Interior to the circle $r = 3 \cos \theta$ and exterior to $r = \cos \theta$.
- (vi) Right to the line $4r \cos \theta = 3$ and within the cardioid $r = 1 + \cos \theta$.
- (vii) Inside the circle $r = 1$ and outside the cardioid $r = 1 + \cos \theta$.
- (viii) Interior to the lemniscate $r^2 = a^2 \sin 2\theta$ and exterior to the circle $r = a\sqrt{2} \cos \theta$.
- (ix) Outside the parabola $r(1 + \cos \theta) = a$ and inside the cardioid $r = a(1 + \cos \theta)$.
- (x) Interior to the circle $r = \cos \theta + \sin \theta$ and exterior to the circle $r = 1$.
- (xi) Interior to the circle $r = \sin \theta$ and exterior to the cardioid $r = 1 - \cos \theta$.

3. Find co-ordinates of the centroid of the following areas : (Zero co-ordinate is not shown in Answers).

- (i) Interior to the circle $r = 2a \cos \theta$ and exterior to the circle $r = a$.
- (ii) Bounded by the cardioid $r = a(1 + \cos \theta)$.
- (iii) Of a loop of $r = a \cos 2\theta$.
- (iv) Of the lemniscate $r^2 = a^2 \cos 2\theta$.
- (v) Inside the cardioid $r = a(1 + \cos \theta)$ and outside the circle $r = a$.
- (vi) Of $r = \sin 2\theta$ in the first quadrant.
- (vii) Inside the circle $r = 4$ and to the right of the line $r = 2 \sec \theta$.
- (viii) Enclosed by the limaçon $r = 3 + \cos \theta$.

4. Find the moment of inertia of the following areas :
- Inside the circle $r = 2a \cos \theta$ and outside the circle $r = a$.
 - Enclosed by the lemniscate $r^2 = a^2 \cos 2\theta$ about y -axis.
 - Enclosed by $r^2 = 2a^2 \sin 2\theta$ in the first quadrant about the origin.
 - Bounded by $r = 2(\sin \theta + \cos \theta)$, $\theta = 0$, $\theta = \pi/2$ about the origin.
 - Of the circle $r = 2a \cos \theta$ about the pole.
 - Of the cardioid $r = a(1 - \cos \theta)$ about the pole.
 - Of the circle $r = 4 \sin \theta$.
5.
 - Find the mass of a thin sheet covering the region bounded by cardioid $r = 2(1 - \sin \theta)$ if the density is kr .
 - Find the mass of lamina coinciding with area between $y = x^3$ and $x = y^2$, $\delta = k(x^2 + y^2)$.
 - Compute the mass of a thin solid lying completely within the outer loop of the curve $r = 1 + 2 \cos \theta$ and of density $\rho = k|y|$.
 - Compute the mass of a plate in the form of $r = 2 + \sin \theta$, the density of a point is proportional to the square of the distance from the point $(2, 0)$.
6.
 - Find I_{xy} for the area bounded by $y = x^2$ and $y = x^3$.
 - Find I_{xy} for the upper half of the cardioid $r = 1 + \cos \theta$.

Answers

- $\int_0^{2\pi} \int_0^a r \, dr \, d\theta = \pi a^2$,
 - $\int_0^{\pi/2} \int_0^a r^3 \, dr \, d\theta = \frac{\pi a^4}{8}$,
 - $\int_0^{\pi/4} \int_0^a r^2 \cos \theta \, dr \, d\theta = \frac{a^3 \sqrt{2}}{8}$,
 - $\int_0^{\pi/2} \int_0^\infty e^{-r^2} r \, dr \, d\theta = \frac{\pi}{4}$,
 - $\int_0^{\pi/4} \int_0^{2 \sec \theta} r^2 \sin \theta \, dr \, d\theta = \frac{4}{3}$,
 - $\int_0^{\pi/2} \int_0^{2a \cos \theta} r^3 \cos^2 \theta \, dr \, d\theta = \frac{5\pi a^4}{8}$,
- 2 sq. units,
 - $\frac{1}{2}(5\pi + 3\sqrt{3})$ sq. units,
 - $3(2\pi + 3\sqrt{3})/8$ sq. units,
 - $(\pi - 2)$ sq. units,
 - 2π sq. units,
 - $\left(\frac{\pi}{2} + \frac{9\sqrt{3}}{16}\right)$ sq. units,
 - $(\pi + 8)/4$ sq. units,
 - $a^2(4 - \pi)/8$ sq. units,
 - $a^2(9\pi + 16)/12$ sq. units,
 - $\frac{1}{2}$ sq. units,
 - $(4 - \pi)/4$ sq. units.
- $\bar{x} = \frac{(8\pi + 3\sqrt{3})a}{2(2\pi + 3\sqrt{3})}$,
 - $\bar{x} = \frac{5a}{6}$,
 - $\bar{x} = \frac{128\sqrt{2}a}{105\pi}$,
 - $\bar{x} = \frac{\pi}{8} a\sqrt{2}$,
 - $\bar{x} = \frac{32 + 15\pi}{48 + 6\pi}$,
 - $\left(\frac{128}{105\pi}, \frac{128}{105\pi}\right)$,
 - $\bar{x} = 12\sqrt{3}/(4\pi - 3\sqrt{3})$,
 - $\bar{x} = \frac{37}{38}$.

4. (i) $I_x = \left(\frac{\pi}{12} + \frac{3\sqrt{3}}{16}\right)a^4$, $I_y = \left(\frac{3\pi}{4} + \frac{11\sqrt{3}}{16}\right)a^4$, (ii) $\frac{A}{48}(3\pi + 8)a^2$, $A = \text{area}$,
 (iii) $\frac{\pi a^4}{4}$, (iv) $3\pi + 8$, (v) $\frac{3\pi a^4}{2}$, (vi) $\frac{35\pi a^4}{16}$, (vii) $I_x = 20\pi$, $I_y = 4\pi$.
 5. (i) $\frac{40\pi k}{3}$, (ii) $\frac{23k}{105}$, (iii) $\frac{27k}{4}$, (iv) $\frac{515k\pi}{16}$,
 6. (i) $\frac{1}{48}$, (ii) $\frac{16}{15}$.

16.3.7 VOLUME UNDER A SURFACE BY DOUBLE INTEGRAL : (CARTESIAN CO-ORDINATES)

Let a solid stands on the xy plane. Its top surface is $z = F(x, y)$ and the base area A is bounded by the curves $y = g(x)$, $y = f(x)$, $x = a$ and $x = b$. The volume contained in the solid is to be determined by double integration.

The side faces are two plane surfaces $x = a$ and $x = b$ perpendicular to x -axis and two cylinders $y = f(x)$ and $y = g(x)$ with generating lines parallel to z -axis (fig. 107). Let the base area A be divided by lines parallel to x - and y -axes with widths Δx and Δy respectively. Let a parallelepiped be constructed with base area $\Delta x \cdot \Delta y$. Let $P(x, y, z)$ be a point on its top surface ΔS of the surface S i.e. the height of the x parallelepiped is z . Since

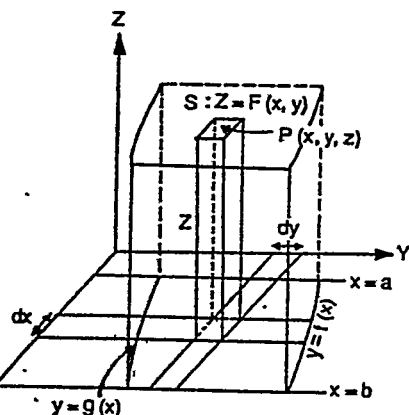


Fig. 107

Volume = (area of the base) $\times$ height

$\therefore$ area of the elementary volume $\doteq$ volume of the constructed

parallelepiped $\Delta V = z \cdot \Delta A$

when $\Delta A \rightarrow 0$, the sum $\Sigma \Delta V = \Sigma z \cdot \Delta A$ becomes

$$V = \text{total volume} \doteq \iint z \, dA \quad (1)$$

Now integral (1) can be evaluated in two ways as in 16.3.2 i.e. either by

$$V = \int_{x=a}^b \int_{y=g(x)}^{f(x)} z \, dy \, dx \quad \text{as in figure}$$

$$\text{or by } V = \int_c^d \int_{x=f_1(y)}^{f_2(y)} z \, dx \, dy \quad \text{if } A \text{ is enclosed by}$$

$$x = f_1(y), x = f_2(y), y = c \text{ and } y = d.$$

16.3.7.1 REVIEW OF THREE DIMENSIONAL SURFACES

The identification of 3-dimensional surfaces are necessary for 16.3.7 and from 16.4 to 16.4.2. So for ready references, a list of those surfaces with identifying characteristics are given below :

(a) **Planes** : These are the simplest of all surfaces. A linear equation in x , y and z represents a plane. The general equation is $ax + by + cz + d = 0$. Missing of one variable in equn. represents a plane parallel to the co-ordinate of the missing variable and perpendicular to the plane of other two. Missing of two variables in equn. represents a plane parallel to the plane of missing variables and perpendicular to the axis of the present variable.

(b) **Cylinders** : The next simplest surfaces after planes are cylinders. Cylinders may be of different types depending on the shape of cross-sections perpendicular to their axes or generating lines. The equations of cylinders are the equations of their cross-sections in two dimensions, the missing variable in the equation represents the generating line of the cylinders parallel to co-ordinate axis corresponding to the missing variable. Thus $y^2 = 4ax$ is a parabolic cylinder with generating lines parallel to the z -axis. It extends to both positive and negative directions of z -axis indefinitely.

(c) **Spheres** : This is the most commons of all second degree surfaces. Its equation with centre (h, k, m) and radius ' a ' is $(x-h)^2 + (y-k)^2 + (z-m)^2 = a^2$. All cross-sections by planes are circles.

(d) **Ellipsoids** : These are melon-shaped figures. General equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ where a, b, c are the lengths of intercepts on the x -, y - and z -axes. All cross-sections perpendicular to co-ordinate axes or parallel to co-ordinate planes are ellipses.

(e) **Hyperboloid of one sheet** : The general equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$. Cross-sections perpendicular to co-ordinate axes namely x, y and z -axes are hyperbola, hyperbola and ellipse respectively (fig. 108) shows a cross-section perpendicular to z -axis at $z = c$, so the equation of ellipse is

$$\frac{x^2}{(a\sqrt{2})^2} + \frac{y^2}{(b\sqrt{2})^2} = 1.$$

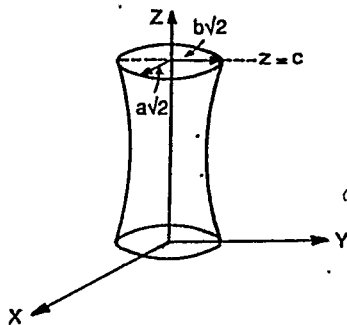


Fig. 108

(f) Elliptic cone :

The general equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0. \quad \text{Cross-}$$

sections perpendicular to x - and y -axes produce lines whereas sections perpendicular to z -axis gives ellipse (fig. 109). It shows cross-section at $z = c$.

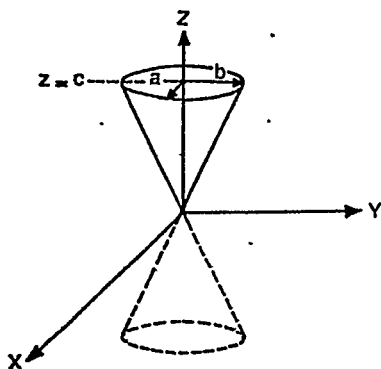


Fig. 109

(g) Hyperboloid of two sheets : The general equation is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

Cross-sections perpendicular to y and z -axes are hyperbolas and that perp. to x -axis is an ellipse (real or imaginary). The surface consists of two unconnected "pieces" or "sheets" and so the name "two sheets" (fig. 110).

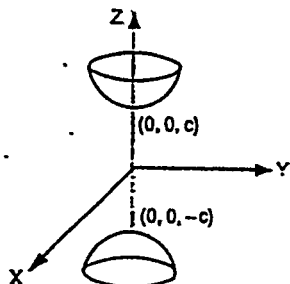


Fig. 110

(h) Hyperbolic paraboloid : The general equation is $\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}$.

Cross-sections perpendicular to z -axis is hyperbola and those to x - and y -axes are parabolas (fig. 111).

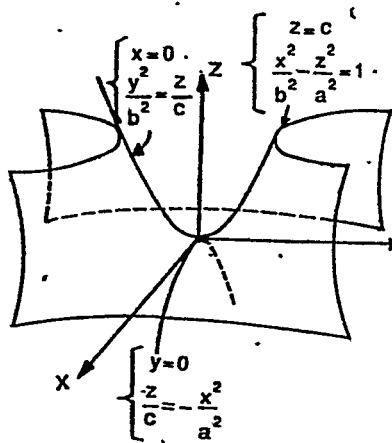


Fig. 111

(i) Elliptic paraboloid : The

general equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$.

Cross-sections perpendicular to x - and y -axes are parabolas and that to z -axis is ellipse (fig. 112). N.B. other forms of hyperbolic and elliptic paraboloid are

$$\frac{z^2}{c^2} - \frac{x^2}{a^2} = \frac{y}{b}, \quad \frac{y^2}{b^2} - \frac{z^2}{c^2} = \frac{x}{a}$$

and $\frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{x}{a}, \quad \frac{z^2}{c^2} + \frac{x^2}{a^2} = \frac{y}{b}$ respectively.

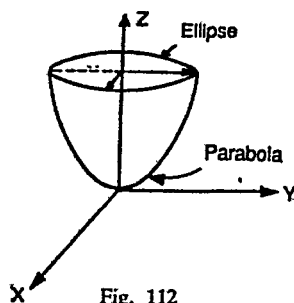


Fig. 112

Examples

1. Find the volume of the solid bounded by the elliptic paraboloid $4z = 16 - 4x^2 - y^2$ and the xy -plane.

Let the projection of the elliptic paraboloid on xy -plane be the curve AB . So, base area is AOB in (fig. 113). Here elementary area can be taken in both ways. Let it be $dA = dy dx$. Also the equation of the base is $\frac{x^2}{4} + \frac{y^2}{16} = 1$ which is an ellipse and is symmetrical about both x - and y -axes. So, the required volume is

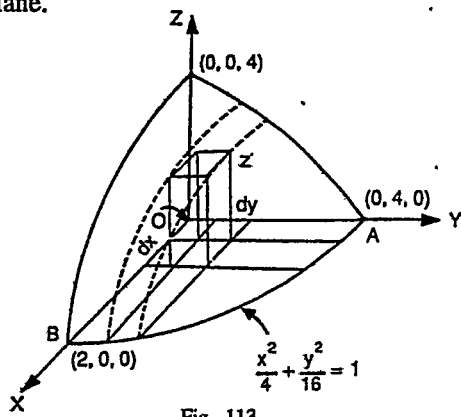


Fig. 113

$$V = 4 \int_0^2 \int_0^{2\sqrt{4-x^2}} \left(4 - x^2 - \frac{1}{4}y^2 \right) dy dx = 16\pi \text{ cubic units.}$$

2. Find the volume common to the elliptic paraboloid $y^2 + z^2 = 4ax$ and cylinder $x^2 + y^2 = 2ax$.

The circular cylinder $x^2 + y^2 = 2ax$ has centre $(a, 0, 0)$ and radius $= a$. It cuts the paraboloid thoroughly. So, volume common to them lies above and below xy and xz -planes. Since the cylinder is symmetrical about x -axis only, the volume is divided into 4 equal parts. The base of the volume above xy -plane is the semicircle ABO (fig. 114) of the circle $(x-a)^2 + y^2 = a^2$ and height is the height z of the surface $z = \sqrt{4ax - y^2}$. Therefore, the required volume

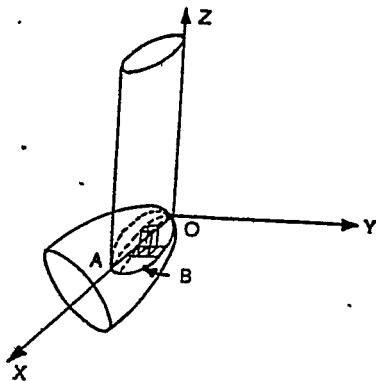


Fig. 114

$$\begin{aligned}
 V &= 4 \int_0^{2a} \int_0^{\sqrt{2ax-x^2}} \sqrt{4ax-y^2} \, dy \, dx \\
 &= 4 \int_0^{2a} \left[\frac{4ax}{2} \sin^{-1} \frac{y}{\sqrt{2ax}} + \frac{1}{2} y \sqrt{4ax-y^2} \right]_0^{\sqrt{2ax-x^2}} dx \\
 &= 4 \int_0^{2a} \left[2ax \sin^{-1} \frac{\sqrt{2ax-x^2}}{\sqrt{2ax}} + \frac{1}{2} \sqrt{2ax-x^2} \sqrt{2ax+x^2} \right] dx \\
 &= 16a^3 \int_0^{\pi/2} [\phi \cos \phi + \sin \phi \cos \phi] \sin \phi \, d\phi \quad \text{when } x = 2a \cos \phi \\
 &= 2(\pi + 8/3) a^3 \text{ cubic units.}
 \end{aligned}$$

3. Find the volume bounded by the plane $2x+y+z=4$, the cylinder $x^2+y^2=1$ and the xy -plane, lying in the first octant.

The volume of cylinder stands on the circular base OAB (fig. 115) whose equation is $x^2+y^2=1$. The required volume is

$$\begin{aligned}
 V &= \int_0^1 \int_0^{\sqrt{1-x^2}} (4-2x-y) \, dy \, dx \\
 &= (\pi - 1) \text{ cubic units.}
 \end{aligned}$$

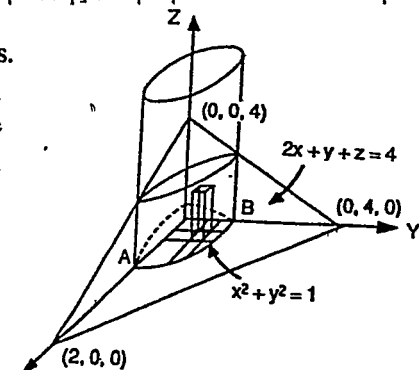


Fig. 115

Exercises 16 (6)

1. Find the volume of the solid bounded by the planes $y=z$, $3x+2y=12$, $z=0$, $x=0$.
2. Find the volume under the parabolic cylinder $z=4-x^2$, above the plane $z=0$ and within the cylinder $y^2=4x$.
3. Find the volume under the plane $x+z=2$, above the plane $z=0$ and within the cylinder $x^2+y^2=4$.
4. Determine the volume bounded by the co-ordinate planes and the plane
$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$
5. Determine the volume enclosed by the surface $z=2xy$, the cylinder $y=\frac{6}{x^2+3}$ and the planes $x=3$, $z=0$, $x=0$.
6. Determine the volume, lying in the first octant, within the cylinder $x^2+y^2=4$ and bounded by the planes $z=4-x-y$, $x=0$, $y=0$, $z=0$.
7. Find the volume within the cylinder $x^2+y^2=a^2$ and bounded above by $y^2=a^2-az$ and below by $z=0$.
8. Find the volume of the elliptic paraboloid $z=1-\frac{1}{4}x^2-\frac{1}{9}y^2$ above the plane $z=0$.
9. Find the volume bounded by the cylindrical surface $x^2+az=a^2$ and the planes $x+y=a$, $y=0$, $z=0$.
10. Find the volume below the cylindrical surface $y^2=a^2-az$, above $z=0$ and within the cylindrical surface $x^2+y^2=ax$.
11. Determine the volume of the solid under the plane surface $z=2x+a$, above the plane $z=0$ and within the cylindrical surface $x^2+y^2=2ax$.
12. A solid is bounded by the paraboloid $y^2=az-x^2$, the cylindrical surface $y^2=a^2-ax$ and the planes $x=0$ and $z=0$. Find the volume.
13. Find the volume within the cylinder $x^2+y^2=2x$ under the surface $4z=16-4x^2-y^2$ and above the plane $z=0$.
14. Find the volumes of the solids enclosed by
 - (i) $x^{2/3}+y^{2/3}+z^{2/3}=a^{2/3}$ and the co-ordinate planes,
 - (ii) $\sqrt{x}+\sqrt{y}+\sqrt{z}=1$ and the co-ordinate planes.
15. Find the volume of the solid bounded by the paraboloid of revolution $x^2+y^2=az$, the XOY plane and the cylinder $x^2+y^2=2ax$.
16. Find the volume bounded by the paraboloid $z=x^2+y^2$, the planes $z=0$,

$$x = 1, x + y = 3 \text{ and } y = 1.$$

Answers

1. 24 c. units; 2. 17.24 c. units; 3. 8π c. units; 4. $\frac{abc}{6}$ c. units; 5. $\frac{9}{2}$ c. units; 6. $\left(4\pi - \frac{16}{3}\right)$ c. units; 7. $\frac{3}{4}\pi a^3$ c. units; 8. 3π c. units; 9. $\frac{4}{3}a^3$ c. units; 10. $\frac{15}{64}a\pi^3$ c. units; 11. $3\pi a^3$ c. units; 12. $\frac{4}{7}a^3$ c. units; 13. $\frac{43}{16}\pi$ c. units; 14. (i) $\frac{4}{35}\pi a^3$ c. units; and (ii) $\frac{1}{90}$ c. units; 15. $\frac{3}{2}\pi a^3$ c. units; 16. $\frac{11}{6}$ c. units.

16.3.7.2 VOLUME UNDER A SURFACE BY DOUBLE INTEGRATION : (CYLINDRICAL CO-ORDINATES)

Three dimensional generalizations of polar co-ordinates, (r, θ) are *cylindrical* and *spherical* co-ordinates. So they are sometimes called cylindrical polar and spherical polar instead of simply cylindrical and spherical co-ordinates. Here discussion will be confined to cylindrical co-ordinates only.

The polar co-ordinates (r, θ) is extended to 3-dimension only by introducing the third co-ordinate z and keeping the old polar (r, θ) as it is. Therefore, the cylindrical co-ordinates of any point is (r, θ, z) . This is obtained by the intersection of (i) a right circular cylinder, $r = c$ with OZ as axis of symmetry and c as radius (ii) the plane $\theta = c_1$ making an angle θ with xz plane and containing z -axis and (iii) the plane $z = c_2$ parallel to xy -plane (fig. 116).

Let $P(r, \theta, z)$ be the point $P(x, y, z)$. So, relationship between the cartesian to cylindrical is

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$z = z$$

and the jacobian of transformation is

$$J \begin{pmatrix} x, y, z \\ r, \theta, z \end{pmatrix} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \cos \theta & -r \sin \theta & 0 \\ \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r \neq 0$$

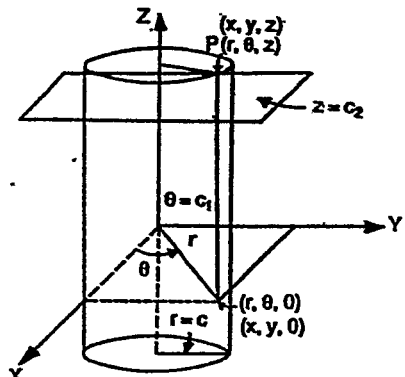


Fig. 116

$$\therefore dA = |J| dr d\theta = r dr d\theta \text{ and } dV = |J| z dA = r z dr d\theta \quad (1)$$

is the elementary area and volume respectively of the region R enclosed by $r = f_1(\theta)$ and $r = f_2(\theta)$ and $\theta = \alpha$ and $\theta = \beta$ and of the solid standing on R under the surface $z = f(r, \theta)$. Therefore volume,

$$V = \iint_R z dA = \int_{\alpha}^{\beta} \int_{f_1(\theta)}^{f_2(\theta)} z r dr d\theta \quad (2)$$

This co-ordinate system is useful to problems involving cylindrical surfaces with axis of symmetry.

Examples

1. Find the volume of the cylinder $x^2 + y^2 = 1$ above $z = 0$ and under the surface $2x + y + z = 4$ using cylindrical co-ordinates.

Equations of the surface and the cylinder in cylindrical co-ordinates are

$z = 4 - 2r \cos \theta - r \sin \theta$ and $r = 1$ respectively.

Note that the cutting face of the cylinder (fig. 117) by the plane is not symmetrical while the base of the cylinder on xy plane is symmetrical about x and y -axes. So, the volume created is not symmetrically divided into 4 parts above $z = 0$. Let us first compute the volume in the first octant between $\theta = 0$ to $\pi/2$. It is

$$\begin{aligned} V_1 &= \int_0^{\pi/2} \int_0^1 z r dr d\theta = \int_0^{\pi/2} \int_0^1 (4 - 2r \cos \theta - r \sin \theta) r dr d\theta \\ &= \int_0^{\pi/2} \left(2 - \frac{2}{3} \cos \theta - \frac{1}{3} \sin \theta \right) d\theta \end{aligned} \quad (1)$$

$$\text{or } V_1 = (\pi - 1) \text{ c. units} \quad (2)$$

Also volume in other quadrants from $\pi/2$ to π , π to $\frac{3\pi}{2}$ and $\frac{3\pi}{2}$ to 2π can be obtained by changing only the limits in (1) which are

$$V_2 = \int_{\pi/2}^{\pi} \left(2 - \frac{2}{3} \cos \theta - \frac{1}{3} \sin \theta \right) d\theta = \left(\pi + \frac{1}{3} \right) \text{ c. units} \quad (3)$$

$$V_3 = \int_{\pi}^{3\pi/2} \left(2 - \frac{2}{3} \cos \theta - \frac{1}{3} \sin \theta \right) d\theta = (\pi + 1) \text{ c. units} \quad (4)$$

$$\text{and } V_4 = \int_{3\pi/2}^{2\pi} \left(2 - \frac{2}{3} \cos \theta - \frac{1}{3} \sin \theta \right) d\theta = \left(\pi - \frac{1}{3} \right) \text{ c. units} \quad (5)$$

$\therefore$ adding (2), (3), (4) and (5), the required volume is $V = 4\pi$ c. units.

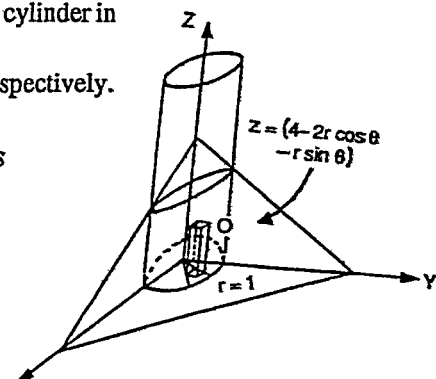


Fig. 117

2. Find the volume of the solid standing on a loop of $r = a \cos 2\theta$ between $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ and under the plane $z = x$.

The base of the solid and the top of the solid are shown in fig. 118 only. The equation of the plane in cylindrical co-ordinates is $z = r \cos \theta$. The volume is symmetrical in xz plane as both base and top are symmetrical about x and z -axes respectively. Therefore the required volume

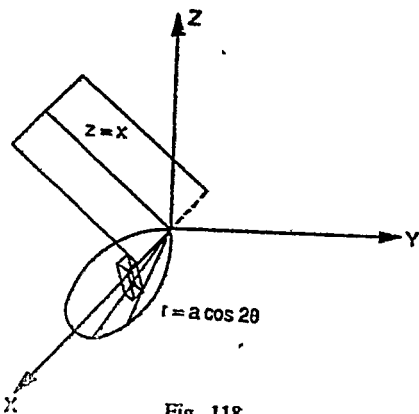


Fig. 118

$$\begin{aligned}
 V &= \int_{-\pi/4}^{\pi/4} \int_0^{a \cos 2\theta} z r dr d\theta \\
 &= \int_{-\pi/4}^{\pi/4} \int_0^{a \cos 2\theta} r^2 \cos \theta dr d\theta = 2 \int_0^{\pi/4} \int_0^{a \cos 2\theta} r^2 \cos \theta dr d\theta \\
 &= \frac{2}{3} \int_0^{\pi/4} a^3 \cos^3 2\theta \cos \theta d\theta = \frac{16a^3 \sqrt{2}}{105} \text{ c. units.}
 \end{aligned}$$

Exercise 16 (7)

1. Show that for the integral

$$\int_0^h \int_{-z \tan \alpha}^{z \tan \alpha} \int_{-\sqrt{z^2 \tan^2 \alpha}}^{\sqrt{z^2 \tan^2 \alpha}} (x^2 + y^2) dx dy dz$$

the region of integration is the region enclosed by a certain right circular cone. Express the integral in terms of cylindrical co-ordinates and hence evaluate the integral.

Find the following volumes of the solid using cylindrical co-ordinates :

- Volume of the solid enclosed by the ellipsoid $x^2 + y^2 + 4z^2 = 36$ and the co-ordinate planes.
- Volume enclosed by the paraboloid $x^2 + y^2 - 6z = 0$, the cylinder $x^2 + y^2 = 36$ and the xy plane.
- Volume bounded by the hyperboloid $x^2 + y^2 - z^2 = 16$ and the cylinder $x^2 + y^2 = 25$.
- Volume outside the conicoid $x^2 + y^2 + z^2 = 16$ inside the cylinder and in the first octant.
- Volume included between the ellipsoid $x^2 + y^2 + z^2 = a^2$ and the cylinder $r = a \sin \theta$.

2. Find the volume of the solid standing on a loop of $r = a \cos 2\theta$ between $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ and under the plane $z = x$.

The base of the solid and the top of the solid are shown in fig. 118 only. The equation of the plane in cylindrical co-ordinates is $z = r \cos \theta$. The volume is symmetrical in xz plane as both base and top are symmetrical about x and z -axes respectively. Therefore the required volume

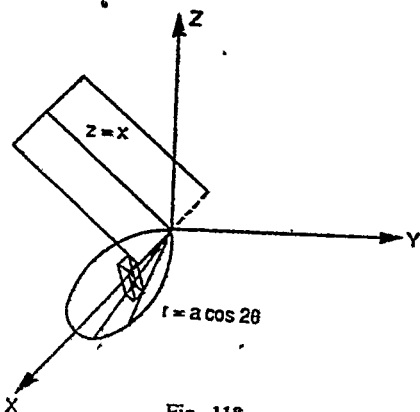


Fig. 118

$$\begin{aligned}
 V &= \int_{-\pi/4}^{\pi/4} \int_0^{a \cos 2\theta} z r dr d\theta \\
 &= \int_{-\pi/4}^{\pi/4} \int_0^{a \cos 2\theta} r^2 \cos \theta dr d\theta = 2 \int_0^{\pi/4} \int_0^{a \cos 2\theta} r^2 \cos \theta dr d\theta \\
 &= \frac{2}{3} \int_0^{\pi/4} a^3 \cos^3 2\theta \cos \theta d\theta = \frac{16a^3 \sqrt{2}}{105} \text{ c. units.}
 \end{aligned}$$

Exercises 16 (7)

1. Show that for the integral

$$\int_0^h \int_{-z \tan \alpha}^{z \tan \alpha} \int_{-\sqrt{(z^2 \tan^2 \alpha - y^2)}}^{\sqrt{(z^2 \tan^2 \alpha - y^2)}} (x^2 + y^2) dx dy dz$$

the region of integration is the region enclosed by a certain right circular cone. Express the integral in terms of cylindrical co-ordinates and hence evaluate the integral.

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- Volume outside the conicoid $z = 2$, and inside the cylinder and in the first octant.
- Volume included between the sphere $x^2 + y^2 + z^2 = a^2$ and the cylinder $r = a \sin \theta$.

2. Find the volume of the solid standing on a loop of $r = a \cos 2\theta$ between $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ and under the plane $z = x$.

The base of the solid and the top of the solid are shown in fig. 118 only. The equation of the plane in cylindrical co-ordinates is $z = r \cos \theta$. The volume is symmetrical in xz plane as both base and top are symmetrical about x and z -axes respectively. Therefore the required volume

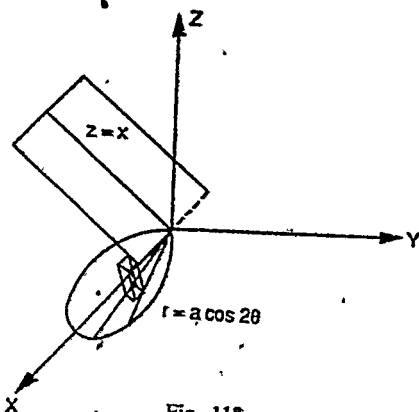


Fig. 118

$$\begin{aligned}
 V &= \int_{-\pi/4}^{\pi/4} \int_0^{a \cos 2\theta} z r dr d\theta \\
 &= \int_{-\pi/4}^{\pi/4} \int_0^{a \cos 2\theta} r^2 \cos \theta dr d\theta = 2 \int_0^{\pi/4} \int_0^{a \cos 2\theta} r^2 \cos \theta dr d\theta \\
 &= \frac{2}{3} \int_0^{\pi/4} a^3 \cos^3 2\theta \cos \theta d\theta = \frac{16a^3 \sqrt{2}}{105} \text{ c. units.}
 \end{aligned}$$

Exercises 16 (7)

1. Show that for the integral

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- Volume bounded by the hyperboloid $x^2 + y^2 - z^2 = 16$ and the cylinder $x^2 + y^2 = 25$.
- Volume outside the conicoid $z = 2r$ and inside the cylinder and in the first octant.
- Volume included between the sphere $x^2 + y^2 + z^2 = a^2$ and the cylinder $r = a \sin \theta$.

7. Volume bounded by $z=0$, cylinder $x^2+y^2=1$ and the plane $x+y+z=3$.
8. Volume bounded by $x^2+y^2-2ax=0$, $z=0$, $x^2+y^2=z^2$.
9. Find the volume under the plane $x+z=2$, above $z=0$ and within the cylinder $x^2+y^2=4$.
10. Find the volume inside the sphere $r^2+z^2=16$ and also inside the cylinder $r=2$.
11. The problems of Exercises 16 (6).
12. The base of a pile of sand covers the region in the xy plane that is bounded by the parabola $x^2+y=6$ and the line $y=x$. The depth of the sand above the point (x, y) is x^2 . Sketch the base of the sand pile and a representative element of volume ΔV , and find the volume of the sand in the pile by double integration.

Answers

1. $\int_0^k \int_0^{2\pi} \int_0^{\tan \alpha} p^3 dp d\phi dz = \left(\frac{1}{10}\right) \pi k^5 \tan^4 \alpha$; 2. 18π c. units; 3. 108π c. units;
5. 96 c. units; 6. $\frac{2}{9}a^2(3\pi-4)$ c. units; 7. 3π c. units; 8. $\frac{32}{9}a^3$ c. units; 9. 8π c. units;
10. $\frac{32\pi}{3}(8-3\sqrt{3})$ c. units.

16.3.8. AREA OF CURVED SURFACES (CARTESIAN CO-ORDINATES) BY DOUBLE INTEGRATION

The volume under a 3-dimensional surface has been discussed in 16.3.7 and general characteristics of some common surfaces of specific nature have been reviewed in 16.3.7.1. Here area of curved surfaces which is pre-requisite for the understanding and problem solving of surface integrals, (Ref. Vector Analysis with Applications by A.K. Hazra & M.A. Ashraf), not a subject matter of this book, will be discussed.

The general equation of a surface is $z=f(x, y)$ explicitly and $F(x, y, z)=0$ implicitly. Here explicit equation will be considered for general discussion. The equation shows that z is a function of two independent variables x and y . It may be interpreted as the height of a point (x, y) from the plane $z=0$ as is considered in 16.3.7.

To find this area of a curved surface, we need to project the surface to a co-ordinate plane which determines limits of the plane area in 2-dimension as in 16.3.2 and hence the surface area is expressed in terms of this projected area. So, a thorough understanding of Secs. 16.3.2 and 16.3.2.1 are prerequisite for this section.

Let the equation of surface S be $z = f(x, y)$ and its projection on the xy plane be the region R of area A bounded by $y = g(x)$, $y = f(x)$, $x = a$ and $x = b$ (fig. 119). Let an elementary area of A be $\Delta A = \Delta x \Delta y$ which contains a point $Q(x, y, 0)$ which is the projection of $P(x, y, z)$ of S . Let a tangent plane to S at P be constructed and let its area be ΔS . This tangent plane makes an angle γ with xy -plane, say. Then ΔA must be equal the product of ΔS and the cosine of the angle γ between them *i.e.*

$$\Delta A = \Delta S \cos \gamma \quad (1)$$

$$\therefore \frac{(\text{projected area})}{(\text{original area})} = (\cosine \text{ of the angle between them})$$

Also (angle between two planes) = (angle between their normals) = (sum of the product of the direction cosines ($d.cs$) of the two normals).

Now normal to the xy -plane is the z -axis whose $d.cs$ are 0, 0, 1 and direction ratios ($d.rs$) of the normal to a part of the surface $z = f(x, y)$ are the partial derivatives in the respective directions *i.e.* $f_x, f_y, 1$ are the $d.rs$ of the normal to ΔS in the outward +ve directions. Therefore,

$$\cos \gamma = \frac{1}{\sqrt{f_x^2 + f_y^2 + 1}},$$

[from analytic geometry or dot product of vectors]

$$= \frac{1}{\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}} \quad (2)$$

$$\therefore \text{from (1), } \Delta S = \Delta A \cos \gamma$$

$$= \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \Delta A$$

$$\text{In the limit when } \Delta S \rightarrow 0, dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dA$$

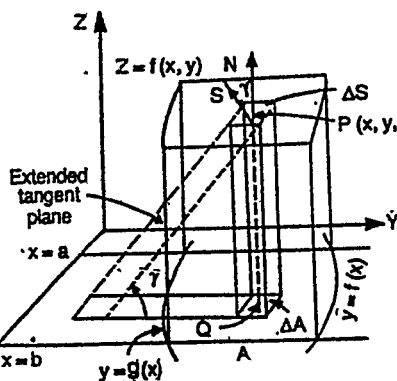


Fig. 119

Now integrating,

$$S = \int_a^b \int_{g(x)}^{f(x)} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dy dx \quad (3)$$

or it may be,
$$S = \int_c^d \int_{\phi(y)}^{\psi(y)} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dx dy \quad (4)$$

when ΔA is bounded by $x = \phi(y)$, $x = \psi(y)$, $y = c$ and $y = d$. If S is projected to either xz or yz planes, then either

$$S = \iint_{R_{xz}} \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2} dA_{xz} \quad (5)$$

or

$$S = \iint_{R_{yz}} \sqrt{1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2} dA_{yz} \quad (6)$$

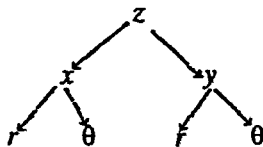
the suffixes indicates the plane to which S is projected.

16.3.8.1 AREA OF CURVED SURFACES IN CYLINDRICAL CO-ORDINATES BY DOUBLE INTEGRATION

The relation between cartesian co-ordinates and cylindrical co-ordinates is $x = r \cos \theta$, $y = r \sin \theta$, $z = z$.

From 16.3.7.2, $dA = r dr d\theta$ (1)

Also, the functional relationship is



$\therefore$ using chain rule of partial differentiation,

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial r} = \frac{\partial z}{\partial x} \cos \theta + \frac{\partial z}{\partial y} \sin \theta$$

and
$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial \theta} = -\frac{\partial z}{\partial x} r \sin \theta + \frac{\partial z}{\partial y} r \cos \theta$$

Squaring and adding,

$$\left(\frac{\partial z}{\partial r}\right)^2 + \left(\frac{1}{r} \frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 \quad (2)$$

$\therefore$ equation (3) of 16.3.8 becomes, on using (1) and (2) above,

$$S = \iint \sqrt{1 + \left(\frac{\partial z}{\partial r}\right)^2 + \left(\frac{1}{r} \frac{\partial z}{\partial \theta}\right)^2} r dr d\theta \quad (3)$$

Similarly, (5) and (6) of sec. 16.3.8 become

$$S = \iint \sqrt{1 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{1}{r} \frac{\partial y}{\partial \theta}\right)^2} r dr d\theta \quad (4)$$

$$\text{and } S = \iint \sqrt{1 + \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{1}{r} \frac{\partial x}{\partial \theta}\right)^2} r dr d\theta \quad (5)$$

where limits of r and θ will be according to the projected area, A in (3), (4) and (5).

Examples

1. Find surface area of the cylinder $z = 4 - x^2$ intersected by the plane $y = x$ and lies in the first octant.

The plane $y = x$ cuts the inverted parabolic cylinder along a parabolic curve $z = 4 - y^2$ (fig. 120). So, the projection of the required surface on the xy -plane is the area AOB bounded by $x=2$, $y=0$ and $y=x$. Now differentiating $z = 4 - x^2$ partially w.r.t. x & y , $\frac{\partial z}{\partial x} = -2x$, $\frac{\partial z}{\partial y} = 0$.

$\therefore \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1 + 4x^2}$. Hence, the required surface area,

$$\begin{aligned} S &= \int_0^2 \int_0^x \sqrt{1 + 4x^2} dy dx = \int_0^2 x \sqrt{1 + 4x^2} dx \\ &= \frac{1}{12} (17\sqrt{17} - 1) \text{ sq. units.} \end{aligned}$$

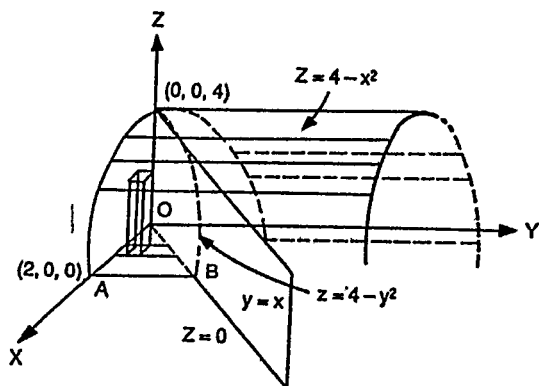


Fig. 120

2. Find the surface area of the sphere $x^2 + y^2 + z^2 = r^2$.

The projection of the sphere on the xy plane is bounded by $x=0$, $y=0$ and $x^2 + y^2 = r^2$ (fig. 121). Also, the partial derivatives of

$x^2 + y^2 + z^2 = a^2$ w.r.t. x and y are

$$\frac{\partial z}{\partial x} = -\frac{x}{z}, \quad \frac{\partial z}{\partial y} = -\frac{y}{z}.$$

$$\begin{aligned} \therefore 1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 \\ = 1 + \frac{x^2}{z^2} + \frac{y^2}{z^2} = \frac{r^2}{r^2 - x^2 - y^2} \end{aligned}$$

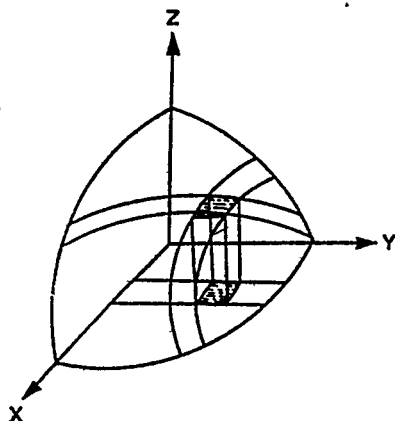


Fig. 121

$\therefore$ the required surface area,

$$\begin{aligned} S &= 8 \int_0^r \int_0^{\sqrt{r^2 - x^2}} \frac{r}{\sqrt{r^2 - x^2 - y^2}} dy dx \\ &= 4\pi r^2 \text{ sq. units.} \end{aligned}$$

Exercises 16 (8)

- Find the surface area of the portion of the cylinder $x^2 + z^2 = a^2$ included between the planes $y = 0$ and $y = x$ and lying in the first octant.
- Find the area of that portion of the sphere $x^2 + y^2 + z^2 = 2ay$ cut out by one nappe of the cone $x^2 + z^2 = y^2$.
- Find the surface of the cylinder $x^2 + y^2 = r^2$ included between the plane $z = mx$ and the XOY plane.
- Find the surface area of the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ intercepted by the co-ordinate planes.

5. Find the surface area of the portion of the sphere $x^2 + y^2 + z^2 = 2ay$ which lies within the paraboloid $by = x^2 + z^2$.
6. Find the surface area of the paraboloid $y^2 + z^2 = 4ax$ intercepted by the parabolic cylinder $y^2 = ax$ and the plane $x = 3a$.
7. Find the area of that portion of the surface of the cylinder $y^{2/3} + z^{2/3} = a^{2/3}$ bounded by a curve whose projection on the xy plane is $x^{2/3} + y^{2/3} = a^{2/3}$.
8. Find the area of that portion of the sphere $x^2 + y^2 + z^2 = 2a^2$ that is cut out by the upper nappe of the cone $x^2 + y^2 = z^2$.
9. Compute the area of that part of the plane $x + y + z = 2a$, which lies in the first octant and is bounded by the cylinder $x^2 + y^2 = a^2$.
10. Find the surface area of the sphere $x^2 + y^2 + z^2 = a^2$ cut out by the cylinder $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b$.
11. Find the surface area of a solid that is the common part of two cylinders $x^2 + y^2 = a^2, y^2 + z^2 = a^2$.
12. Compute the area of that part of the surface of the paraboloid $y^2 + z^2 = 2ax$, which lies between the parabolic cylinder $y^2 = ax$ and the plane $x = a$.
13. Find the portion of the surface $z = xy$ inside the cylinder $x^2 + y^2 = 1$.
[Hints : Change to polar co-ordinates]
14. Find the portion of the surface of the paraboloid $y^2 + z^2 = 4ax$ above the region bounded by $y^2 = ax$ and $x = 3a$.

Answers

1. a^2 sq. units 2. $2\pi a^2$ sq. units 3. $4r^2 m$ sq. units
4. $\left(\frac{1}{2}\sqrt{b^2c^2 + c^2a^2 + a^2b^2}\right)$ sq. units 5. $2\pi ab$ sq. units 6. $\frac{56}{9}\pi a^2$ sq. units
7. $\frac{12}{5}a^2$ sq. units 8. $2\pi a^2(2 - \sqrt{2})$ sq. units 9. $\frac{\pi a^2}{4}\sqrt{3}$ sq. units
10. $\left[4\pi a^2 - 8a^2 - \sin^{-1}(\sqrt{a^2 - b^2}/a)\right]$ sq. units 11. $16a^2$ sq. units
12. $\frac{1}{3}\pi a^2(3\sqrt{3} - 1)$ sq. units 13. $(2\sqrt{2} - 1)\frac{2\pi}{3}$ sq. units 14. $\frac{28\pi a^2}{9}$ sq. units.

164 TRIPLE INTEGRAL AND ITERATED TRIPLE INTEGRALS

Triple integral : A triple integral is another extension of the single integral. In this case, the number of variables is three and the corresponding function is defined in a 3-dimensional space V of these three variables. Let $f(x, y, z)$ be a function defined in V . Let V be subdivided in any manner into n sub-spaces as in the cases of one or two variables. Let $P_i(x_i, y_i, z_i)$ be an arbitrary point in ΔV_i and the sum of the products be $\sum_{i=1}^n f(x_i, y_i, z_i) \Delta V_i$. As

$n \rightarrow \infty$ i.e. $\Delta V_i \rightarrow 0$, if $\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i, y_i, z_i) \Delta V_i$ exists, then this limiting value is called the triple integral of $f(x, y, z)$ w.r.t. dV and is denoted either by the symbol $\iiint_V f(x, y, z) dV$ or by the symbol $\int_V f(x, y, z) dV$ (1)

Triple iterated integrals : As in the case of double integral, triple integral is computed or evaluated by using triple iterated integrals. Here we have to integrate from innermost to uttermost.

Let $S : f(x, y, z) = 0$ be a closed surface with enclosing volume V . Let every line parallel to z -axis cuts the surface at most at two points. Reconstruction of S may be necessary to satisfy this assumption. Let the entire region with volume V be projected to xy plane into a regular region R bounded by $y = f_1(x)$, $y = f_2(x)$, $x = a$ and $x = b$. Let the surface, S be bounded below by $z = \phi_1(x, y)$ and above by $z = \phi_2(x, y)$ (fig. 122). Let S be divided into a set of parallelepipeds whose faces are parallel to the co-ordinate planes. Then the volume of a typical i th cell

$\Delta V_i = \Delta x_i \Delta y_j \Delta z_k$, $i = 1, 2, \dots, m$; $j = 1, 2, \dots, n$; $k = 1, 2, \dots, p$ (say). Hence

$$\iiint_V f(x, y, z) dV = \lim_{m, n, p \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^p f(x_i, y_j, z_k) \Delta x_i \Delta y_j \Delta z_k$$

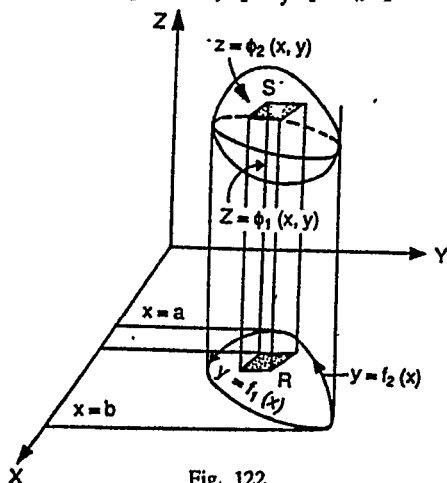


Fig. 122

Let the limits be considered separately. Then

$$\begin{aligned}\iiint_V f(x, y, z) dV &= \lim_{m, n \rightarrow \infty} \left[\int_{z=\phi_1(x, y)}^{\phi_2(x, y)} f(x_i, y_j, z) dz \right] \Delta y_j \Delta x_i \\ &= \lim_{m \rightarrow \infty} \left[\int_{y=f_1(x)}^{f_2(x)} \int_{z=\phi_1(x, y)}^{\phi_2(x, y)} f(x_i, y, z) dz dy \right] \Delta x_i \\ &= \int_{x=a}^b \int_{y=f_1(x)}^{f_2(x)} \int_{z=\phi_1(x, y)}^{\phi_2(x, y)} f(x, y, z) dz dy dx \quad (2)\end{aligned}$$

The order of integration may be changed depending on the nature of curves enclosing region R and also the nature of projection of V within S . For example if S enclosing V be projected either to xz or to yz planes, the base area will be changed along with the height of the parallelopiped constructed. So, there are 6 possible forms of iterated integrals, since dx, dy, dz can be arranged in 3 ways and with one fixed other two can be arranged in 2 ways giving a total of 6.

16.4.1 CHANGE OF VARIABLES IN TRIPLE INTEGRAL

The process of computation in changing the variables in a double as well as in a triple integral remains analogous. Only difference arise in their dimensions. Here an elementary cell becomes a parallelopiped instead of a parallelogram in double integral. In vector notation, here scalar triple product or box product gives the volume whereas scalar double product gave area in double integration.

Let $\iiint_V F(x, y, z) dV$ be a triple integral in the xyz space. Its equivalent form is to be determined under the transformation or substitution $x = f(u, v, w)$, $y = g(u, v, w)$ and $z = h(u, v, w)$. Let the transformation be (1-1) and vice-versa. Let, due to subdivision, the four vertices of a parallelopiped like cell in uvw system be $B'(u, v, w)$, $D'(u + du, v, w)$, $C'(u, v + dv, w)$ and $E'(u, v, w + dw)$. So, the corresponding vertices of the region in xyz system are $B(x, y, z)$

$$\begin{aligned}&C(f(u, v + dv, w), g(u, v + dv, w), h(u, v + dv, w)) \\ &= C\left(x + \frac{\partial f}{\partial v} dv, y + \frac{\partial g}{\partial v} dv, z + \frac{\partial h}{\partial v} dv\right), \text{ neglecting higher powers in}\end{aligned}$$

Taylor series,

$$D\left(x + \frac{\partial f}{\partial u} du, y + \frac{\partial g}{\partial u} du, z + \frac{\partial h}{\partial u} du\right)$$

$$\text{and } E\left(x + \frac{\partial f}{\partial w} dw, y + \frac{\partial g}{\partial w} dw, z + \frac{\partial h}{\partial w} dw\right) \quad (\text{fig. 123 a, b})$$

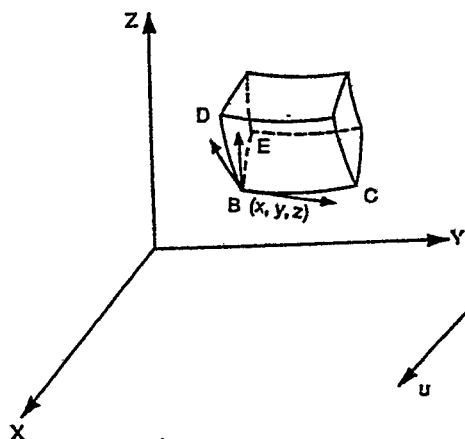


Fig. 123 (a)

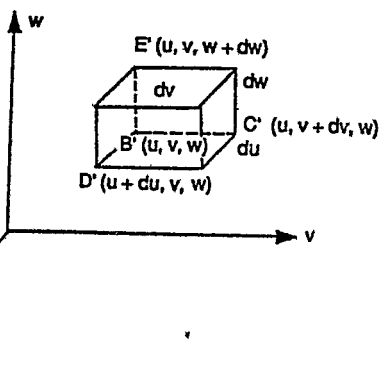


Fig. 123 (b)

$\therefore$ the tangential vectors at BC , BD and BE are respectively given by (in component form)

$$\left(\frac{\partial f}{\partial v} dv, \frac{\partial g}{\partial v} dv, \frac{\partial h}{\partial v} dv \right), \left(\frac{\partial f}{\partial u} du, \frac{\partial g}{\partial u} du, \frac{\partial h}{\partial u} du \right)$$

and

$$\left(\frac{\partial f}{\partial w} dw, \frac{\partial g}{\partial w} dw, \frac{\partial h}{\partial w} dw \right)$$

Their scalar triple product gives the volume element dV i.e.

$$dV = \begin{vmatrix} \frac{\partial f}{\partial u} du & \frac{\partial g}{\partial u} du & \frac{\partial h}{\partial u} du \\ \frac{\partial f}{\partial v} dv & \frac{\partial g}{\partial v} dv & \frac{\partial h}{\partial v} dv \\ \frac{\partial f}{\partial w} dw & \frac{\partial g}{\partial w} dw & \frac{\partial h}{\partial w} dw \end{vmatrix} = \left| J \begin{pmatrix} f, g, h \\ u, v, w \end{pmatrix} \right| du dv dw \quad (1)$$

But $dV' = du dv dw$. Therefore,

$$dV = |J| dV' \\ \therefore \iiint_V F(x, y, z) dV = \iiint_{V'} F(f(u, v, w), g(u, v, w), h(u, v, w)) |J| dV' \quad (2)$$

16.42 TRIPLE INTEGRAL IN CYLINDRICAL AND SPHERICAL CO-ORDINATES

Cylindrical Co-ordinates : In 16.3.7.2, Jacobian of transformation was obtained as $|J| = r$. As the relation between cylindrical and cartesian co-ordinates is $x = r \cos \theta$, $y = r \sin \theta$ and $z = z$, so the triple integral (2) in

16.4.1 becomes

$$\begin{aligned}
 & \iiint_V F(x, y, z) dV \\
 &= \int_{\theta=\alpha}^{\beta} \int_{r=f_1(\theta)}^{f_2(\theta)} \int_{z=\phi_1(r, \theta)}^{\phi_2(r, \theta)} F(r \cos \theta, r \sin \theta, z) dz \cdot r dr d\theta \\
 &= \int_{\theta=\alpha}^{\beta} \int_{r=f_1(\theta)}^{f_2(\theta)} \int_{z=\phi_1(r, \theta)}^{\phi_2(r, \theta)} f(r, \theta, z) r dz dr d\theta \quad (1)
 \end{aligned}$$

where $z = \phi_1(r, \theta)$ and $z = \phi_2(r, \theta)$ are the lower and upper boundaries of the surface, S and its projection on $r\theta$ -plane is bounded by $r = f_1(\theta)$, $r = f_2(\theta)$ and $\theta = \alpha$ and $\theta = \beta$.

Spherical Co-ordinates : It is another generalization of the polar co-ordinates to 3-dimensions. The system is useful when there is symmetry w.r.t. a point which may be convenient to choose it as the origin. Normally, this is useful for cylindrical and spherical problems.

It is obtained by the intersection of (i) a sphere $r = \text{const.}$, (ii) a plane $\theta = \text{const.}$ and (iii) a half cone $\phi = \text{const.}$ (fig. 124). The coordinates are written as an ordered triple (r, θ, ϕ) where $r \geq 0$, $0 \leq \phi \leq \pi$ and $0 \leq \theta \leq 2\pi$ (fig. 124).

Superimposing the point (r, θ, ϕ) on (x, y, z) their relationship can be easily obtained as

$$x = r \cos \theta \sin \phi$$

$$y = r \sin \theta \sin \phi$$

$$z = r \cos \phi$$

$\therefore$ the jacobian of transformation is

$$J \begin{pmatrix} x, y, z \\ r, \theta, \phi \end{pmatrix} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \cos \theta \sin \phi & -r \sin \theta \sin \phi & r \cos \theta \cos \phi \\ \sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \cos \phi \\ \cos \phi & 0 & -r \sin \phi \end{vmatrix}$$

$$= -r^2 \sin \phi \neq 0$$

$\therefore dV = |J| d\phi dr d\theta$. So, equation (2) of 16.4.1 becomes

$$\iiint_V F(x, y, z) dV = \int_{\theta=\alpha}^{\beta} \int_{r=f_1(\theta)}^{f_2(\theta)} \int_{\phi=\phi_1(r, \theta)}^{\phi_2(r, \theta)} H(r, \theta, \phi) |J| d\phi dr d\theta$$

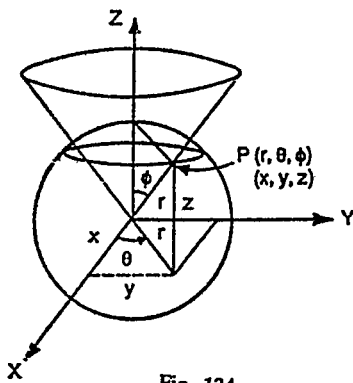


Fig. 124

$$= \int_{\theta=\alpha}^{\beta} \int_{r=f_1(\theta)}^{f_2(\theta)} \int_{\phi=\phi_1(r,\theta)}^{\phi_2(r,\theta)} H(r, \theta, \phi) r^2 \sin \phi \, d\phi \, dr \, d\theta \quad (2)$$

where lower and upper part of surface S have equation $\phi = \phi_1(r, \theta)$, $\phi = \phi_2(r, \theta)$ and the projection of S on $r\theta$ -plane is bounded by $r = f_1(\theta)$, $r = f_2(\theta)$, $\theta = \alpha$ and $\theta = \beta$.

Note that unlike polar co-ordinates, here r is non-negative.

16.4.3 VOLUME USING TRIPLE INTEGRALS

Volume of any solid = (area of the base) $\times$ height. So, elementary volume, dV in cartesian form = $dA \times dz = dz \, dA$. Hence using $f(x, y, z) = 1$ in equation (1) of 16.4, the volume in cartesian system is given by

$$V = \int_a^b \int_{f_1(x)}^{f_2(x)} \int_{\phi_1(x,y)}^{\phi_2(x,y)} dz \, dy \, dx \quad (1)$$

and five other equivalent forms of (1), similarly in cylindrical and spherical co-ordinates, the volume is given by equations (1) and (2) of 16.4.2 and equation (2) of 16.4.1 in general for other co-ordinate systems with $f = 1$ and also by their equivalent forms. Hence equations (2) of 16.4.1 & (1) and (2) of 16.4.2 for volume stand as

$$V = \iiint_{V'} |J| \, dV', \text{ (in general)} \quad (2)$$

$$V = \int_{\theta=\alpha}^{\beta} \int_{r=f_1(\theta)}^{f_2(\theta)} \int_{z=\phi_1(r,\theta)}^{\phi_2(r,\theta)} r \, dz \, dr \, d\theta, \text{ (in cylindrical)} \quad (3)$$

$$V = \int_{\theta=\alpha}^{\beta} \int_{r=f_1(\theta)}^{f_2(\theta)} \int_{\phi=\phi_1(r,\theta)}^{\phi_2(r,\theta)} r^2 \sin \phi \, d\phi \, dr \, d\theta, \text{ (in spherical)} \quad (4)$$

Examples

1. Find the volume of the solid bounded by the surface $z = x^2 + y^2$, the cylinders $y = 2\sqrt{x}$, $x = 2\sqrt{y}$ and the xy -plane using triple integral.

The cylinders $y = 2\sqrt{x}$ and $x = 2\sqrt{y}$ intersect at $(4, 4, 0)$ on the xy -plane. The volume stands on the area shown in fig. 125. Its height is from xy plane to the surface of the paraboloid $z = x^2 + y^2$.

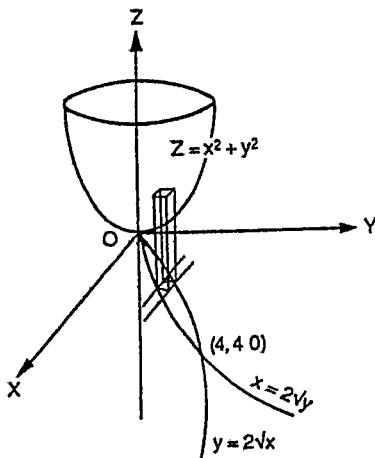


Fig. 125

Therefore, volume is given by

$$\begin{aligned}
 V &= \int_0^4 \int_{y=x^2/4}^{2\sqrt{x}} \int_0^{x^2+y^2} dz \, dy \, dx \\
 &= \int_0^4 \int_{x^2/4}^{2\sqrt{x}} (x^2+y^2) \, dy \, dx = 43 \frac{93}{105} \text{ c. unit.}
 \end{aligned}$$

2. Find the volume of the region common to both the sphere $x^2+y^2+z^2=4$ and the cylinder $x^2+y^2=1$ using both cylindrical and spherical co-ordinate systems.

Cylindrical co-ordinates : The equations of the sphere and the cylinder in cylindrical co-ordinates are $r^2+z^2=4$ and $r^2=1$ i.e. $z=\sqrt{4-r^2}$

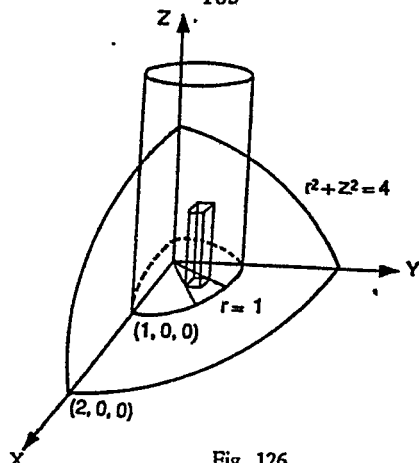


Fig. 126

and $r=1$ (fig. 126) respectively. Since the sphere is symmetric about all the 3-axes and cylinder is symmetrical to x - and y -axes i.e. about z -axis. So, the common volume is divided into 8 equal parts. Considering the first octant, the required volume is

$$\begin{aligned}
 V &= 8 \int_0^{\pi/2} \int_0^1 \int_{z=0}^{\sqrt{4-r^2}} dz \, dr \, d\theta \\
 &= 4\pi \left(\frac{8}{3} - \sqrt{3} \right) \text{ c. units.}
 \end{aligned}$$

This system is very convenient for this problem.

Spherical co-ordinates : The equation of the sphere and the cylinder in this system are $r^2=4$ and $r^2 \sin^2 \phi = 1$ respectively.

$\therefore r=2$ on the sphere and $r = \operatorname{cosec} \phi$ on the cylinder. Also at the point of intersection, $z^2=3$ or $r^2 \cos^2 \phi = 3$ or $r \cos \phi = \sqrt{3}$.

$$\therefore \cos \phi = \frac{\sqrt{3}}{2} \text{ w.r.t. sphere and } \tan \phi = \frac{1}{\sqrt{3}} \text{ w.r.t. cylinder}$$

$\therefore r=2$ and $r = \operatorname{cosec} \phi$ respectively.

$\therefore \phi = \pi/6$ in both cases. Figure shows that the cone $\phi = \frac{\pi}{6}$ divides the volume into two unequal parts — (i) in one ϕ varies from 0 to $\pi/6$ and (ii) in other

ϕ varies from $\pi/6$ to $\pi/2$ and r varies from 0 to $\operatorname{cosec} \phi$ (fig. 127). Therefore, the required volume is

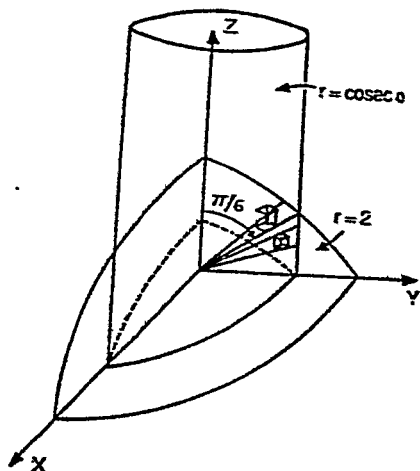


Fig. 127

$$\begin{aligned}
 V &= 8 \int_0^{\pi/2} \int_0^{\pi/6} \int_0^2 r^2 \sin \phi \, dr \, d\phi \, d\theta \\
 &+ 8 \int_0^{\pi/2} \int_{\pi/6}^{\pi/2} \int_0^{\operatorname{cosec} \phi} r^2 \sin \phi \, dr \, d\phi \, d\theta \\
 &= 4\pi \left(\frac{8}{3} - \sqrt{3} \right) \text{ c. units.}
 \end{aligned}$$

Exercises 16 (9)

Use Triple integrals to

1. find the volume bounded by the paraboloid $z = x^2 + y^2$, the cylinder $y = x^2$, the planes $y = x$ and $x = 0$ and lying in the first octant;
2. find the volume in the first octant bounded by the cylinder $z = \frac{8}{x^2 + 4}$ and the planes $y = x$, $x = 2$, $z = 0$;
3. find the volume of the solid bounded by the cylinders $z = 2/x^2$, $y = 2x - x^2$ and the planes $x = \frac{1}{2}$, $x = \frac{3}{2}$, $y = 0$, $z = 0$;
4. find the volume enclosed by the planes $x + y + z = 9$, $x + y - z = 0$, $x = 0$, $y = 0$;
5. find the volume of the solid bounded by the surface $z = x + y^2$, the cylinders $y = 2\sqrt{x}$, $x = 2\sqrt{y}$, and the plane xy and the volume lies in the first octant;

6. find the volume enclosed by $z = 4 - x^2$, $x = 0$, $y = 0$, $y = 6$, $z = 0$.

Use cylindrical and spherical co-ordinates for the following :

7. To find the volume above the xy -plane and enclosed by $x^2 + y^2 + z^2 = 25$ and $16z^2 = 9x^2 + 9y^2$.

8. Find the volume of the solid whose upper surface is on the sphere $x^2 + y^2 + z^2 = 8$ and whose lower surface is on the paraboloid of revolution $x^2 + y^2 = 2z$.

9. Find the volume of the solid bounded by the cylinder $x^2 + z = 4$, the planes $x + z = 2$, $y = 0$ and $y = 3$.

10. Find the volume of the solid bounded above by the cylinder $z = 4 - x^2$ and below by the elliptic paraboloid $z = 3x^2 + y^2$.

11. Find the volume below the plane $z = x$ and above the elliptic paraboloid $z = x^2 + y^2$.

12. Find the volume bounded by the sphere $r^2 + z^2 = a^2$ within the cylinder $r = a \cos \theta$.

13. A hole is bored through a sphere, the axis of the hole being a diameter of the sphere. The volume of the solid is given by

$$V = 2 \int_0^{2\pi} \int_0^{\sqrt{3}} \int_1^{\sqrt{4-z^2}} r \, dr \, dz \, d\theta.$$

(a) By inspecting the given integral, determine the radius of the hole and the radius of the sphere.

(b) Calculate the value of the integral.

Answers

- | | | | |
|-------------------------------|-----------------------|---------------------------------|--|
| 1. $3/35$ c. units | 2. $\log 16$ c. units | 3. $(4 \log 3 - 2)$ c. units | 4. $243/8$ c. units |
| 5. $\frac{1104}{35}$ c. units | 6. 32 c. units | 7. $\frac{100}{3} \pi$ c. units | 8. 18.1 c. units |
| 9. $13 \frac{1}{2}$ c. units | 10. 4π c. units | 11. $\frac{1}{32} \pi$ c. units | 12. $\frac{2}{3} a^3 \left(\pi - \frac{4}{3} \right)$ c. units. |

13. (a) radius of sphere = 2, of the hole = 1, (b) $4\pi\sqrt{3}$ c. units.

16.4.4 MASS, CENTROIDS, MOMENTS OF INERTIA ETC. BY TRIPLE INTEGRALS

The previous section 16.4.3 shows one of the applications of triple integral in computing volume of solids. Also triple integral can be used to compute mass, moments about co-ordinate planes, centre of mass and centre of gravity, moments and products of inertia, radius of gyration, force of attraction, potentials etc.

Mass : Since density is the mass per unit volume, so, the density

$$\delta(x, y, z) = \frac{dm}{dv}$$

or

$$dm = \delta(x, y, z) dv$$

$$\text{Integrating, mass, } m = \iiint_V \delta(x, y, z) dv \quad (1)$$

If $\delta(x, y, z) = 1$, then mass, $m = \iiint_V dv$ only.

Moments : The moments of mass m about co-ordinate planes are given by

$$\left. \begin{aligned} \text{moments about } xy \text{ plane} &= M_{xy} = \iiint_R z \delta(x, y, z) dv \\ \text{moments about } yz \text{ plane} &= M_{yz} = \iiint_R x \delta(x, y, z) dv \\ \text{and moments about } zx \text{ plane} &= M_{zx} = \iiint_R y \delta(x, y, z) dv \end{aligned} \right\} \quad (2)$$

Centroid or centre of mass $(\bar{x}, \bar{y}, \bar{z})$ is defined by

$$m\bar{x} = M_{yz}, \quad m\bar{y} = M_{zx} \quad \text{and} \quad m\bar{z} = M_{xy} \quad (3)$$

Moments of inertia about co-ordinate axes are defined by

$$\left. \begin{aligned} I_x &= \iiint_V (y^2 + z^2) \delta(x, y, z) dv \\ I_y &= \iiint_V (x^2 + z^2) \delta(x, y, z) dv \\ I_z &= \iiint_V (x^2 + y^2) \delta(x, y, z) dv \end{aligned} \right\} \quad (4)$$

and about the co-ordinate planes are defined by

$$\left. \begin{aligned} I_{xy} &= \iiint_V \delta z^2 dz dy dx, \quad I_{yz} = \iiint_V \delta x^2 dz dy dx, \\ I_{zx} &= \iiint_V \delta y^2 dz dy dx \end{aligned} \right\} \quad (5)$$

$$\therefore I_x = I_{xx} + I_{xy}, \quad I_y = I_{xy} + I_{yz} \quad \text{and} \quad I_z = I_{yz} + I_{zx}$$

$$\text{Radius of gyration: } R_z = \sqrt{I_z/m}, \quad R_y = \sqrt{I_y/m} \quad \text{and} \quad R_x = \sqrt{I_x/m} \quad (6)$$

$$\left. \begin{aligned} \text{Product of inertia: } U_{xy} &= \iiint_V xy \delta(x, y, z) dV, \\ U_{yz} &= \iiint_V yz \delta(x, y, z) dv, \quad U_{zx} = \iiint_V zx \delta(x, y, z) dV \end{aligned} \right\} \quad (7)$$

Potential : The potential at a point P due to a mass Δm at Q is $\frac{\Delta m}{r}$,

$$\text{where } r \text{ is the distance from } P \text{ to } Q \text{ i.e. potential } u(P) = \iiint_V \frac{1}{r} dV \quad (8)$$

Attraction : Force of attraction at Q is

$$\text{Force} = \iiint_V \frac{\hat{u}}{r^2} dV \quad (9)$$

where r is the distance from P to Q and $\hat{u}$ is unit vector in the direction of PQ .

All these applications and the volume by triple integrals are the physical interpretations of the most general triple integral $\iiint_V f(x, y, z) dV$ in 16.4 equation (1). Note that with each definite form of $f(x, y, z)$ the name of the triple integral becomes an application or interpretation although equation (1) in 16.4 seems arbitrary at the beginning.

Equations (1) through (9) can be expressed and used in both cylindrical and spherical using the usual transformation relation.

Examples

1. Use triple integration to find the moments of inertia of the tetrahedron bounded by the plane $x + y + z = 1$ and the co-ordinate planes.

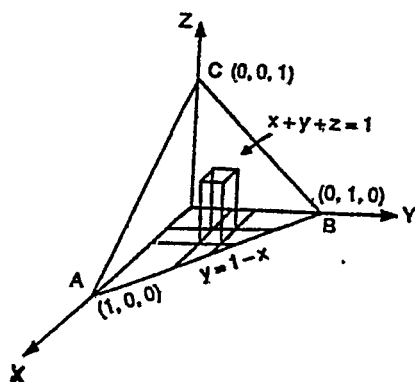


Fig. 128

The tetrahedron, in question, $OABC$ is shown in fig. 128. Now the required moments of inertia are the moments of inertia about the co-ordinate axes. So, considering density $\delta = 1$,

$$I_x = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (y^2 + z^2) dz dy dx$$

$$= \frac{1}{30}$$

$$I_y = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x^2 + z^2) dz dy dx$$

$$= \int_0^1 \int_0^{1-x} \left[x^2 z + \frac{1}{3} z^3 \right]_{z=0}^{1-x-y} dy dx$$

$$= \int_0^1 \left[\frac{1}{2} x^2 (1-x)^2 + \frac{1}{12} (1-x)^4 \right] dx$$

$$= \frac{1}{30}$$

$$I_z = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x^2 + y^2) dz dy dx = \frac{1}{30}.$$

2. Find the mass, moments and the centroid of the solid bounded by the cylinder $z = 4 - x^2$ and the planes $x = 0$, $y = 0$, $y = 6$ and $z = 0$.

The solid is shown in the fig. 129. Now, considering density $\delta = \text{const.}$

$$\begin{aligned} \text{mass, } m &= \int_0^2 \int_0^6 \int_0^{4-x^2} \delta dz dy dx \\ &= 32 \delta. \end{aligned}$$

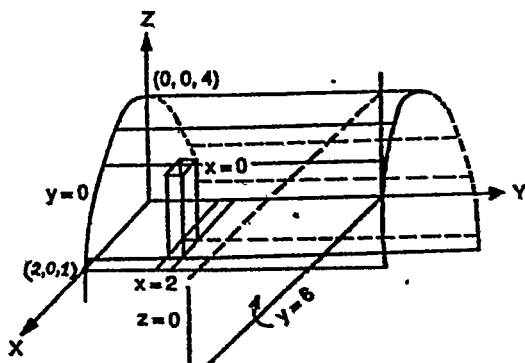


Fig. 129

Also moment about yz -plane is

$$\begin{aligned} M_{yz} &= \int_0^2 \int_0^6 \int_0^{4-x^2} \delta x dz dy dx \\ &= 24 \delta \end{aligned}$$

moment about xz -plane is

$$M_{xz} = \int_0^2 \int_0^6 \int_0^{4-x^2} \delta y dz dy dx = 96 \delta$$

and moment about xy -plane is

$$M_{xy} = \int_0^2 \int_0^6 \int_0^{4-x^2} \delta z dz dy dx = \frac{256}{5} \delta.$$

$\therefore$ co-ordinates of the centroid of the solid or centre of mass are

$$\bar{x} = \frac{M_{yz}}{m} = \frac{24\delta}{32\delta} = \frac{3}{4}$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{96\delta}{32\delta} = 3$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{256\delta}{5 \times 32} = \frac{8}{5}$$

i.e., the centroid is $\left(\frac{3}{4}, 3, \frac{8}{5}\right)$

Exercises 16 (10)

- Find the moment of inertia of the solid enclosed in the paraboloid $z = x^2 + y^2$ and the cylinder $x^2 + y^2 = a^2$. Use cylindrical and spherical co-ordinates.
- Find the centroid of the solid bounded by the sphere $x^2 + y^2 + z^2 = a^2$ and the cone $z^2 \sin^2 \alpha = (x^2 + y^2) \cos^2 \alpha$ where α is a constant and $0 \leq \alpha \leq \pi$. Use spherical co-ordinates.
- Find the centroid of the tetrahedron whose faces are $4x + 3y + z = 12$ and the co-ordinate planes.
- If the density of the tetrahedron in prob. 3 above is proportional to the distance of any point on it from the xy -plane, then determine the z -co-ordinate of the centroid.
- The density ρ of a solid bounded by $z = 4 - x^2 - y^2$ and the xy plane is constant. Find its centre of mass.
- Find the moments of inertia about co-ordinate planes as well as axes of the homogeneous solid bounded by the surface $z = 2xy$ and the planes $z = 0, x = 2, y = 2$.
- Using triple integral in cylindrical co-ordinates find the centroid of the solid bounded by the sphere $x^2 + y^2 + z^2 = 9$, the paraboloid $6z = 18 - x^2 - y^2$ and the plane $z = 0$.
- If the density of the solid in prob. 7 is proportional to the sum of the square of the distances of any point on it from the z -axis, then find the moments of inertia (i) about z -axis, (ii) about a line passing through $(1, 2, 0)$ and (iii) about zx plane.
- Find the centroid of the solid bounded by xy -plane, $r = z(1 - \cos \theta)$ and $r = 4(1 - \cos \theta)$.
- Find the mass and the moment of inertia w.r.t. x -axis of the solid with density $\mu(x, y, z) = kz$ lying in the first octant and bounded by the cylinders $z^2 = y$ and $x^2 = y$ and the plane $y = 1$.

11. Find the centre of mass of the solid bounded by the parabolic cylinder $y^2 = x$ and the planes $x = 0$, $y = 1$, $z = 0$ and $z = y$ in the first octant with density $\mu = (x + 2z)$.
12. A shell roof has the form of the hyperbolic paraboloid $z = kxy$ and covers the region enclosed by the planes $x = 0$, $x = a$, $y = 0$ and $y = b$. If the mass of unit area of the roof is cxy at a typical point (x, y, z) , find the total mass.

Answers

1. $\frac{\pi a^6}{3}$ 2. $\left(0, 0, \frac{3a}{8}(1 + \cos \alpha)\right)$ 3. $\left(\frac{3}{4}, 1, 3\right)$ 4. $\frac{24}{5}$ 5. $\left(0, 0, \frac{4}{3}\right)$
6. $I_{xy} = \frac{128}{3} \rho$, $I_{xz} = I_{yz} = 16 \rho$, $I_x = I_y = \frac{176}{3} \rho$, $I_z = 32 \rho$ 7. $\left(0, 0, \frac{3}{4}\right)$
8. (i) $\frac{39,366}{35} k\pi$ (ii) $\frac{56,376}{35} k\pi$ (iii) $\frac{19,683}{35} k\pi$ 9. $\frac{3}{2}$
10. mass = $k/5$ units, $I_x = 23k/126$ 11. $\left(\frac{95}{238}, \frac{100}{119}, \frac{185}{357}\right)$
12. $(c/15k^4) \{(1 + k^2(a^2 + b^2))^{5/2} - (1 + k^2a^2)^{5/2} - (1 + k^2b^2)^{5/2} + 1\}$.

PART III

Elliptic Integrals and Elliptic Functions

CH. 17

ELLIPTIC INTEGRALS AND ELLIPTIC FUNCTIONS

17.1. INTRODUCTION

The integral of rational functions of the form $\int R(x, \sqrt[n]{ax+b}) dx$ and $\int R(x, \sqrt{ax^2+bx+c}) dx$ integrable in terms of elementary functions are discussed in chapter 4. But the integrals of the rational function of x and the square root of a cubic or a quartic without multiple zeros i.e. $\int R(x, \sqrt{a_0x^3+a_1x^2+a_2x+a_3}) dx$ or $\int R(x, \sqrt{a_0x^4+a_1x^3+a_2x^2+a_3x+a_4}) dx$ cannot be integrated in terms of elementary functions as before. These integrals are known as *elliptic integrals*. The name was derived from the fact that the perimeter of an ellipse produces an integral of this form (ch. 12). The inverse of elliptic integral is called *elliptic functions*. This chapter will discuss both the topics in brief.

The general form of elliptic integral is

$$\int \frac{A(x) + B(x)\sqrt{S(x)}}{C(x) + D(x)\sqrt{S(x)}} dx$$

where A, B, C, D are arbitrary polynomials and $S(x)$ is either a cubic or a quartic polynomial with no multiple zeros real or imaginary.

Both elliptic integral and elliptic function are used in the theory of numbers, algebra, geometry, differential equations, theoretical mechanics and physics, Schwarz-Christoffel transformation and magnetic field of dynamo-electric machine etc.

17.2. KINDS OF ELLIPTIC INTEGRALS

There are various kinds of elliptic integrals and with various notations for each. We are mainly concerned here with Legendre's elliptic integrals.

A useful starting point is to define the *general elliptic integral of the second kind* which bears relation with three Legendre's elliptic integrals. It is defined as

$$el\ 2(y, k_c, a, b) \equiv \int_0^y \frac{(a + bx^2) dx}{(1+x^2) \sqrt{(1+x^2)(1+k_c^2 x^2)}} \quad (1)$$

where $y \geq 0$ and a, b, k_c are any real numbers.

Let $x = \tan \phi$, so, (1) becomes

el 2 $(\tan^{-1} y, k_c, a, b)$

$$\equiv \int_0^{\tan^{-1} y} \frac{(a + b \tan^2 \phi) d\phi}{\sqrt{(1 + \tan^2 \phi)(1 + k_c^2 \tan^2 \phi)}} \quad (2)$$

Again let $k^2 = 1 - k_c^2$. Therefore, (2) becomes

$$\text{el 2 } (\tan^{-1} y, k, a, b) \equiv \int_0^{\tan^{-1} y} \frac{a + (b - a) \sin^2 \phi}{\sqrt{1 - k^2 \sin^2 \phi}} d\phi \quad (3)$$

General Complete elliptic integral: It is defined as
cel (k_c, p, a, b)

$$\equiv \int_0^{\infty} \frac{(a + bx^2) dx}{(1 + px^2) \sqrt{(1 + x^2)(1 + k_c^2 x^2)}} \quad (4)$$

Let $x = \tan \phi$, so (4) becomes

cel (k_c, p, a, b)

$$\equiv \int_0^{\pi/2} \frac{(a \cos^2 \phi + b \sin^2 \phi) d\phi}{(\cos^2 \phi + p \sin^2 \phi) \sqrt{\cos^2 \phi + k_c^2 \sin^2 \phi}} \quad (5)$$

In the limit when $y \rightarrow \infty$ in (3) and $p = 1, k^2 = 1 - k_c^2$ in (5) then

$$\text{el 2 } \left(\frac{\pi}{2}, k, a, b \right) = \text{cel } (k, 1, a, b).$$

Legendre elliptic integral of the first kind: It is defined as

$$u = F(k, \phi) = \int_0^{\phi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}, \quad 0 < k < 1 \quad (6)$$

Here k is called the modulus of u , written mod $u = k$ and ϕ is called amplitude of u , written $\text{am } u = \phi$. The integral (6) is also called the *incomplete elliptic integral of the first kind*.

If $\phi = \pi/2$ the integral (6) is called the *complete elliptic integral of the first kind* and is denoted by $K(k)$ or simply K i. e.

$$K = K(k) = F(k, \pi/2)$$

$$= \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} = \text{cel } (k, 1, 1, 1) \quad (7)$$

Other Equivalent forms: The integral (6) can be represented in two more equivalent forms as follows:—

(i) Let $\sin \theta = v$ in (6) and $x = \sin \phi$. Then (6) becomes

$$F_1(k, x) = \int_0^x \frac{dv}{\sqrt{(1-v^2)(1-k^2v^2)}}, \quad 0 < k < 1 \quad (8)$$

For the complete elliptic integral $\theta = \pi/2$. So $x = \sin \pi/2 = 1$. Hence the corresponding complete elliptic integral is

$$F_1(k, 1) = \int_0^1 \frac{dv}{\sqrt{(1-v^2)(1-k^2v^2)}}, \quad 0 < k < 1 \quad (9)$$

$= \text{cel}(k, 1, 1, 1)$ by (5) when $\sin \phi = v$.

The integral (9) is one of the equivalent forms of integral (6).

(ii) The integral (6) can be easily simplified to

$$F(k_c, \phi) = \int_0^\phi \frac{d\theta}{\cos \theta \sqrt{1+k_c^2 \tan^2 \theta}} \quad (10)$$

Let $x = \tan \theta$ in (10) and $\tan \phi = y$. So the next equivalent form of (6) is

$$F_2(k_c, y) = \int_0^y \frac{dx}{\sqrt{(1+x^2)(1+k_c^2x^2)}} \quad (11)$$

$$= \text{el } 2(y, k_c, 1, 1) \quad \text{by (1)}$$

For the complete elliptic integral, $\phi = \pi/2$ and so $y = \tan \pi/2 = \infty$. Hence the corresponding complete elliptic integral is

$$F_2(k_c, \infty) = \int_0^\infty \frac{dx}{\sqrt{(1+x^2)(1+k_c^2x^2)}} \quad (12)$$

$$= \text{cel}(k_c, 1, 1, 1) \quad \text{by (4)}$$

Legendre elliptic integral of second kind: It is defined as

$$E(k, \phi) = \int_0^\phi \sqrt{1-k^2 \sin^2 \theta} d\theta, \quad 0 < k < 1 \quad (13)$$

This is also known as *incomplete elliptic integral of the second kind*. Here also k is the modulus & ϕ is the amplitude.

If $\phi = \pi/2$, the integral (13) is called the *complete elliptic integral of the second kind*, and is denoted by $E(k)$ or simply E i.e.

$$E = E(k) = E(k, \pi/2) = \int_0^{\pi/2} \sqrt{1-k^2 \sin^2 \theta} d\theta \quad (14)$$

$$= \text{cel}(k_c, 1, 1, k_c^2) \quad \text{by (5)}$$

This integral arises as the one quarter perimeter of an ellipse which derives the name elliptic integral as mentioned in sec. 17.1.

Other equivalent forms : The integral (13) can be represented in two more equivalent forms described below :—

(i) Let $\sin \theta = v$ in (13) and $x = \sin \phi$. Then (13) becomes

$$E_1(k, x) = \int_0^x \frac{\sqrt{1-k^2 v^2}}{\sqrt{1-v^2}} dv \quad (15)$$

and the corresponding complete elliptic integral obtained by taking $\theta = \pi/2$ is

$$E_1(k, 1) = \int_0^1 \frac{\sqrt{1-k^2 v^2}}{\sqrt{1-v^2}} dv, \quad 0 < k < 1 \quad (16)$$

= $cel(k, 1, 1, k^2)$ by (5) when $\sin \phi = v$.

The integral (16) is one of the equivalent forms of integral (13).

(ii) The integral (13) can be easily simplified to

$$E_2(k_c, \phi) = \int_0^\phi \cos \theta \sqrt{1+k_c^2 \tan^2 \theta} d\theta \quad (17)$$

Let $x = \tan \theta$ in (17) and $\tan \theta = y$ then (17) becomes

$$E_2(k_c, \phi) = \int_0^y \frac{\sqrt{1+k_c^2 x^2}}{(1+x^2)\sqrt{1+x^2}} dx \quad (18)$$

= $el\ 2(y, k_c, 1, k_c^2)$ by (1)

and the corresponding complete integral is given by

$$E_2(k_c, \infty) = \int_0^\infty \frac{\sqrt{1+k_c^2 x^2}}{(1+x^2)\sqrt{1+x^2}} dx \quad (19)$$

(when $\phi = \pi/2$ i.e. $y = \infty$)

= $cel(k_c, 1, 1, k_c^2)$ by (4)

Legendre elliptic integral of the first kind in terms of general elliptic integral of the second kind :

Let

$$B(k, \phi) = \int_0^\phi \frac{\cos^2 \theta}{\sqrt{1-k^2 \sin^2 \theta}} d\theta = el\ 2(y, k, 1, 0) \text{ by (3)} \quad (20)$$

and

$$D(k, \phi) = \int_0^\phi \frac{\sin^2 \theta}{\sqrt{1-k^2 \sin^2 \theta}} d\theta = el\ 2(y, k, 0, 1) \text{ by (3)} \quad (21)$$

$$\begin{aligned} \therefore B(k, \phi) + D(k, \phi) \\ &= el\,2(y, k, 1, 0) + el\,2(y, k, 0, 1) \\ &= el\,2(y, k, 1, 1) = F(k, \phi) \end{aligned} \quad (22)$$

Hence Legendre elliptic integral of the first kind is the linear combination (sum in this case) of two general elliptic integrals of the second kind.

Legendre elliptic integral of third kind : This type of elliptic integral occurs less frequently in practice. It cannot be written in terms of $el\,2$ defined above. It is defined as

$$\begin{aligned} \Pi(\phi, n, k) \\ &= \int_0^\phi \frac{d\theta}{(1+n\sin^2\theta)\sqrt{1-k^2\sin^2\theta}}, \quad 0 < k < 1 \end{aligned} \quad (23)$$

Some authors define this integral as

$$\Pi(\phi, n^2, k) = \int_0^\phi \frac{d\theta}{(1-n^2\sin^2\theta)\sqrt{1-k^2\sin^2\theta}} \quad (23a)$$

Both the definitions (23) and (23 a) will be used in this chapter as and when it is necessary.

This is also known as *incomplete elliptic integral of the third kind*. Here k is the *modulus*, ϕ is the *amplitude* & $n \neq 0$ is called the *characteristic* of this elliptic integral.

If $\phi = \pi/2$, the integral (23) becomes the complete elliptic integral of the third kind and is denoted by $\Pi(n, k)$ or simply Π i.e.

$$\begin{aligned} \Pi(n, k) &= \Pi(\pi/2, n, k) \\ &= \int_0^{\pi/2} \frac{d\theta}{(1+n\sin^2\theta)\sqrt{1-k^2\sin^2\theta}}, \quad 0 < k < 1 \end{aligned} \quad (24)$$

$$= cel(k, n, 1, 1) \quad \text{by (5)} \quad (25)$$

Other equivalent forms: As before the integral (23) has two more equivalent forms given below.

(i) Let $\sin\theta = v$ in (23) and $x = \sin\phi$. Then (23) takes the form $\Pi_1(x, n, k)$

$$= \int_0^x \frac{dv}{(1+n v^2)\sqrt{(1-v^2)(1-k^2 v^2)}}, \quad 0 < k < 1 \quad (26)$$

and the corresponding complete elliptic integral, obtained by taking $\theta = \pi/2$ i.e. $x = 1$, is

$$\begin{aligned}\Pi_1(1, n, k) &= \int_0^1 \frac{dv}{(1+n v^2) \sqrt{(1-v^2)(1-k^2 v^2)}}, 0 < k < 1 \quad (27) \\ &= \text{cel}(k, n, 1, 1) \text{ by (5) when } \sin \phi = v.\end{aligned}$$

(ii) The integral (23) can be simplified to

$$\Pi_2(\theta, n, k_c) = \int_0^\theta \frac{d\theta}{(1+n \sin^2 \theta) \cos \theta \sqrt{1+k_c^2 \tan^2 \theta}} \quad (28)$$

Let $v = \tan \theta$ in (28) and $\tan \theta = y$ then (28) becomes the equivalent form

$$\begin{aligned}\Pi_2(\theta, n, k_c) &= \int_0^y \frac{dv}{(1+n v^2/(1+v^2)) \sqrt{1+v^2} \sqrt{1+k_c^2 v^2}} \\ &= \int_0^y \frac{(1+v^2) dv}{(1+p v^2) \sqrt{(1+v^2)(1+k_c^2 v^2)}}, p = n+1, \quad (29)\end{aligned}$$

The corresponding complete elliptic integral is obtained when $\phi = \pi/2$ i.e. $y = \infty$. Therefore, from (29) we get,

$$\begin{aligned}\Pi_2(\infty, n, k_c) &= \int_0^\infty \frac{(1+v^2) dv}{(1+p v^2) \sqrt{(1+v^2)(1+k_c^2 v^2)}}, p = n+1 \quad (30) \\ &= \text{cel}(k_c, p, 1, 1) \text{ by (4)}\end{aligned}$$

17.3. ASSOCIATED ELLIPTIC INTEGRALS

Since $k_c^2 = 1 - k^2$, k_c and k are called complementary modulus of elliptic integrals. Again $K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2 \sin^2 \theta}}$, the associated elliptic integral of the first kind is defined as

$$K'(k_c) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k_c^2 \sin^2 \theta}} = K(k_c). \text{ Also}$$

$$K'(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-(1-k^2) \sin^2 \theta}} = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k_c^2 \sin^2 \theta}} = K(k_c)$$

Similarly, $E(k)$ and $E'(k) \equiv E(k_c)$ are called the associated elliptic integrals of the second kind, $\Pi(n, k)$ and $\Pi'(n, k) \equiv \Pi(n, k_c)$ are associated elliptic integrals of the third kind.

The following identities easily follow from the above definitions :

$$K(0) = K'(1) = \pi/2 \text{ and } K(1) = K'(0) = \infty.$$

17.4 REDUCTION OF ELLIPTIC INTEGRALS TO STANDARD FORM

Appropriate substitution reduces a great variety of definite and improper integrals into standard elliptic integrals. Substitution should be selected in such a way that the modulus of elliptic integral is less than unity under all circumstances for the prescribed domain of zeros. Some of them are discussed below just to give an idea.

(A) TYPE $\int_0^u \frac{dx}{\sqrt{(x^2+a^2)(x^2+b^2)}} \quad , a > b > 0.$

This is a definite integral but cannot be approached by the procedures of earlier chapters. Let $x = b \tan \alpha$.

Therefore,

$$\begin{aligned} \int_0^u \frac{dx}{\sqrt{(x^2+a^2)(x^2+b^2)}} &= \int_0^{\alpha=\tan^{-1} \frac{u}{b}} \frac{b \sec^2 \alpha d\alpha}{\sqrt{(b^2 \tan^2 \alpha + a^2) b^2 \sec^2 \alpha}} \\ &= \int_0^\alpha \frac{\sec \alpha d\alpha}{\sqrt{a^2 + b^2 \tan^2 \alpha}} = \int_0^\alpha \frac{d\alpha}{\sqrt{a^2 - (a^2 - b^2) \sin^2 \alpha}} \\ &= \frac{1}{a} \int_0^\alpha \frac{d\alpha}{\sqrt{1 - q^2 \sin^2 \alpha}} \quad , q = \frac{\sqrt{a^2 - b^2}}{a} < 1 \\ &= \frac{1}{a} F(\alpha, q) \text{ where } \alpha = \tan^{-1} \frac{u}{b} \text{ and } q^2 = \frac{a^2 - b^2}{a^2}. \end{aligned}$$

(B) TYPE $\int_u^\infty \frac{dx}{\sqrt{(x^2+a^2)(x^2+b^2)}} \quad , a > b > 0.$

Let $x = \cot \beta$. Proceeding as in (A), we have

$$\begin{aligned} \int_u^\infty \frac{dx}{\sqrt{(x^2+a^2)(x^2+b^2)}} &= \frac{1}{a} F(\beta, q) \text{ where} \\ \beta &= \cot^{-1} \frac{u}{a} \text{ and } q^2 = \frac{a^2 - b^2}{a^2}. \end{aligned}$$

(C) TYPE $\int \frac{1}{\sqrt{P_3(x)}} dx, P_3(x)$ IS A CUBIC

Numerous cases arise in this form. Let us consider some of them.

(i) TYPE $\int_{-\infty}^u \frac{dx}{\sqrt{(a-x)(b-x)(c-x)}} \quad , a > b > c \geq u$

This is an improper integral of the first kind with lower limit $(-\infty)$. So let us substitute for factor corresponding to the biggest zero i.e.

$a - x = \frac{1}{y^2}$ to get zero for the lower limit value of the new variable, y . Therefore

$$\begin{aligned} \int_{-\infty}^u \frac{dx}{\sqrt{(a-x)(b-x)(c-x)}} &= 2 \int_0^{\sqrt{a-u}} \frac{dy}{y^3 \sqrt{\frac{1}{y^2} \left(b-a + \frac{1}{y^2}\right) \left(c-a + \frac{1}{y^2}\right)}} \\ &= 2 \int_0^{\sqrt{a-u}} \frac{dy}{\sqrt{(1-\frac{a-b}{a-c}y^2)(1-\frac{a-c}{a-b}y^2)}} \\ &= \frac{2}{\sqrt{a-c}} \int_0^{\theta=\sin^{-1}\sqrt{\frac{a-c}{a-u}}} \frac{\cos \theta d\theta}{\sqrt{\left(1-\frac{a-b}{a-c}\sin^2 \theta\right) \cos^2 \theta}} \\ &\quad \text{where } \sqrt{a-c} y = \sin \theta \\ &= \frac{2}{\sqrt{a-c}} F(\theta, p) \text{ where } p^2 = \frac{a-b}{a-c} < 1 \text{ and } \theta = \sin^{-1} \sqrt{\frac{a-c}{a-u}} \end{aligned}$$

(ii) TYPE $\int_u^c \frac{dx}{\sqrt{(a-x)(b-x)(c-x)}}$, $a > b > c > u$.

This is an improper integral of the second kind. Its upper limit 'c' : the smallest singular point of the integrand i.e. the smallest zero of the cubic. So, let us substitute $c - x = y^2$. Hence,

$$\begin{aligned} \int_u^c \frac{dx}{\sqrt{(a-x)(b-x)(c-x)}} &= 2 \int_0^{\sqrt{c-u}} \frac{dy}{\sqrt{(y^2+a-c)(y^2+b-c)}} \\ &= 2 \int_0^{\theta=\tan^{-1}\sqrt{\frac{c-u}{b-c}}} \frac{\sec \theta d\theta}{\sqrt{(b-c) \tan^2 \theta + (a-c)}} \text{ where } y = \sqrt{b-c} \tan \theta. \\ &= 2 \int_0^{\theta} \frac{d\theta}{\sqrt{(a-c) - (a-b) \sin^2 \theta}} = \frac{2}{\sqrt{a-c}} \int_0^{\theta} \frac{d\theta}{\sqrt{1-p^2 \sin^2 \theta}}, p^2 = \frac{a-b}{a-c} < 1 \\ &= \frac{2}{\sqrt{a-c}} F(\beta, p) \text{ where } \beta = \sin^{-1} \sqrt{\frac{c-u}{b-u}} \text{ \& } p^2 = \frac{a-b}{a-c}. \end{aligned}$$

(iii) TYPE $\int_c^u \frac{dx}{\sqrt{(a-x)(b-x)(x-c)}}$, $a > b \geq u > c$.

This is an improper integral of the second kind has the lower limit as the smallest singular point of the integrand. The procedure remains same as (ii) and same substitution gives

$$\int_c^u \frac{dx}{\sqrt{(a-x)(b-x)(x-c)}} = \frac{2}{\sqrt{a-c}} F(\gamma, q), a > b \geq u > c$$

$$\text{where } \gamma = \sin^{-1} \sqrt{\frac{u-c}{b-c}} \text{ and } q^2 = \frac{b-c}{a-c}.$$

$$(iv) \text{ TYPE } \int_c^u \frac{x dx}{\sqrt{(a-x)(b-x)(x-c)}}, a > b \geq u > c.$$

The integrand is a rational function of x and $P_3(x)$. The integral is still second kind improper with singular point $x=c$, the smallest zero of $P_3(x)$. So, substitution is $x-c=y^2$. Hence

$$\begin{aligned} \int_c^u \frac{x dx}{\sqrt{(a-x)(b-x)(x-c)}} &= 2 \int_0^{\gamma = \sin^{-1} \sqrt{\frac{u-c}{b-c}}} \frac{(c + \overline{b-c} \sin^2 \gamma)}{\sqrt{(a-c) - (b-c) \sin^2 \gamma}} d\gamma \\ &= \frac{2}{\sqrt{a-c}} \int_0^{\gamma} \frac{a-a+c+\overline{b-c} \sin^2 \gamma}{\sqrt{1-q^2 \sin^2 \gamma}} d\gamma, q^2 = \frac{b-c}{a-c} < 1 \\ &= \frac{2}{\sqrt{a-c}} \int_0^{\gamma} \left[a \cdot \frac{1}{\sqrt{1-q^2 \sin^2 \gamma}} - (a-c) \frac{1-q^2 \sin^2 \gamma}{\sqrt{1-q^2 \sin^2 \gamma}} \right] d\gamma \\ &= \frac{2}{\sqrt{a-c}} F(\gamma, q) - 2\sqrt{a-c} E(\gamma, q), a > b \geq u > c \end{aligned}$$

$$\text{where } \gamma = \sin^{-1} \sqrt{\frac{u-c}{b-c}} \text{ and } q^2 = \frac{b-c}{a-c} < 1.$$

$$(v) \text{ TYPE } \int_u^b \sqrt{\frac{a-x}{(b-x)(x-c)}} dx, a > b > u \geq c.$$

This improper integral of the second kind has singularity at the second zero between the largest and the smallest zeros of the cubic $P_3(x)$. In such case substitution is sought as a bi-linear (or linear fractional) transform (BLT). The transformation is linear in two variables – the new and the old ones or one is expressed as a fraction of two linear functions of the other. Let the bi-linear transform be $x = \frac{At+B}{Ct+D}$. The constants A, B, C, D are so chosen that $x=b, c, a$ corresponds to $t=0, 1, \infty$. This choice of x -values are

not arbitrary. The upper limit $x = b$ is the middle between the largest and the smallest and it always corresponds to $t = 0$. Then the remaining zeros in ascending order corresponds to $t = 1, \infty$ respectively.

$$b = \frac{B}{D} \text{ when } x = b, t = 0 \therefore B = bD \quad (1)$$

$$c = \frac{A+B}{C+D} \text{ when } x = c, t = 1 \therefore c(C+D) = A+B \quad (2)$$

$$a = \frac{A}{C} \text{ when } x = a, t = \infty \therefore A = aC \quad (3)$$

Expressing A, B, C in terms of D from (1), (2), (3) we get,

$$x = \frac{-a(b-c)t + b(a-c)}{-(b-c)t + (a-c)} \text{ and } dx = \frac{-(a-b)(b-c)(a-c)}{[-(b-c)t + (a-c)]^2} dt$$

Also

x	u	b
t	$\frac{b-u}{q^2(a-u)}$	0

when $q^2 = \frac{b-c}{a-c} < 1$

$$\begin{aligned} \therefore I &= \int_u^b \sqrt{\frac{a-x}{(b-x)(x-c)}} dx = \int_u^b \frac{a-x}{\sqrt{(a-x)(b-x)(x-c)}} dx \\ &= a \int_u^b \frac{dx}{\sqrt{(a-x)(b-x)(x-c)}} - \int_u^b \frac{x dx}{\sqrt{(a-x)(b-x)(x-c)}} \\ &= -a \int_t^0 \frac{(a-b)(b-c)(a-c) dt}{\sqrt{-(b-c)t + (a-c)} \sqrt{[(a-c)(a-b)(c-b)(b-a) + t(a-c)(a-c)(t-1)]}} \\ &\quad + \int_t^0 \frac{(a-b)(b-c)(a-c)[-a(b-c)t + b(a-c)] dt}{[-(b-c)t + (a-c)]^{3/2} \sqrt{[(a-c)(a-b)(c-b)(b-a) + t(a-c)(a-c)(t-1)]}} \\ &= \frac{1}{\sqrt{a-c}} \left[\int_0^t \frac{a dt}{\sqrt{(1-q^2 t)t(1-t)}} - \int_0^t \frac{[-a(b-c)t + (a-c)b] dt}{(a-c)(1-q^2 t)^{3/2} \sqrt{t(1-t)}} \right] \\ &= \frac{2}{\sqrt{a-c}} \left[a \int_0^{\delta = \sin^{-1} \sqrt{t}} \frac{d\delta}{\sqrt{1-q^2 \sin^2 \delta}} - \int_0^{\delta = \sin^{-1} \sqrt{t}} \frac{(b-aq^2 \sin^2 \delta) d\delta}{(1-q^2 \sin^2 \delta)^{3/2}} \right] \end{aligned}$$

where $t = \sin^2 \delta$

$$\begin{aligned}
 \therefore I &= \frac{2}{\sqrt{a-c}} \left[a F(\delta, q) - \int_0^\delta \frac{a(1-q^2 \sin^2 \delta) - a + b}{(1-q^2 \sin^2 \delta) \sqrt{1-q^2 \sin^2 \delta}} d\delta \right] \\
 &= \frac{2}{\sqrt{a-c}} \left[a F(\delta, q) - a F(\delta, q) + (a-b) \Pi(\delta, q^2, q) \right] \text{ by (23 a)} \\
 &= \frac{2(a-b)}{\sqrt{a-c}} \Pi(\delta, q^2, q).
 \end{aligned}$$

(vi) TYPE $\int_u^v \frac{dx}{\sqrt{(x-a)(x-b)(x-c)}}$, $u > a > b > c$.

This definite integral can be reduced to elliptic integral by the substitution $x - a = y^2$. Therefore,

$$\begin{aligned}
 I &= \int_u^v \frac{dx}{\sqrt{(x-a)(x-b)(x-c)}} = 2 \int_{\sqrt{u-a}}^{\sqrt{v-a}} \frac{dy}{\sqrt{[(a-b)+y^2][(a-c)+y^2]}} \\
 &= 2 \int_{\theta_1 = \tan^{-1} \sqrt{\frac{u-a}{a-b}}}^{\theta_2 = \tan^{-1} \sqrt{\frac{v-a}{a-b}}} \frac{d\theta}{\sqrt{(a-c) \cos^2 \theta + (a-b) \sin^2 \theta}} \\
 &= \frac{2}{\sqrt{a-c}} \int_{\theta_1}^{\theta_2} \frac{d\theta}{\sqrt{1-q^2 \sin^2 \theta}}
 \end{aligned}$$

$$\text{where } y = \sqrt{a-b} \tan \theta, q^2 = \frac{b-c}{a-c} < 1$$

$$= \frac{2}{\sqrt{a-c}} \left[\int_0^{\theta_2} \frac{d\theta}{\sqrt{1-q^2 \sin^2 \theta}} - \int_0^{\theta_1} \frac{d\theta}{\sqrt{1-q^2 \sin^2 \theta}} \right]$$

$$\therefore I = \frac{2}{\sqrt{a-c}} [F(\theta_2, q) - F(\theta_1, q)]$$

(D) TYPE $\int \frac{1}{\sqrt{P_4(x)}} dx$, $P_4(x)$ IS A QUARTIC of the various forms. Some of them are discussed below.

(i) TYPE $\int_u^v \frac{dx}{\sqrt{(x-a)(x-b)(x-c)(x-d)}}$, $a > b > c > d$.

This is a definite integral (or an improper integral of the first kind) whose limits are not the zeros of the quartic with non-repeated real zeros.

This integral is reduced to standard elliptic integrals using bilinear transformation (BLT) used in C (iv) above. Let the BLT be $x = \frac{At+B}{Ct+D}$.

The constants A, B, C, D are determined by choosing three x -values in ascending order of the zeros starting from smallest zero which corresponds to $t = 0, 1, \infty$ i.e. $x = d, c, b$ corresponds to $t = 0, 1, \infty$ respectively.

Let us calculate the constants A, B, C, D .

$$d = \frac{B}{D} \text{ when } x = d, t = 0, \therefore B = dD \quad (1)$$

$$c = \frac{A+B}{C+D} \text{ when } x = c, t = 1, \therefore c(C+D) = A+B \quad (2)$$

$$b = \frac{A}{C} \text{ when } x = b, t = \infty, \therefore A = bC \quad (3)$$

$\therefore$ the appropriate BLT is then

$$x = \frac{b \cdot \frac{c-d}{b-c}t + d}{\frac{c-d}{b-c}t + 1} = \frac{b(c-d)t + d(b-c)}{(c-d)t + (b-c)}$$

$$\therefore dx = \frac{(b-c)(c-d)(b-d)}{[(c-d)t + (b-c)]^2} dt \text{ and } t = \frac{(b-c)(d-x)}{(c-d)(x-b)}$$

$$\begin{aligned} \therefore I &= \int_u^v \frac{dx}{\sqrt{(x-a)(x-b)(x-c)(x-d)}} \\ &= \int_{t_1}^{t_2} \frac{\frac{(b-c)(d-v)}{(c-d)(v-b)} \cdot \frac{(b-c)(c-d)(b-d) dt}{\sqrt{[(c-d)(a-b)t + (a-d)(b-c)]} \cdot \frac{(b-c)^2(c-d)^2}{\sqrt{(b-d)^2 t(t-1)}}}}{\frac{(b-c)(d-u)}{(c-d)(u-d)}} \\ &= \int_{t_1}^{t_2} \frac{dt}{\sqrt{[(c-d)(a-b)t + (a-d)(b-c)]t(t-1)}} \end{aligned}$$

If both the limits t_1 and t_2 are +ve then substitution of $t = y^2$ is suitable, otherwise we need another BLT as follows to make the new limits +ve. Therefore, let us assume

$$t = \frac{A_1 u + B_1}{C_1 u + D_1} \text{ and calculate } A_1, B_1, C_1, D_1 \text{ when}$$

$t = -\frac{(a-d)(b-c)}{(c-d)(a-b)} = -q^2$ (say), $0, 1$ (zeros in ascending order) corresponds to $u = 0, 1, \infty$.

$$\therefore -q^2 = \frac{B_1}{D_1} \text{ giving } B_1 = -q^2 D_1$$

$$0 = \frac{A_1 + B_1}{C_1 + D_1} \text{ giving } A_1 + B_1 = 0 \text{ i.e. } A_1 = q^2 D_1$$

$$\text{and } 1 = \frac{A_1}{C_1} \text{ giving } C_1 = A_1 = q^2 D_1$$

$$\therefore t = \frac{q^2 u - q^2}{q^2 u + 1}. \text{ Hence } dt = \frac{q^2 (q^2 + 1) du}{(q^2 u + 1)^2} \text{ and}$$

$$\begin{array}{c|c|c} t & t_1 & t_2 \\ \hline u & u_1 = -\frac{(c-d)(u-a)}{(a-d)(c-u)} & u_2 = -\frac{(c-d)(v-a)}{(a-d)(c-v)} \end{array}$$

Therefore,

$$I = \int_{u_1}^{u_2} \frac{q^2 (q^2 + 1) du}{u_1 \sqrt{(a-b)(c-d)} (q^2 u + 1)^2 \sqrt{\left(q^2 + \frac{q^2 u - q^2}{q^2 u + 1} \right) \left(\frac{q^2 u - q^2}{q^2 u + 1} \right) \left(\frac{q^2 u - q^2}{q^2 u + 1} - 1 \right)}}$$

$$= \frac{1}{\sqrt{(a-b)(c-d)}} \int_{u_1}^{u_2} \frac{du}{\sqrt{1 + q^2 u} \sqrt{u(1-u)}}$$

$$= \frac{-2}{\sqrt{(a-b)(c-d)}} \int_{\theta_2 = \cos^{-1} \sqrt{u_1}}^{\theta_1 = \cos^{-1} \sqrt{u_2}} \frac{d\theta}{\sqrt{1 + q^2 \cos^2 \theta}}, \text{ where } u = \cos^2 \theta$$

$$= \frac{2}{\sqrt{(a-b)(c-d)}} \int_{\theta_1}^{\theta_2} \frac{d\theta}{\sqrt{(1 + q^2) - q^2 \sin^2 \theta}}$$

$$= \frac{2}{\sqrt{(a-b)(c-d)(1+q^2)}} \left[\int_0^{\theta_2 = \cos^{-1} \sqrt{u_1}} \frac{d\theta}{\sqrt{1 - Q^2 \sin^2 \theta}} - \int_0^{\theta_1 = \cos^{-1} \sqrt{u_2}} \frac{d\theta}{\sqrt{1 - Q^2 \sin^2 \theta}} \right],$$

$$\text{where } Q^2 = \frac{q^2}{1+q^2} < 1$$

$$= \frac{2}{\sqrt{(a-b)(c-d)(1+q^2)}} [F(\theta_2, Q) - F(\theta_1, Q)]$$

N. B. The improper integral of second kind.

$\int_b^c \frac{dx}{\sqrt{(x-a)(x-b)(x-c)(x-d)}}$ can be expressed in terms of elliptic integral in the above BLT process. Note problem 6 (iv) in the exercise.

(ii) TYPES $\int_u^v \frac{dx}{\sqrt{P_4(x)}}$, $\int_u^v \frac{x dx}{\sqrt{P_4(x)}}$, $\int_u^v \frac{dx}{x \sqrt{P_4(x)}}$ improper in-

tegral of second kind with one singular point.

In all the cases, BLT is used to reduce the quartic to the cubic and then a trigonometric substitution is required to reduce each of them to standard elliptic integral. Let the zeros of $P_4(x)$ be $a > b > c > d$ in descending order with order positions 1, 2, 3, 4 respectively i.e. "a and c" are in odd ordered position and "b and d" are in even ordered position. It is noticed that the singular point (S.P.) of the integrand always corresponds $t = 0$ of the BLT $x = \left(\frac{A_1 t + B_1}{C_1 t + D_1} \right)$ and if the upper limit, s.p. occupies the one odd (or even) ordered position say 'a' (or 'b') then starting with 'a' (or 'b') the zeros of $P_4(x)$ except other odd (or even) ordered positioned zero 'c' (or 'd') in anti-clockwise order i.e. $x = a, b, d$ (or b, c, a) corresponds to $t = 0, 1, \infty$ (fig. 130).

In case the s.p. is the lower limit of the integral, the zeros of $P_4(x)$ in clockwise order will correspond to $t = 0, 1, \infty$. With this rule 64 improper integral of second kind obtained from these three general types can be reduced to standard elliptic integrals. Of these 64 integrals, only one in each with upper limit as s.p. and lower limit as s.p. will be discussed in full and formulae for the determination of their amplitude and modulus will be given. From the remaining integrals, one $5 + 5 = 10$ integrals will be discussed briefly. If 'd' is replaced in these ten integrals by a, b, c then all the 64 integrals can be obtained.

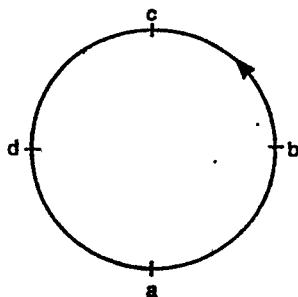


Fig. 130

1 (a) TYPE $\int_u^d \frac{dx}{\sqrt{(a-x)(b-x)(c-x)(d-x)}}$, $a > b > c > d > u$

Here upper limit 'd' is the smallest singular point. It occupies 4th ordered position i.e. second even ordered position. Therefore, according to the stated rule $x = d, a, c$ will correspond to $t = 0, 1, \infty$ in the BLT

$$x = \frac{A_1 t + B_1}{C_1 t + D_1}$$

$$\text{Therefore, } x = \frac{-c(a-d)t + d(a-c)}{-(a-d)t + (a-c)}, t = \frac{(a-c)(d-x)}{(a-d)(c-x)}$$

$$\text{and } dx = -\frac{(a-c)(a-d)(c-d)}{[-(a-d)t + (a-c)]^2} dt. \text{ The new limits are given}$$

$$\text{by } t = \frac{(a-c)(d-u)}{(a-d)(c-u)}, 0 \text{ corresponding to } x = u, d. \text{ Hence,}$$

$$\begin{aligned} I &= \int_u^d \frac{dx}{\sqrt{(a-x)(b-x)(c-x)(d-x)}} \\ &= - \int_{t=\frac{(a-c)(d-u)}{(a-d)(c-u)}}^{t=0} \frac{(a-c)(a-d)(c-d) dt}{\sqrt{(a-c)^2(a-d)^2(1-t)[(a-d)(c-b)t + (a-c)(b-d)](c-d)^2 t}} \\ &= \int_0^1 \frac{dt}{\sqrt{t(1-t)[(a-d)(c-b)t + (a-c)(b-d)]}} \\ &= \frac{1}{\sqrt{(a-c)(b-d)}} \int_0^1 \frac{dt}{\sqrt{t(1-t)(1-q^2 t)}}, q^2 = \frac{(a-d)(b-c)}{(a-c)(b-d)} < 1 \\ &= \frac{2}{\sqrt{(a-c)(b-d)}} \int_0^{\alpha = \sin^{-1} \sqrt{t}} \frac{d\alpha}{\sqrt{1-q^2 \sin^2 \alpha}}, t = \sin^2 \alpha. \\ &= \frac{2}{\sqrt{(a-c)(b-d)}} F(\alpha, q) \text{ where } \alpha = \sin^{-1} \sqrt{\frac{(a-c)(d-u)}{(a-d)(c-u)}} \\ &\quad \text{and } q^2 = \frac{(a-d)(b-c)}{(a-c)(b-d)} < 1. \end{aligned}$$

Observe that $\sin^2 \alpha$ and q^2 can be stated from the accepted x -values for the BLT. Let $A \equiv$ Absent; not accepted; $M \equiv$ middle, $L \equiv$ left, $R \equiv$ right denote three accepted x -values and variable limit, $u \equiv u$. Thus

$$\sin^2 \alpha = \frac{M-R}{M-L} \cdot \frac{L-u}{R-u} \text{ and } q^2 = \frac{A-R}{M-R} \cdot \frac{M-L}{A-L}$$

For other s.p.s use these rules and then arrange a, b, c, d in order of magnitude i.e. write $b-a = -(a-b)$ and the like. Use these rules for

the remaining 24 cases of the same type with upper limit as a s.p. or a zero as in (c) below to determine $\sin^2$ (amplitude) and (modulus)²

(b) TYPE $\int_u^d \frac{x dx}{\sqrt{(a-x)(b-x)(c-x)(d-x)}}, a > b > c > d > u.$

Proceeding as in (a) with same BLT, we have

$$\begin{aligned} I &= \int_u^d \frac{x dx}{\sqrt{(a-x)(b-x)(c-x)(d-x)}} \\ &= \int_0^t \frac{-(a-d)ct + (a-c)d}{-(a-d)t + (a-c)} \cdot \frac{dt}{\sqrt{t(1-t)[(a-d)(c-b)t + (a-c)(b-d)]}} \\ &= \frac{2}{\sqrt{(a-c)(b-d)}} \int_0^{\alpha=\sin^{-1}\sqrt{t}} \frac{-c(a-d)\sin^2\alpha + (a-c)d}{-(a-d)\sin^2\alpha + (a-c)} \cdot \frac{d\alpha}{\sqrt{1-q^2\sin^2\alpha}}, \\ &\quad \text{where } t = \sin^2\alpha. \end{aligned}$$

$$= \frac{2}{\sqrt{(a-c)(b-d)}} \int_0^\alpha \frac{d - \frac{a-d}{a-c} \sin^2\alpha}{1 - \frac{a-d}{a-c} \sin^2\alpha} \cdot \frac{d\alpha}{\sqrt{1-q^2\sin^2\alpha}}$$

$$= \frac{2}{\sqrt{(a-c)(b-d)}} \int_0^\alpha \frac{c\left(1 - \frac{a-d}{a-c} \sin^2\alpha\right) - (c-d)}{1 - \frac{a-d}{a-c} \sin^2\alpha} \cdot \frac{d\alpha}{\sqrt{1-q^2\sin^2\alpha}}$$

$$= \frac{2}{\sqrt{(a-c)(b-d)}} \left[cF(\alpha, q) - (c-d) \Pi\left(\alpha, \frac{a-d}{a-c}, q\right) \right].$$

(c) TYPE $\int_u^d \sqrt{\frac{d-x}{(a-x)(b-x)(c-x)}} dx, a > b > c > d > u$

This is a definite integral and upper limit 'd' is a zero of the integrand as it makes the integrand zero instead of being a s.p.

Here

$$\begin{aligned} \int_u^d \sqrt{\frac{d-x}{(a-x)(b-x)(c-x)}} dx &= d \int_u^d \frac{dx}{\sqrt{(a-x)(b-x)(c-x)(d-x)}} \\ &\quad - \int_u^d \frac{x dx}{\sqrt{(a-x)(b-x)(c-x)(d-x)}}, \text{ by rationalisation.} \end{aligned}$$

$$= \frac{2(c-d)}{\sqrt{(a-c)(b-d)}} \left[\Pi \left(\alpha, \frac{a-d}{a-c}, q \right) - F(\alpha, q) \right]$$

There are three more types as (c) obtained by replacing $(d-x)$ by other three factors. In those three cases upper limit 'd' is a s.p.

$$(d) \text{ TYPE } \int_u^d \frac{dx}{x \sqrt{(a-x)(b-x)(c-x)(d-x)}},$$

$$a > b > c > d > u$$

Proceeding as in (a)

$$\begin{aligned} I &= \int_u^d \frac{dx}{x \sqrt{(a-x)(b-x)(c-x)(d-x)}} \\ &= \int_0^t \frac{[-(a-d)t + (a-c)] dt}{[-c(a-d)t + (a-c)d] \sqrt{t(1-t)} [(a-d)(c-b)t + (a-c)(b-d)]} \\ &= \frac{2}{\sqrt{(a-c)(b-d)}} \int_0^{\alpha = \sin^{-1} \sqrt{t}} \frac{[(a-c) - (a-d) \sin^2 \alpha] d\alpha}{[(a-c)d - c(a-d) \sin^2 \alpha] \sqrt{1 - q^2 \sin^2 \alpha}} \\ &= \frac{2}{\sqrt{(a-c)(b-d)}} \int_0^\alpha \frac{\frac{1}{d} - \frac{a-d}{(a-c)d} \sin^2 \alpha}{1 - \frac{c}{d} \cdot \frac{a-d}{a-c} \sin^2 \alpha} \cdot \frac{d\alpha}{\sqrt{1 - q^2 \sin^2 \alpha}} \\ &= \frac{2}{c\sqrt{(a-c)(b-d)}} \int_0^\alpha \frac{\frac{c}{d} - \frac{c}{d} \cdot \frac{a-d}{a-c} \sin^2 \alpha}{1 - \frac{c}{d} \cdot \frac{a-d}{a-c} \sin^2 \alpha} \cdot \frac{d\alpha}{\sqrt{1 - q^2 \sin^2 \alpha}} \\ &= \frac{2}{cd\sqrt{(a-c)(b-d)}} \int_0^\alpha \frac{\left[(c-d) + d \left(1 - \frac{c}{d} \cdot \frac{a-d}{a-c} \sin^2 \alpha \right) \right] d\alpha}{\left(1 - \frac{c}{d} \cdot \frac{a-d}{a-c} \sin^2 \alpha \right) \sqrt{1 - q^2 \sin^2 \alpha}} \\ &= \frac{2}{cd\sqrt{(a-c)(b-d)}} \left[(c-d) \Pi \left(\alpha, \frac{c}{d} \cdot \frac{a-d}{a-c}, q \right) + dF(\alpha, q) \right] \end{aligned}$$

$$(e) \text{ TYPE } \int_u^d \frac{dx}{(p-x) \sqrt{(a-x)(b-x)(c-x)(d-x)}},$$

$$a > b > c > d > u, p \neq d.$$

Using the same BLT as in (a), we have

$$\begin{aligned} I &= \int_u^d \frac{dx}{(p-x) \sqrt{(a-x)(b-x)(c-x)(d-x)}} \\ &= \int_0^1 \frac{[-(a-d)t + (a-c)] dt}{[(a-d)(c-p)t + (a-c)(p-d)] \sqrt{t(1-t)[(a-d)(c-b)t + (a-c)(b-d)]}} \\ &= \frac{2}{\sqrt{(a-c)(b-d)}} \int_0^{\alpha = \sin^{-1} \sqrt{t}} \frac{[(a-c) - (a-d) \sin^2 \alpha] d\alpha}{[(a-c)(p-d) - (a-d)(p-c) \sin^2 \alpha] \sqrt{1-q^2 \sin^2 \alpha}} \\ &= \frac{2}{(p-d) \sqrt{(a-c)(b-d)}} \int_0^\alpha \frac{\left(1 - \frac{a-d}{a-c} \sin^2 \alpha\right) d\alpha}{\left(1 - \frac{a-d}{a-c} \frac{p-c}{p-d} \sin^2 \alpha\right) \sqrt{1-q^2 \sin^2 \alpha}} \\ &= \frac{2}{(p-d) \sqrt{(a-c)(b-d)}} \int_0^\alpha \frac{\left[\left(\frac{p-d}{p-c}\right) \left(1 - \frac{a-d}{a-c} \frac{p-c}{p-d} \sin^2 \alpha\right) + 1 - \frac{p-d}{p-c}\right] d\alpha}{\left(1 - \frac{a-d}{a-c} \frac{p-c}{p-d} \sin^2 \alpha\right) \sqrt{1-q^2 \sin^2 \alpha}} \\ &= \frac{2}{(p-c)(p-d) \sqrt{(a-c)(b-d)}} \left[(p-d) F(\alpha, q) + (d'-c) \Pi \left(\alpha, \frac{a-d}{a-c} \frac{p-c}{p-d}, q \right) \right] \end{aligned}$$

$$2. (a) \text{ TYPE } \int_d^u \frac{dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}},$$

$$a > b > c \geq u > d$$

Here s.p. is the lower limit 'd'. Note that last factor corresponding to s.p. is $(x-d)$ instead of $(d-x)$ as in 1 (a)–1(e). This type of change

always happens for other lower limits as s.p. As per established rule $x = d, c, a$ corresponds to $t = 0, 1, \infty$ in the BLT

$$x = \frac{A_1 t + B_1}{C_1 t + D_1}. \text{ Therefore, } x = \frac{a(c-d)t + d(a-c)}{(c-d)t + (a-c)} \text{ and}$$

$$t = \frac{(a-c)(x-d)}{(c-d)(a-x)}. \text{ The result of reduction is}$$

$$\int_d^u \frac{dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\beta, r), \begin{matrix} a > b > c \\ \geq u > d, \end{matrix}$$

$$\text{where } \beta = \sin^{-1} \sqrt{\frac{(a-c)(u-d)}{(c-d)(a-u)}} \text{ and } r^2 = \frac{(a-c)(c-d)}{(a-c)(b-d)} < 1$$

With the same symbols as 1 (a) we can establish rules for the $\sin^2 \beta$ and (modulus)² as follows :

$$\sin^2 \beta = \frac{M-R}{R-L} \cdot \frac{u-L}{M-u} \text{ and } r^2 = \frac{M-A}{M-R} \cdot \frac{R-L}{A-L}.$$

For other s.p. s use these rules and then arrange a, b, c, d in order of magnitude to make the factors positive i.e. $b-a = -(a-b)$ and the like. Use these rules for the remaining cases of the same type with lower limit as a s.p. or a zero as in (c) below to determine $\sin^2$ (amplitude) and (modulus)².

Hence follows other modifications.

$$(b) \text{ TYPE } \int_d^u \frac{x dx}{\sqrt{(a-x)(b-x)(c-x)(d-x)}}, \quad a > b > c \geq u > d$$

$$= \frac{2}{\sqrt{(a-c)(b-d)}} \left[(d-a) \Pi\left(\beta, \frac{d-c}{a-c}, r\right) + a F(\beta, r) \right]$$

$$(c) \text{ TYPE } \int_d^u \sqrt{\frac{x-d}{(a-x)(b-x)(c-x)}} dx$$

$$= \frac{2(d-a)}{\sqrt{(a-c)(b-d)}} \left[\Pi\left(\beta, \frac{d-c}{a-c}, r\right) - F(\beta, r) \right], \quad a > b > c \geq u > d.$$

There are three more types as in (c) obtained by replacing $(x-d)$ by other three factors. In those three cases the lower limit 'd' is a s.p.

(d) TYPE

$$\int_d^u \frac{dx}{x \sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{ad \sqrt{(a-c)(b-d)}} \left[(a-d) \Pi \left(\beta, \frac{a(d-c)}{d(a-c)}, r \right) + d F(\beta, r) \right], a > b > c \geq u > d,$$

$$\begin{aligned} \text{(e) TYPE } \int_d^u \frac{dx}{(p-x) \sqrt{(a-x)(b-x)(c-x)(x-d)}} &= \frac{2}{(p-a)(p-d) \sqrt{(a-c)(b-d)}} \\ &\left[(d-a) \Pi \left(\beta, \frac{(d-c)(p-c)}{(a-c)(p-d)}, r \right) + (p-d) F(\beta, r) \right] \\ &a > b > c \geq u > d, p \neq d. \end{aligned}$$

$$\text{(E) TYPE } \int_u^v \frac{dx}{\sqrt{(ax^2+bx+c)(dx^2+ex+f)}}$$

The quadratics have no real zeros. So, BLT cannot be applicable. Each of the general quadratics can be expressed as the sum of two squares by the following procedures :

Let $Q_1 = ax^2 + bx + c$ and $Q_2 = dx^2 + ex + f$. Now choose λ in $(Q_1 - \lambda Q_2)$ such that the expression becomes a perfect square. Therefore, $(b - e\lambda)^2 = 4(a - d\lambda)(c - f\lambda)$ which gives two values of λ , say λ_1 and λ_2 . Let

$$Q_1 - \lambda_1 Q_2 = (Ax + B)^2 \text{ and } Q_1 - \lambda_2 Q_2 = (Cx + D)^2.$$

Therefore, solving for Q_1 and Q_2 , we have

$$Q_1 = \frac{1}{\lambda_2 - \lambda_1} \left[\lambda_2 (Ax + B)^2 - \lambda_1 (Cx + D)^2 \right]$$

and

$$Q_2 = \frac{1}{\lambda_2 - \lambda_1} \left[(Ax + B)^2 - (Cx + D)^2 \right]$$

$$\text{Hence } I = \int_u^v \frac{dx}{\sqrt{(ax^2+bx+c)(dx^2+ex+f)}} = \int_u^v \frac{dx}{\sqrt{Q_1 Q_2}}$$

$$\text{Now the substitutions } t = \frac{Cx + D}{Ax + B} \text{ and } t = \sin^2 \theta$$

will reduce I to standard elliptic integral. The procedure will be more clear from the example 3 given below.

17.5. ELLIPTIC FUNCTIONS

Definition. Jacobian elliptic functions are (i) the inverse of Legendre elliptic integral of the first kind (this sec.), (ii) the generalization of plane trigonometric functions (cor. sec 17.11), (iii) doubly periodic functions (sec. 17.11), (iv) continuous (sec. 17.9) and (v) satisfy non-linear ordinary first order differential equation (cor. sec. 17.9) [For definition ch. 18].

Abel defines elliptic function as the inversion of Legendre elliptic integrals of the first kind

$$u = F(k, \phi) = \int_0^\phi \frac{d\theta}{\sqrt{1-k^2 \sin^2 \theta}} \text{ and } u = \int_0^x \frac{dv}{\sqrt{(1-v^2)(1-k^2 v^2)}}$$

where $x = \sin \phi$ (eqn. (6), (8) of sec. 17.2) by considering x and its associated functions $\sqrt{1-x^2}$, $\sqrt{1-k^2 x^2}$ etc. as function of u .

Since the upper limits of the aforesaid integrals are connected by

$$x = \sin \phi, \text{ so } \phi = \sin^{-1} x \quad (1)$$

$$\text{Also by definition (sec. 17.2), } \phi = \text{am } u \quad (2)$$

Hence from (1) and (2), $\phi = \sin^{-1} x = \text{am } u$ i.e., $x = \sin (\text{am } u)$ which is presently written as $x = \text{sn } u$. This $\text{sn } u$ is the abbreviation of "sinus amplitudinis of u " or "sine-amplitude of u " and is called a Jacobian elliptic function under the name of Jacobi who exhibit it. Henceforth, by elliptic function we shall mean Jacobian elliptic function and write as J. elliptic function.

Analogous to plane trigonometry, other J. elliptic function are $\text{cn } u$ (cosinus amplitudinis of u or cosine-amplitude of u) and $\text{dn } u$ (delta amplitudinis of u or delta-amplitude of u) which are defined as

$$\sqrt{1-x^2} = \sqrt{1-\sin^2 (\text{am } u)} = \cos (\text{am } u) = \text{cn } u \quad \text{and} \quad \sqrt{1-k^2 x^2} = \sqrt{1-k^2 \sin^2 (\text{am } u)} = \sqrt{1-k^2 \sin^2 u} = \text{dn } u, 0 < k < 1.$$

Glaisher defines other J. elliptic functions by the reciprocals and the quotients of these three fundamental functions $\text{sn } u$, $\text{cn } u$ and $\text{dn } u$ as follows:

$$\text{ns } u = \frac{1}{\text{sn } u}, \text{ nc } u = \frac{1}{\text{cn } u}, \text{ nd } u = \frac{1}{\text{dn } u}.$$

Thus the reciprocals reverse the order of letters denoting the functions. Again,

$$\begin{array}{l} \text{sc } u = \frac{\text{sn } u}{\text{cn } u} \quad \left| \quad \text{sd } u = \frac{\text{sn } u}{\text{dn } u} \quad \left| \quad \text{cd } u = \frac{\text{cn } u}{\text{dn } u} \right. \right. \\ \text{cs } u = \frac{\text{cn } u}{\text{sn } u} \quad \left| \quad \text{ds } u = \frac{\text{dn } u}{\text{sn } u} \quad \left| \quad \text{dc } u = \frac{\text{dn } u}{\text{cn } u} \right. \right. \end{array}$$

The functions defined by quotients of fundamental functions use the first letters of both numerator and denominator in order. Some authors still use Gudermann's notation u for $\text{sc } u$. Although $\text{tn } u$ finds analogy with plane trigonometry, yet it destroys the harmonious nomenclature of Glaisher.

17.6. VALUES OF J. ELLIPTIC FUNCTIONS AT 0 AND K

$\phi = 0$ in eqn. (6) of sec. 17.2 implies $u = 0$; so $\text{am } 0 = 0$. Hence $\text{sn } 0 = \sin(\text{am } 0) = 0$, $\text{cn } 0 = \cos 0 = 1$, $\text{dn } 0 = 1 - k^2 \text{sn}^2 0 = 1$. Thus,

$$\begin{array}{l|l|l} \text{ns } 0 = \infty & \text{nc } 0 = 1 & \text{nd } 0 = 1 \\ \text{sc } 0 = 0 & \text{cd } 0 = 0 & \text{cd } 0 = 1 \\ \text{cs } 0 = \infty & \text{dc } 0 = \infty & \text{dc } 0 = 1 \end{array}$$

Again,

$$K = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}. \text{ So, } \text{am } K = \frac{\pi}{2}. \text{ Therefore,}$$

$$\text{sn } K = \sin(\text{am } K) = \sin \frac{\pi}{2} = 1. \text{ Hence,}$$

$$\text{cn } K = \sqrt{1 - \text{sn}^2 K} = 0 \text{ and } \text{dn } K = \sqrt{1 - k^2 \text{sn}^2 K} = \sqrt{1 - k^2} = k_c$$

Therefore,

$$\begin{array}{l|l|l} \text{ns } K = 1 & \text{nc } K = \infty & \text{nd } K = \frac{1}{k_c} \\ \text{sc } K = \infty & \text{sd } K = \frac{1}{k_c} & \text{cd } K = 0 \\ \text{cs } K = 0 & \text{ds } K = k_c & \text{dc } K = \infty \end{array}$$

17.7. IDENTITIES OF J. ELLIPTIC FUNCTIONS

From sec. 17.5, $x = \text{sn } u$ and $\sqrt{1 - x^2} = \text{cn } u$.

$$\therefore \text{sn}^2 u + \text{cn}^2 u = x^2 + 1 - x^2 = 1$$

$$\text{Also, } k^2 \text{sn}^2 u + \text{dn}^2 u = k^2 \text{sn}^2 u + 1 - k^2 \text{sn}^2 u = 1.$$

$$\text{Similarly, } \text{ns}^2 u + \text{nc}^2 u = 1/\text{sn}^2 u + 1/\text{cn}^2 u = 1/(\text{sn}^2 u \text{cn}^2 u) \\ = \text{ns}^2 u \text{nc}^2 u$$

$$\text{ns}^2 u + k^2 \text{nd}^2 u = 1/\text{sn}^2 u + k^2/\text{dn}^2 u = 1/(\text{sn}^2 u \text{dn}^2 u) = \text{ns}^2 u \text{nd}^2 u.$$

Now from quotient functions,

$$k^2 \text{sc}^2 u + \text{dc}^2 u = k^2 (\text{sn}^2 u/\text{cn}^2 u) + (\text{du}^2 u/\text{cn}^2 u) = 1/\text{cn}^2 u = \text{nc}^2 u;$$

$$1 + \text{sc}^2 u = 1 + (\text{sn}^2 u/\text{cn}^2 u) = 1/\text{cn}^2 u = \text{nc}^2 u$$

$$\therefore k^2 \text{sc}^2 u + \text{dc}^2 u = 1 + \text{sc}^2 u$$

$$\text{Similarly, } \text{sd}^2 u + \text{cd}^2 u = 1/\text{dn}^2 u = \text{nd}^2 u.$$

17.8. EVENNESS PROPERTIES OF J. ELLIPTIC FUNCTIONS

Evenness property of a function implies whether the function is even, odd or "neither even nor odd" (ch. 8 sec. 8.7 for definitions and properties). Since,

$$u = \int_0^x \frac{dv}{\sqrt{(1-v^2)(1-k^2v^2)}}; \text{ therefore, replacing } v \text{ by } (-v)$$

$$\text{we get, } u = - \int_0^{-x} \frac{dv}{\sqrt{(1-v^2)(1-k^2v^2)}} \text{ Hence}$$

$(-x) = \text{am } (-u)$ and as $x = \text{am } u$, i.e. $(-x) = -(\text{am } u)$, so,

$\text{am } (-u) = -\text{am } u$. Therefore, $\text{sn } (-u) = \sin(\text{am } (-u)) = \sin(-\text{am } u) = -\sin(\text{am } u) = -\text{sn } u$. Hence $\text{sn } u$ is an odd function. Similarly,

$$\text{cn } (-u) = \sqrt{1 - \text{sn}^2(-u)} = \sqrt{1 - \text{sn}^2 u} = \text{cn } u.$$

$\therefore \text{cn } u$ is an even function. Again, $\text{dn } (-u) = \sqrt{1 - k^2 \text{sn}^2(-u)} = \sqrt{1 - k^2 \text{sn}^2 u} = \text{dn } u$ i.e., $\text{dn } u$ is also an even function. Then it follows that

$\text{ns } (-u) = -\text{ns } u$, $\text{sc } (-u) = -\text{sc } u$, $\text{sd } (-u) = -\text{sd } u$, $\text{ds } (-u) = -\text{ds } u$ are all odd functions while $\text{nc } (-u) = \text{nc } u$, $\text{nd } (-u) = \text{nd } u$, $\text{cd } (-u) = \text{cd } u$, $\text{dc } (-u) = \text{dc } u$ are all even functions.

17.9. DIFFERENTIATIONS OF J. ELLIPTIC FUNCTIONS

$$\text{By definition, if } u = \int_0^\phi \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \quad (1)$$

then $\text{sn } u = \sin \theta = \sin(\text{am } u)$.

$$\text{therefore, } \frac{d}{du}(\text{sn } u) = \frac{d}{du}(\sin \phi) = \frac{d}{d\phi}(\sin \phi) \cdot \frac{d\phi}{du} = \cos \phi \frac{d\phi}{du} \quad (2)$$

Now, differentiating (1) w.r.t. ϕ under the integral sign, we get

$$\begin{aligned} \frac{du}{d\phi} &= \int_0^\phi \frac{\partial}{\partial \phi} \left(\frac{1}{\sqrt{1 - k^2 \sin^2 \theta}} \right) d\theta + \frac{1}{\sqrt{1 - k^2 \sin^2 \phi}} \cdot 1 - \frac{1}{\sqrt{1 - k^2 \sin^2 \theta}} \cdot 0 \\ &= \frac{1}{\sqrt{1 - k^2 \sin^2 \phi}} \end{aligned} \quad (3)$$

$\therefore$ using (3) in (2), we get

$$\begin{aligned} \frac{d}{du}(\text{sn } u) &= \cos \phi \sqrt{1 - k^2 \sin^2 \phi} \\ &= \cos(\text{am } u) \sqrt{1 - k^2 \sin^2(\text{am } u)} = \text{cn } u \text{ dn } u \end{aligned} \quad (4)$$

Similarly,

$$\begin{aligned}\frac{d}{du}(\operatorname{cn} u) &= \frac{d}{du}(\cos(\operatorname{am} u)) = \frac{d}{du}(\cos \phi) \\ &= -\sin \phi \frac{d\phi}{du} \\ &= -\sin \phi \sqrt{1 - k^2 \sin^2 \phi} \quad \text{by (3)} \\ &= -\operatorname{sn} u \operatorname{dn} u\end{aligned}\quad (5)$$

$$\begin{aligned}\text{and } \frac{d}{du}(\operatorname{dn} u) &= \frac{d}{du} \sqrt{1 - k^2 \operatorname{sn}^2 u} = \frac{(-2k^2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u)}{2\sqrt{1 - k^2 \operatorname{sn}^2 u}} \\ &= \frac{-k^2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u}{\operatorname{dn} u} = -k^2 \operatorname{sn} u \operatorname{cn} u\end{aligned}\quad (6)$$

Using quotient rules of differentiation and the above three fundamental results, the differential coefficients of the remaining J. elliptic functions can be determined easily as follows —

$$\frac{d}{du}(\operatorname{ns} u) = \frac{d}{du} \left(\frac{1}{\operatorname{sn} u} \right) = -\frac{1}{\operatorname{sn}^2 u} \cdot \operatorname{cn} u \operatorname{dn} u = -\operatorname{cs} u \operatorname{ds} u$$

$$\frac{d}{du}(\operatorname{nc} u) = \frac{d}{du} \left(\frac{1}{\operatorname{cn} u} \right) = -\frac{1}{\operatorname{cn}^2 u} (-\operatorname{sn} u \operatorname{dn} u) = \operatorname{sc} u \operatorname{dc} u$$

$$\frac{d}{du}(\operatorname{nd} u) = \frac{d}{du} \left(\frac{1}{\operatorname{dn} u} \right) = -\frac{1}{\operatorname{dn}^2 u} (-k^2 \operatorname{sn} u \operatorname{cn} u) = k^2 \operatorname{sd} u \operatorname{cd} u$$

$$\frac{d}{du}(\operatorname{sc} u) = \frac{d}{du} \left(\frac{\operatorname{sn} u}{\operatorname{cn} u} \right) = \frac{\operatorname{cn}^2 u \operatorname{dn} u + \operatorname{sn}^2 u \operatorname{dn} u}{\operatorname{cn}^2 u} = \operatorname{dc} u \operatorname{nc} u$$

$$\frac{d}{du}(\operatorname{cs} u) = -\operatorname{ds} u \operatorname{ns} u, \quad \frac{d}{du}(\operatorname{cd} u) = -k_c^2 \operatorname{sd} u \operatorname{nd} u,$$

$$\frac{d}{du}(\operatorname{sd} u) = \operatorname{cd} u \operatorname{nd} s, \quad \frac{d}{du}(\operatorname{ds} u) = -\operatorname{cs} u \operatorname{ns} u,$$

$$\frac{d}{du} \left(\frac{\operatorname{dn} u}{\operatorname{cn} u} \right) = k_c^2 \operatorname{sc} u \operatorname{nc} u.$$

Note that the differential coefficients of all the twelve J. elliptic functions exist for all u except that of only three functions $\operatorname{ns} u$, $\operatorname{cs} u$ and $\operatorname{ds} u$ are indeterminate at $u = 0$. Therefore, fundamental J. elliptic functions are *continuous functions* except those with reciprocal of $\operatorname{sn} u$ which is zero at $u = 0$.

Cor. If we let $y = \operatorname{sn} u$, then from equation (4)

$$\frac{dy}{du} = \sqrt{1 - \operatorname{sn}^2 u} \sqrt{1 - k^2 \operatorname{sn}^2 u} = \sqrt{(1 - y^2)(1 - k^2 y^2)}$$

which shows that $\operatorname{sn} u$ satisfies non-linear first order differential equation.

Similarly, If $y = \operatorname{cn} u$ then $dy/du = -\sqrt{(1-y^2)(k_c^2 + k^2 y^2)}$

If $y = \operatorname{dn} u$ then $dy/du = -\sqrt{(1-y^2)(y^2 - k_c^2 y^2)}$

If $y = \operatorname{sc} u$ then $dy/du = \sqrt{(1+y^2)(1+k_c^2)}$

If $y = \operatorname{sd} u$ then $dy/du = \sqrt{(1-k_c^2 y^2)(1+k^2 y^2)}$

If $y = \operatorname{cd} u$ then $dy/du = -\sqrt{(1-y^2)(1-k^2 y^2)}$
etc.

17.10 ADDITION FORMULAE

The three fundamental addition formulae for two arguments u and v are

$$\operatorname{sn}(u+v) = \frac{\operatorname{sn} u \operatorname{cn} v \operatorname{dn} v + \operatorname{sn} v \operatorname{cn} u \operatorname{dn} u}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v} \quad (\text{A})$$

$$\operatorname{cn}(u+v) = \frac{\operatorname{cn} u \operatorname{cn} v - \operatorname{sn} u \operatorname{sn} v \operatorname{dn} u \operatorname{dn} v}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v} \quad (\text{B})$$

$$\operatorname{dn}(u+v) = \frac{\operatorname{dn} u \operatorname{dn} v - k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{cn} u \operatorname{cn} v}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v} \quad (\text{C})$$

Proof of (A) : For the sake convenience, let

$U = \operatorname{sn} u$, $V = \operatorname{sn} v$ and u, v vary in such a way that $u+v = \text{constant}$
 $= b$ (say). Hence,

$$\frac{dv}{du} = -1, \text{ Also, } \dot{U} = \frac{dU}{du} = \frac{d}{du}(\operatorname{sn} u) = \operatorname{cn} u \operatorname{dn} u$$

$$\text{and } \dot{V} = \frac{dV}{du} = \frac{dV}{dv} \frac{dv}{du} = -\frac{d(\operatorname{sn} v)}{dv} = -\operatorname{cn} v \operatorname{dn} v$$

where dots denote differentiation w.r.t. u . Therefore,

$$\begin{aligned} \dot{U}^2 &= \operatorname{cn}^2 u \operatorname{dn}^2 u = (1 - \operatorname{sn}^2 u)(1 - k^2 \operatorname{sn}^2 u) \\ &= (1 - U^2)(1 - k^2 U^2) \end{aligned} \quad (1)$$

$$\text{and } \dot{V}^2 = \operatorname{cn}^2 v \operatorname{dn}^2 v = (1 - V^2)(1 - k^2 V^2) \quad (2)$$

Now differentiating (1), we have

$$U \dot{U} = -2 U \dot{U} (1 - k^2 U^2) + (1 - U^2) (-2k^2 U \dot{U})$$

$$\text{or } \dot{U} = 2k^2 U^3 - (1 + k^2) U \quad (3)$$

Similarly, differentiating (2), we get

$$\ddot{V} = 2k^2 v^3 - (1 + k^2) V \quad (4)$$

Multiplying (3) by V , (4) by U and subtracting, we have,

$$\dot{U}V - U\dot{V} = -2k^2 UV(U^2 - V^2) \quad (5)$$

Also, multiplying (1) by V^2 , (2) by U^2 and subtracting,

$$\dot{U}^2 V^2 - U^2 \dot{V}^2 = (1 - k^2 U^2 V^2)(V^2 - U^2) \quad (6)$$

$$\text{or, } (\dot{U}V - U\dot{V})(\dot{U}V + U\dot{V}) = (1 - k^2 U^2 V^2)(V^2 - U^2)$$

$$\text{or, } \dot{U}V - U\dot{V} = \frac{(1 - k^2 U^2 V^2)(V^2 - U^2)}{\dot{U}V + U\dot{V}} \quad (7)$$

Dividing (5) by (7) we get

$$\frac{\dot{U}V - U\dot{V}}{\dot{U}V + U\dot{V}} = \frac{-2k^2 UV(\dot{U}V - U\dot{V})}{1 - k^2 U^2 V^2}$$

$$\text{or, } \frac{d(\dot{U}V - U\dot{V})}{\dot{U}V - U\dot{V}} = \frac{d(1 - k^2 U^2 V^2)}{1 - k^2 U^2 V^2}$$

where $d(\dots)$ stands for differential coefficients.

Integrating,

$$\log(\dot{U}V - U\dot{V}) = \log(1 - k^2 U^2 V^2) + \text{const.}$$

$$\text{or, } \frac{\dot{U}V - U\dot{V}}{1 - k^2 U^2 V^2} = \text{const.} = C \text{ (say)} \quad (8)$$

Reviving u, v equation (8) becomes,

$$\frac{\text{sn } v \text{ cn } u \text{ dn } u + \text{cn } v \text{ dn } v \text{ sn } u}{1 - k^2 \text{sn}^2 u \text{sn}^2 v} = C \quad (9)$$

That is to say, we have two solutions of the differential equation $dv/du = -1$ namely (i) $u + v = b$ and (ii) solution (9) which contain arbitrary constants ' b ' and ' c ' respectively. By the general theory of first order differential equations, these solutions cannot be functionally independent. So, the left hand side of equation (9) is expressible as a function of $(u + v)$ i.e. let

$$\frac{\text{sn } v \text{ cn } u \text{ dn } u + \text{cn } v \text{ dn } v \text{ sn } u}{1 - k^2 \text{sn}^2 u \text{sn}^2 v} = f(u + v) \quad (10)$$

Putting $v = 0$ in (10), we get $\text{sn } u = f(u)$

(11)

Replacing u by $(u + v)$ in (11), we get

$$\text{sn } (u + v) = f(u + v)$$

(12)

Using (12) in (10) we have (A).

N.B. Let $\text{sn } u = s_1, \text{sn } v = s_2; \text{cn } u = c_1, \text{cn } v = c_2$

and

$$\text{dn } u = d_1, \text{dn } v = d_2.$$

So formulae (A), (B), (C) can be symbolised as

$$\text{sn } (u + v) = \frac{s_1 c_2 d_2 + s_2 c_1 d_1}{1 - k^2 s_1^2 s_2^2} \quad (A_1)$$

$$\text{cn } (u + v) = \frac{c_1 c_2 - s_1 s_2 d_1 d_2}{1 - k^2 s_1^2 s_2^2} \quad (B_1)$$

$$\text{dn } (u + v) = \frac{d_1 d_2 - k^2 s_1 s_2 c_1 c_2}{1 - k^2 s_1^2 s_2^2} \quad (C_1)$$

Proof of (B) : Using (A_1) the formulae (B_1) and (C_1) can be easily established. First establish (B_1) . We know,

$$\text{cn}^2(u + v) = 1 - \text{sn}^2(u + v)$$

$$\text{or, } (1 - k^2 s_1^2 s_2^2)^2 \text{cn}^2(u + v) = (1 - k^2 s_1^2 s_2^2)^2 (1 - \text{sn}^2(u + v))$$

$$= (1 - k^2 s_1^2 s_2^2)^2 - (s_1 c_2 d_2 + s_2 c_1 d_1)^2 \text{ by } (A_1)$$

$$= 1 - 2k^2 s_1^2 s_2^2 + k^4 s_1^4 s_2^4 - s_1^2 c_2^2 d_2^2 - s_2^2 c_1^2 d_1^2 - 2s_1 s_2 c_1 c_2 d_1 d_2$$

$$= 1 - 2k^2 s_1^2 s_2^2 + k^4 s_1^4 s_2^4 - s_1^2 (1 - s_2^2) (1 - k^2 s_2^2)$$

$$- s_2^2 (1 - s_1^2) (1 - k^2 s_1^2) - 2s_1 s_2 c_1 c_2 d_1 d_2$$

$$= (1 - s_1^2) (1 - s_2^2) + s_1^2 s_2^2 (1 - k^2 s_1^2) (1 - k^2 s_2^2) - 2s_1 s_2 c_1 c_2 d_1 d_2$$

$$= c_1^2 c_2^2 + s_1^2 s_2^2 d_1^2 d_2^2 - 2s_1 s_2 c_1 c_2 d_1 d_2$$

$$= (c_2 c_2 - r_1 r_2 d_1 d_2)^2$$

or,

$$\text{cn } (u + v) = \pm \frac{c_1 c_2 - s_1 s_2 d_1 d_2}{1 - k^2 s_1^2 s_2^2}$$

Either $u = 0$ or $v = 0$, shows that +ve sign is true. So, (B_1) and hence (B) is established.

Proof of (C) : Using (A_1) and the identity

$$\operatorname{dn}^2(u+v) = 1 - k^2 \operatorname{sn}^2(u+v)$$

with some algebraic simplifications as (B), (C) can be easily established.

Cor. 1. As in plane trigonometry, J. elliptic functions for multiple and submultiple arguments can be deduced easily. Thus, using $v=u$ in (A), (B), (C) we get

$$S = \operatorname{sn} 2u = \frac{2 \operatorname{sn} u - \operatorname{cn} u \operatorname{dn} u}{1 - k^2 \operatorname{sn}^4 u} = \frac{2 s_1 c_1 d_1}{1 - k^2 s_1^4} \quad (13)$$

where

$$s_1 = \operatorname{sn} u, c_1 = \operatorname{cn} u \text{ and } d_1 = \operatorname{dn} u.$$

$$\begin{aligned} C = \operatorname{cn} 2u &= \frac{\operatorname{cn}^2 u - \operatorname{sn}^2 u \operatorname{dn}^2 u}{1 - k^2 \operatorname{sn}^4 u} \\ &= \frac{1 - 2 \operatorname{sn}^2 u + k^2 \operatorname{sn}^4 u}{1 - k^2 \operatorname{sn}^4 u} \\ &= (1 - 2s_1^2 + k^2 s_1^4)/(1 - k^2 s_1^4) \end{aligned} \quad (14)$$

$$\begin{aligned} D = \operatorname{dn} 2u &= \frac{\operatorname{dn}^2 u - k^2 \operatorname{sn}^2 u \operatorname{cn}^2 u}{1 - k^2 \operatorname{sn}^4 u} \\ &= \frac{1 - 2k^2 s_1^2 + k^2 s_1^4}{1 - k^2 s_1^4} \end{aligned} \quad (15)$$

Also using $\frac{1}{2}u$ for u in (13), (14), (15) we get

$$\operatorname{sn} u = \frac{2 \operatorname{sn}(u/2) \operatorname{cn}(u/2) \operatorname{dn}(u/2)}{1 - k^2 \operatorname{sn}^4(u/2)} \quad (16)$$

$$\operatorname{cn} u = \frac{1 - 2 \operatorname{sn}^2(u/2) + k^2 \operatorname{sn}^4(u/2)}{1 - k^2 \operatorname{sn}^4(u/2)} \quad (17)$$

$$\operatorname{dn} u = \frac{1 - 2k^2 \operatorname{sn}^2(u/2) + k^2 \operatorname{sn}^4(u/2)}{1 - k^2 \operatorname{sn}^4(u/2)} \quad (18)$$

Cor. 2. From (16), (17), (18)

$$\operatorname{sn}^2(u/2) = \frac{1 - \operatorname{cn} u}{1 + \operatorname{dn} u}, \quad (19)$$

$$\operatorname{cn}^2(u/2) = \frac{\operatorname{dn} u + \operatorname{cn} u}{1 + \operatorname{dn} u}, \quad (20)$$

$$\text{and} \quad \operatorname{dn}^2(u/2) = \frac{k_c^2 + \operatorname{dn} u + k^2 \operatorname{cn} u}{1 + \operatorname{dn} u} \quad (21)$$

These can also be derived directly from (13), (14), (15) and using $\frac{1}{2}u$ for u .

Cor. 3. From (17) and (16), we have,

$$\begin{aligned}\frac{1 - \operatorname{cn} u}{\operatorname{sn} u} &= \frac{2 \operatorname{sn}^2(u/2) (1 - k^2 \operatorname{sn}^2(u/2))}{2 \operatorname{sn}(u/2) \operatorname{cn}(u/2) \operatorname{dn}(u/2)} = \frac{\operatorname{sn}(u/2) \operatorname{dn}^2(u/2)}{\operatorname{cn}(u/2) \operatorname{dn}(u/2)} \\ &= \operatorname{sn}(u/2) \operatorname{dc}(n/2)\end{aligned}\quad (22)$$

Similarly, from (16), (18) and from (16), (17) and (18).

$$\frac{1 + \operatorname{dn} u}{\operatorname{sn} u} = \operatorname{ds}(u/2) \operatorname{nc}(u/2) \quad (23)$$

$$\frac{\operatorname{dn} u + \operatorname{cn} u}{\operatorname{cn} u} = \operatorname{cn}(u/2) \operatorname{ds}(u/2) \quad (24)$$

Cor. 4. From sec. 17.6, we know $\operatorname{sn} K = 1$, $\operatorname{cn} K = 0$, $\operatorname{dn} K = k_c$.

Therefore, using $v = K$ in (A), (B), (C), we get,

$$\operatorname{sn}(u + K) = \frac{\operatorname{cn} u \operatorname{dn} u}{1 - k^2 \operatorname{sn}^2 u} = \frac{\operatorname{cn} u \operatorname{dn} u}{\operatorname{dn}^2 u} = \frac{\operatorname{cn} u}{\operatorname{dn} u} \quad (25)$$

$$\operatorname{cn}(u + K) = \frac{-\operatorname{sn} u \operatorname{dn} u}{\operatorname{dn}^2 u} = -k_c \frac{\operatorname{sn} u}{\operatorname{dn} u} \quad (26)$$

$$\operatorname{dn}(u + K) = k_c / \operatorname{dn} u \quad (27)$$

17.11. PERIODS OF J. ELLIPTIC FUNCTIONS

In the formulae (25), (26), (27) of cor. 4 sec. 17.10, let us replace u by $(u + K)$, then we find, by using them again,

$$\operatorname{sn}(u + 2K) = \frac{\operatorname{cn}(u + K)}{\operatorname{dn}(u + K)} = -k_c \frac{\operatorname{sn} u}{\operatorname{dn} u} \cdot \frac{\operatorname{dn} u}{k_c} = -\operatorname{sn} u \quad (1)$$

$$\operatorname{cn}(u + 2K) = -k_c \frac{\operatorname{sn}(u + K)}{\operatorname{dn}(u + K)} = -k_c \frac{\operatorname{cn} u \operatorname{dn} u}{\operatorname{dn} u} \cdot \frac{\operatorname{dn} u}{k_c} = -\operatorname{cn} u \quad (2)$$

$$\operatorname{dn}(u + 2K) = \frac{k_c}{\operatorname{dn}(u + K)} = k_c \frac{\operatorname{dn} u}{k_c} = \operatorname{dn} u \quad (3)$$

Equation (3) shows that $\operatorname{dn} u$ is periodic with first fundamental period $2K$. Replacing u by $(u + 2K)$ in (1), (2), (3) above and using them again, we find,

$$\operatorname{sn}(u + 4K) = -\operatorname{sn}(u + 2K) = \operatorname{sn} u \quad (4)$$

$$\operatorname{cn}(u + 4K) = -\operatorname{cn}(u + 2K) = \operatorname{cn} u \quad (5)$$

$\therefore \operatorname{sn} u$ and $\operatorname{cn} u$ are each a periodic function with first fundamental period $4K$.

Again,

$$K = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \quad (6)$$

and
$$K_c = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k_c^2 \sin^2 \theta}} \quad (7)$$

Let $\sqrt{1 - k_c^2 \sin^2 \theta} = \operatorname{cosec} \beta$ in (7). Then differentiating,

$$\frac{-2k_c^2 \sin \theta \cos \theta}{2\sqrt{1 - k_c^2 \sin^2 \theta}} d\theta = -\operatorname{cosec} \beta \cot \beta d\beta$$

or
$$\frac{d\theta}{\sqrt{1 - k_c^2 \sin^2 \theta}} = \frac{\cos \beta d\beta}{k_c^2 \sqrt{\left(\frac{1 - \operatorname{cosec}^2 \beta}{k_c^2} \cdot \frac{1 - (1 - k_c^2) \sin^2 \beta}{k_c^2 \sin^2 \beta} \right) \sin^2 \beta}}$$

$$= \frac{d\beta}{i \sqrt{1 - k^2 \sin^2 \beta}}$$

and

θ	0	$\pi/2$
β	$\pi/2$	$\sin \beta = 1/k$

$\therefore$ From equation (7)

$$K_c = \frac{1}{i} \int_{\beta=\pi/2}^{\sin \beta = 1/k} \frac{d\beta}{\sqrt{1 - k^2 \sin^2 \beta}} \quad (8)$$

$\therefore$ From (6) and (8)

$$K + iK_c = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} + \int_{\pi/2}^{1/k} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}$$

$$= \int_0^{1/k} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}$$

Hence $\operatorname{sn}(K + iK_c) = 1/k$, (by def. of sn) (9)

and $\operatorname{cn}(K + iK_c) = -i \frac{k_c}{k}$ (10)

[as $0 < k < 1$, so, $1/k > 1$. $\operatorname{cn}(K + iK_c)$ is the value of $\sqrt{1 - x^2}$ when x follows the path joining 0 to $1 - \delta$ along x -axis, the upper semi-circle with radius δ and centre at 1, and again $1 + \delta$ to $1/k$ along x -axis. Its value is $-ik_c/k$ determined by contour integration of complex variable, beyond the scope of this book. This result can also be obtained from equation (19) of this see easily. Before that we need the value.]

and
$$\begin{aligned} \operatorname{dn}(K + iK_c) &= \sqrt{1 - k^2 \operatorname{sn}^2(K + iK_c)} \\ &= \sqrt{1 - k^2 \cdot 1/k^2} = 0 \end{aligned} \quad (11)$$

Now using (9) to (11) in addition formulae (A), (B), (C) i.e., replacing v by $(K + iK_c)$ in (A), (B), (C), we get

$$\operatorname{sn}(u + K + iK_c) = \frac{(1/k) \operatorname{cn} u \operatorname{dn} u}{1 - \operatorname{sn}^2 u} = \frac{\operatorname{dn} u}{k \operatorname{cn} u} \quad (12)$$

$$\operatorname{cn}(u + K + iK_c) = \frac{\operatorname{cn} u (-i k_c/k)}{1 - \operatorname{sn}^2 u} = -\frac{i k_c}{k \operatorname{cn} u} \quad (13)$$

$$\begin{aligned} \operatorname{dn}(u + K + iK_c) &= \frac{-k^2 \operatorname{sn} u (1/k) (-i k_c/k) \operatorname{cn} u}{1 - \operatorname{sn}^2 u} \\ &= \frac{i k_c \operatorname{sn} u}{\operatorname{cn} u} \end{aligned} \quad (14)$$

and replacing u by $(u + K + iK_c)$ in (12) to (14) and using them again we find,

$$\begin{aligned} \operatorname{sn}(u + 2K + 2iK_c) &= \frac{\operatorname{dn}(u + K + iK_c)}{k \operatorname{cn}(u + K + iK_c)} \\ &= \frac{i k_c \operatorname{sn} u}{k \operatorname{cn} u} \cdot \frac{k \operatorname{cn} u}{-i k_c} \\ &= -\operatorname{sn} u \end{aligned} \quad (15)$$

$$\begin{aligned} \operatorname{cn}(u + 2K + 2iK_c) &= -\frac{i k_c}{k \operatorname{cn}(u + K + iK_c)} \\ &= -\frac{i k_c}{k} \cdot \frac{k \operatorname{cn} u}{-i k_c} = \operatorname{cn} u \end{aligned} \quad (16)$$

$\therefore \operatorname{cn} u$ is periodic with *second fundamental period* $(2K + 2iK_c)$

$$\begin{aligned} \operatorname{dn}(u + 2K + 2iK_c) &= \frac{i k_c \operatorname{sn}(u + K + iK_c)}{\operatorname{cn}(u + K + iK_c)} \\ &= i k_c \frac{\operatorname{dn} u}{k \operatorname{cn} u} \cdot \frac{k \operatorname{cn} u}{-i k_c} = -\operatorname{dn} u \end{aligned} \quad (17)$$

By a similar method, we can prove that

$$\operatorname{sn}(u + iK_c) = \operatorname{sn}\left(\overline{u - K} + \overline{K + iK_c}\right)$$

$$\begin{aligned}
& [\operatorname{sn}(u-K) \operatorname{cn}(K+iK_c) \operatorname{dn}(K+iK_c) + \operatorname{sn}(K+iK_c)] \\
&= \frac{\operatorname{cn}(u-K) \operatorname{dn}(u-K)}{[1-k^2 \operatorname{sn}^2(u-K) \operatorname{sn}^2(K+iK_c)]} \quad \text{by (A)} \\
&= \frac{(1/k) \operatorname{cn}(u-K) \operatorname{dn}(u-K)}{1-\operatorname{sn}^2(u-K)} = \frac{\operatorname{dn}(u-K)}{k \operatorname{cn}(u-K)} \quad \text{by (9) - (11)} \\
&= \frac{\operatorname{dn} u \operatorname{dn} K + k^2 \operatorname{sn} u \operatorname{sn} K \operatorname{cn} u \operatorname{cn} K}{k(\operatorname{cn} u \operatorname{cn} K + \operatorname{sn} u \operatorname{sn} K \operatorname{dn} u \operatorname{dn} K)} \quad \text{by (A) and by sec. 17.8} \\
&= \frac{k_c \operatorname{dn} u + k^2 \operatorname{sn} u \operatorname{cn} u \cdot 0}{k(\operatorname{cn} u \cdot 0 + \operatorname{sn} u \cdot 1 \operatorname{dn} u \cdot k_c)} \quad \text{by sec. 17.6} \\
&= \frac{1}{k \operatorname{sn} u} \tag{18}
\end{aligned}$$

$$\begin{aligned}
\operatorname{cn}(u+iK_c) &= \operatorname{cn}(\overline{u-K} + K+iK_c) \\
&= \frac{\operatorname{cn}(u-K) \operatorname{cn}(K+iK_c) - \operatorname{sn}(u-K) \operatorname{sn}(K+iK_c) \operatorname{dn}(u-K) \operatorname{dn}(K+iK_c)}{1-k^2 \operatorname{sn}^2(u-K) \operatorname{sn}^2(K+iK_c)} \\
&= \frac{-(ik_c/k) \operatorname{cn}(u-K)}{\operatorname{cn}^2(u-K)} = \frac{-ik_c}{k \operatorname{cn}(u-K)} \quad \text{by (10), (11)} \\
&= \frac{-ik_c(1-k^2 \operatorname{sn}^2 u \operatorname{sn}^2 K)}{k(\operatorname{cn} u \operatorname{cn} K + \operatorname{sn} u \operatorname{sn} K \operatorname{dn} u \operatorname{dn} K)} \\
&= \frac{-ik_c \operatorname{dn}^2 u}{k(\operatorname{sn} u \operatorname{dn} u) k_c} = -\frac{i \operatorname{dn} u}{k \operatorname{sn} u} \tag{19}
\end{aligned}$$

$$\begin{aligned}
\operatorname{dn}(u+iK_c) &= \operatorname{dn}(\overline{u-K} + K+iK_c) \\
&= \frac{\operatorname{dn}(u-K) \operatorname{dn}(K+iK_c) - k^2 \operatorname{sn}(u-K) \operatorname{sn}(K+iK_c) \operatorname{cn}(u-K) \operatorname{cn}(K+iK_c)}{1-k^2 \operatorname{sn}^2(u-K) \operatorname{sn}^2(K+iK_c)} \\
&= \frac{-k^2 \operatorname{sn}(u-K) (1/k) \operatorname{cn}(u-K) (-ik_c/k)}{\operatorname{cn}^2(u-K)} \\
&= ik_c \frac{\operatorname{sn}(u-K)}{\operatorname{cn}(u-K)} = -i \frac{\operatorname{cn} u}{\operatorname{sn} u}
\end{aligned}$$

$$\text{Also, } \operatorname{sn}(u+2iK_c) = \frac{1}{k_{\operatorname{sn}}(u+iK_c)} = \frac{k \operatorname{sn} u}{k} = \operatorname{sn} u \tag{21}$$

$\therefore \operatorname{sn} u$ is periodic with a second fundamental period $2iK_c$.

$$\begin{aligned}
\operatorname{cn}(u+2iK_c) &= -\frac{i \operatorname{dn}(u+iK_c)}{k \operatorname{sn}(u+iK_c)} = \frac{i}{k} \cdot \frac{-i \operatorname{cn} u}{\operatorname{sn} u} \cdot k \operatorname{sn} u \\
&= -\operatorname{cn} u \tag{22}
\end{aligned}$$

$$\begin{aligned} \operatorname{dn}(u + 2iK_c) &= -\frac{i \operatorname{cn}(u + iK_c)}{\operatorname{sn}(u + iK_c)} = i \left(-\frac{i}{k} \frac{\operatorname{dn} u}{\operatorname{sn} u} \right) \cdot k \operatorname{sn} u \\ &= -\operatorname{dn} u \end{aligned} \quad (23)$$

$$\text{Again, } \operatorname{sn}(u + 4iK_c) = \operatorname{sn} u \quad (24)$$

$$\operatorname{cn}(u + 4iK_c) = \operatorname{cn} u \quad (25)$$

$$\operatorname{dn}(u + 4iK_c) = -\operatorname{dn}(u + 2iK_c) = \operatorname{dn} u \quad (26)$$

$\therefore$ the second fundamental period of $\operatorname{dn} u$ is $4iK_c$.

Hence in the conclusion, the functions $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$ are **DOUBLY** periodic, meaning two fundamental periods; $\operatorname{sn} u$ has first and second fundamental periods $4K$ and $2iK_c$ respectively; $\operatorname{cn} u$ has first and second fundamental periods $4K$ and $(2K_c + 2iK_c)$ respectively; $\operatorname{dn} u$ has first and second fundamental periods $2K$ and $4iK_c$ respectively. This is the *principle difference* of J. elliptic functions with plane trigonometric functions.

Cor : If $k = 1$ in equation (8) of sec. 17.2, then

$$u = \int_0^x \frac{dv}{1-v^2} = \tanh^{-1} x$$

Thus $x = \operatorname{sn} u = \tanh u$. Therefore

$$\operatorname{cn} u = \sqrt{1 - \operatorname{sn}^2 u} = \sqrt{1 - \tanh^2} = \operatorname{sech} u,$$

$$\operatorname{dn} u = \operatorname{sech} u \text{ and } k_c = \pi/2 \text{ i.e., period } 4iK_c = 2\pi i$$

If $k = 0$ in equation (8) of sec. 17.2, then

$$u = \int_0^x \frac{dv}{\sqrt{1-v^2}} = \sin^{-1} x$$

$\therefore x = \operatorname{sn} u = \sin u$, $\operatorname{cn} u = \cos u$, $\operatorname{dn} u = 1$ and $K = \pi/2$ i.e., period $4K = 2\pi$.

These show that J. elliptic functions are the generalization of plane trigonometric functions – circular and hyperbolic.

17.12. INTEGRATION OF J. ELLIPTIC FUNCTIONS

Integration of J. elliptic fundamental functions is not so easy as the elementary plane trigonometric functions. Because the differential coefficients of a J. elliptic fundamental function does not produce another such function (sec. 17.9). Let us first integrate the three fundamental J. elliptic functions. Now,

$$\begin{aligned} \int \operatorname{sn} u \, du &= \int \sin \phi \, d\phi \quad \because \operatorname{sn} u = \phi = \sin(\operatorname{am} u) \\ &= \int \frac{\sin \phi}{\sqrt{1 - k^2 \sin^2 \phi}} \, d\phi, \text{ by equation (3) sec. 17.9} \end{aligned}$$

$$\begin{aligned}
&= \int \frac{\sin \phi}{\sqrt{k^2 c^2 + k^2 \cos^2 \phi}} d\phi \\
&= -(1/k) \int \sec x dx \text{ when } k \cos \phi = k_c \tan x \\
&= -(1/k) \log (\sec x + \tan x) + c_1 \\
&= -(1/k) \log (\sqrt{1 - k^2 \sin^2 \phi} + \tan \phi) + c \\
&= -(1/k) \log (\sqrt{1 - k^2 \sin^2 u} + k \operatorname{cn} u) + c \\
&= -1/k \log (\operatorname{dn} u + k \operatorname{cn} u) + c \quad (1)
\end{aligned}$$

This result can be simplified to

$$\int \operatorname{sn} u du = (1/k) \log (\operatorname{dn} u - k \operatorname{cn} u) + c \quad (1a)$$

Similarly,

$$\begin{aligned}
\int \operatorname{cn} u du &= \int \cos \phi du = \int \frac{\cos \phi d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} \\
&= \frac{1}{k} \int \frac{dx}{\sqrt{1 - x^2}} \text{ when } k \sin \phi = x \\
&= (1/k) \sin^{-1} x + c = (1/k) \sin^{-1} (k \sin \phi) + c \\
&= (1/k) \sin^{-1} (k \operatorname{sn} u) + c \quad (2)
\end{aligned}$$

$$\begin{aligned}
\text{And } \int \operatorname{dn} u du &= \int \frac{\operatorname{cn} u \operatorname{dn} u}{\operatorname{cn} u} du = \int \frac{\operatorname{cn} u \operatorname{dn} u}{\sqrt{1 - \operatorname{sn}^2 u}} du \\
&= \int \frac{dx}{\sqrt{1 - x^2}} \text{ when } x = \operatorname{sn} u \\
&= \sin^{-1} x + c = \sin^{-1} (\operatorname{sn} u) + c.
\end{aligned}$$

Also from equations (4), (5), (6) of sec. 17.9, it is easily seen that

$$\begin{aligned}
\int \operatorname{cn} u \operatorname{dn} u du &= \operatorname{sn} u + c \\
\int \operatorname{sn} u \operatorname{dn} u du &= -\operatorname{cn} u + c \\
\int \operatorname{sn} u \operatorname{cn} u du &= (-1/k^2) \operatorname{dn} u + c.
\end{aligned}$$

Similarly, integral of other products of J. elliptic functions can be obtained.

Examples

1. Evaluate in terms of elliptic integrals :

$$\int_8^{12} \frac{dx}{\sqrt{(x-2)(x-4)(x-6)}}.$$

This is of the type 17.4 (c) (vi), so, let $(x - 6) = y^2$

$$\begin{aligned} \therefore \int_8^{12} \frac{dx}{\sqrt{(x-2)(x-4)(x-6)}} &= 2 \int_{\sqrt{2}}^{\sqrt{6}} \frac{y dy}{\sqrt{(y^2+4)(y^2+2)y^2}} \\ &= 2 \int_{\pi/4}^{\pi/3} \frac{\sqrt{2} \sec^2 \theta d\theta}{\sqrt{(2 \tan^2 \theta + 4) 2 \sec^2 \theta}} \quad \text{where } y = \sqrt{2} \tan \theta \\ &= \sqrt{2} \int_{\pi/4}^{\pi/3} \frac{d\theta}{\sqrt{2 \sin^2 \theta + 4 \cos^2 \theta}} = \sqrt{2} \int_{\pi/4}^{\pi/3} \frac{d\theta}{\sqrt{4 - 2 \sin^2 \theta}} \\ &= \frac{1}{\sqrt{2}} \int_{\pi/4}^{\pi/3} \frac{d\theta}{\sqrt{1 - (1/\sqrt{2}) \sin^2 \theta}} = \frac{1}{\sqrt{2}} \int_0^{\pi/3} \frac{d\theta}{\sqrt{1 - (1/\sqrt{2})^2 \sin^2 \theta}} \\ &\quad - \int_0^{\pi/4} \frac{d\theta}{\sqrt{1 - (1/\sqrt{2})^2 \sin^2 \theta}} \\ &= (1/\sqrt{2}) [F(\pi/3, 1/\sqrt{2}) - F(\pi/4, 1/\sqrt{2})] \end{aligned}$$

2. Show that
$$\int_0^\infty \frac{dx}{\sqrt{(x+1)(x+3)(x+5)(x+7)}}$$

$$= \frac{1}{2} \left[F\left(\sqrt{3}/2, \cos^{-1} \frac{1}{\sqrt{15}}\right) - F\left(\sqrt{3}/2, \cos^{-1} 1/\sqrt{3}\right) \right]$$

This is of type 17.4 (D) (i). Let the BLT be $x = (At + B)/(Ct + D)$. Three x -values in ascending order are $x = -7, -5, -3$ which must corresponds to $t = 0, 1, \infty$ in order to determine the constants. Let us calculate A, B, C, D .

$$-7 = B/D \quad \text{when } x = -7, t = 0 \text{ giving } B = -7D \quad (1)$$

$$-5 = (A + B)/(C + D) \quad \text{when } x = -5, t = 1$$

$$\text{giving } -5(C + D) = A + B \quad (2)$$

$$-3 = A/C \quad \text{when } x = -3, t = \infty \text{ giving } A = -3C \quad (3)$$

Using (1), (2), (3) we get

$$x = \frac{-3t - 7}{t + 1} \quad \text{and} \quad dx = \frac{4}{(t + 1)^2} dt$$

Also,

x	0	∞
t	$-7/3$	-1

$$\therefore I = \int_0^\infty \frac{dx}{\sqrt{(x+1)(x+3)(x+5)(x+7)}}$$

$$\begin{aligned}
 &= 4 \int_{-7/3}^{-1} \frac{dt}{\sqrt{(-2t-6)(-4)(2t-2)(4t)}} \\
 &= \frac{1}{2} \int_{-7/3}^{-1} \frac{dt}{\sqrt{(t+3)t(t-1)}}
 \end{aligned}$$

Since the limits are negative, let us use BLT.

$t = (A_1 u + B_1)/(C_1 u + D_1)$ where ascending orders of zeros

$t = -3, 0, 1$ corresponds to $u = 0, 1, \infty$. Therefore

$$-3 = B_1/D_1 \text{ when } t = -3, u = 0 \text{ giving } B_1 = -3D_1 \quad (4)$$

$$0 = (A_1 + B_1)/(C_1 + D_1) \text{ when } t = 0, u = 1$$

$$\text{giving } A_1 = -B_1 = 3D_1 \quad (5)$$

$$1 = A_1/C_1 \text{ when } t = 1, u = \infty \text{ giving } C_1 = A_1 = 3D_1 \quad (6)$$

Using (4), (5), (6) the new BLT is

$$t = (3u - 3)/(3u + 1). \text{ Also, } dt = \frac{12}{(3u + 1)^2} du \text{ and}$$

x	$-7/3$	-1
t	$1/5$	$1/3$

Therefore,

$$\begin{aligned}
 I &= \frac{1}{2} \int_{1/5}^{1/3} \frac{du}{\sqrt{3u+1} \sqrt{u(1-u)}} \\
 &= -\frac{1}{2} \int_{\theta = \cos^{-1} \frac{1}{\sqrt{15}}}^{\cos^{-1} \frac{1}{\sqrt{3}}} \frac{d\theta}{\sqrt{1 - (\sqrt{3}/2)^2 \sin^2 \theta}}, \text{ when } u = \cos^2 \theta \\
 &= \frac{1}{2} \int_{\theta = \cos^{-1} \frac{1}{\sqrt{3}}}^{\cos^{-1} \frac{1}{\sqrt{15}}} \frac{d\theta}{\sqrt{1 - (\sqrt{3}/2)^2 \sin^2 \theta}} \\
 &= \frac{1}{2} \left[\int_0^{\cos^{-1} \frac{1}{\sqrt{15}}} \frac{d\theta}{\sqrt{1 - (\sqrt{3}/2)^2 \sin^2 \theta}} - \int_0^{\cos^{-1} \frac{1}{\sqrt{3}}} \frac{d\theta}{\sqrt{1 - (\sqrt{3}/2)^2 \sin^2 \theta}} \right] \\
 &= \frac{1}{2} \left[F(\cos^{-1}(1/\sqrt{15}), (\sqrt{3}/2)) - F(\cos^{-1}(1/\sqrt{3}), (\sqrt{3}/2)) \right].
 \end{aligned}$$

$$3. \text{ Show that } \int_0^\infty \frac{dx}{\sqrt{x^4 + x^2 + 1}} = \frac{4}{3} F\left(\frac{2\sqrt{2}}{3}, \frac{\pi}{3}\right).$$

$$\text{Let } I = \int_0^{\infty} \frac{dx}{\sqrt{x^4 + x^2 + 1}} = \int_0^{\infty} \frac{dx}{\sqrt{(x^2 - x + 1)(x^2 + x + 1)}}$$

Let $Q_1 = x^2 + x + 1$ and $Q_2 = x^2 - x + 1$. Therefore choose λ such that

$$Q_1 - \lambda Q_2 = (1 - \lambda)x^2 + (1 + \lambda)x + (1 - \lambda) \text{ is a perfect square.}$$

The required condition is

$$(1 + \lambda)^2 = 4(1 - \lambda)^2$$

$$\text{or, } 3\lambda^2 - 10\lambda - 3 = 0 \quad \text{or, } \lambda = -1/3, 3.$$

$$\therefore Q_1 - \frac{1}{3}Q_2 = \frac{2}{3}x^2 + \frac{4}{3}x + \frac{2}{3} = \frac{2}{3}(x + 1)^2$$

$$\text{and } Q_1 - 3Q_2 = -2x^2 + 4x - 2 = -2(x - 1)^2$$

Solving for Q_1 and Q_2 , we have

$$Q_1 = \frac{1}{4}[(x - 1)^2 + 3(x + 1)^2] \quad \text{and}$$

$$Q_2 = \frac{1}{4}[3(x - 1)^2 + (x + 1)^2]$$

$$\begin{aligned} \therefore I &= 4 \int_0^{\infty} \frac{dx}{\sqrt{[(x - 1)^2 + 3(x + 1)^2][3(x - 1)^2 + (x + 1)^2]}} \\ &= 4 \int_0^{\infty} \frac{dx}{(x + 1)^2 \sqrt{\left[\left(\frac{x - 1}{x + 1}\right)^2 + 3\right] \left[3\left(\frac{x - 1}{x + 1}\right)^2 + 1\right]}} \end{aligned}$$

$$\text{Let } t = \frac{x - 1}{x + 1} \quad \therefore dt = \frac{2}{(x + 1)^2} dx$$

and

x	0	∞
t	-1	1

 . Therefore,

$$\begin{aligned} I &= 2 \int_{-1}^1 \frac{dt}{\sqrt{(t^2 + 3)(3t^2 + 1)}} = \frac{2}{3} \int_{-\pi/3}^{\pi/3} \frac{d\theta}{\sqrt{3 - (8/3)\sin^2 \theta}} \\ &\quad \text{where } t = (1/\sqrt{3}) \tan \theta \\ &= \frac{2}{3} \int_{-\pi/3}^{\pi/3} \frac{d\theta}{\sqrt{1 - (2\sqrt{2}/3)^2 \sin^2 \theta}} = (4/3) \int_0^{\pi/3} \frac{d\theta}{\sqrt{1 - (2\sqrt{2}/3)^2 \sin^2 \theta}} \\ &= \frac{4}{3} F\left(\frac{2\sqrt{2}}{3}, \frac{\pi}{3}\right) \end{aligned}$$

4. Prove that

$$\int_0^1 \sin^{-1} x^2 dx = \frac{\pi}{2} - 2\sqrt{2} E\left(\frac{1}{\sqrt{2}}, \frac{\pi}{2}\right) + \sqrt{2} F\left(\frac{1}{\sqrt{2}}, \frac{\pi}{2}\right)$$

$$\begin{aligned}
 \text{Let } I &= \int_0^1 \sin^{-1} x^2 \, dx \\
 &= x \sin^{-1} x^2 \Big|_{x=0}^1 - \int_0^1 \frac{2x^2}{\sqrt{1-x^4}} \, dx, \text{ integrating by parts} \\
 &= (\pi/2) - I_1 \quad (1)
 \end{aligned}$$

Here $I_1 = \int_0^1 \frac{2x^2}{\sqrt{(1-x^2)(1+x^2)}} \, dx$. It is an improper integral of the second kind. It is of the type with result :

$$\int_u^b \frac{x^2 \, dx}{\sqrt{(a^2+x^2)(b^2-x^2)}} = \sqrt{a^2+b^2} E(\gamma, \gamma) - \frac{a^2}{\sqrt{a^2+b^2}} E(\gamma, \gamma),$$

$b > u \geq 0$.

where $\cos \gamma = \frac{u}{b}$, $r = \frac{b}{\sqrt{a^2+b^2}}$ (Not derived in the book).

Now to evaluate I_1 ,

Let $x = \cos \gamma$

$$dx = -\sin \gamma \, d\gamma$$

$$\begin{aligned}
 \therefore I_1 &= - \int_{\pi/2}^0 \frac{2 \cos^2 \gamma \sin \gamma \, d\gamma}{\sqrt{(1-\cos^2 \gamma)(1+\cos^2 \gamma)}} = 2 \int_0^{\pi/2} \frac{\cos^2 \gamma}{\sqrt{1+\cos^2 \gamma}} \, d\gamma \\
 &= 2 \int_0^{\pi/2} \frac{(1+\cos^2 \gamma) - 1}{\sqrt{1+\cos^2 \gamma}} \, d\gamma \\
 &= 2 \int_0^{\pi/2} \sqrt{1+\cos^2 \gamma} \, d\gamma - 2 \int_0^{\pi/2} \frac{1}{\sqrt{1+\cos^2 \gamma}} \, d\gamma \\
 &= 2 \int_0^{\pi/2} \sqrt{2-\sin^2 \gamma} \, d\gamma - 2 \int_0^{\pi/2} \frac{d\gamma}{\sqrt{2-\sin^2 \gamma}} \\
 &= 2\sqrt{2} \int_0^{\pi/2} \sqrt{1-(1/\sqrt{2})^2 \sin^2 \gamma} \, d\gamma \\
 &\quad - \sqrt{2} \int_0^{\pi/2} \frac{d\gamma}{\sqrt{1-(1/\sqrt{2})^2 \sin^2 \gamma}} \\
 &= 2\sqrt{2} E(1/\sqrt{2}, \pi/2) - \sqrt{2} F(1/\sqrt{2}, \pi/2) \\
 \therefore I &= \pi/2 - 2\sqrt{2} E(1/\sqrt{2}, \pi/2) + \sqrt{2} F(1/\sqrt{2}, \pi/2)
 \end{aligned}$$

5. Evaluate $\int_0^x \frac{F(x, k)}{\sqrt{1-k^2 \sin^2 x}} \, dx$, $0 < x \leq \pi/2$.

$$\begin{aligned}
 \text{Let } I &= \int_0^x \frac{F(x, k)}{\sqrt{1-k^2 \sin^2 x}} dx, \quad 0 < x \leq \pi/2 \\
 &= F(x, k) \int_0^x \frac{dx}{\sqrt{1-k^2 \sin^2 x}} \\
 &\quad - \int_0^x \left[F'(x, k) \int_0^x \frac{dx}{\sqrt{1-k^2 \sin^2 x}} \right] dx \\
 &\quad \text{integrating by parts} \\
 &= [F(x, k)]^2 - \int_0^x \frac{F(x, k)}{\sqrt{1-k^2 \sin^2 x}} dx \\
 &\quad \therefore \text{by definition } F(x, k) = \int_0^x \frac{dx}{\sqrt{1-k^2 \sin^2 x}} \\
 &\quad \therefore F'(x, k) = 1/\sqrt{1-k^2 \sin^2 x}
 \end{aligned}$$

$$\therefore I = [F(x, k)]^2 - I$$

$$\text{or } I = \frac{1}{2} [F(x, k)]^2$$

$$6. \text{ Prove that } \int_0^1 F(x, k) k dk = \frac{1 - \cos x}{\sin x} = \tan(x/2)$$

$$\begin{aligned}
 \int_0^1 F(x, k) k dk &= \int_0^1 \left(\int_0^x \frac{d\theta}{\sqrt{1-k^2 \sin^2 \theta}} \right) k dk, \\
 &\quad \text{by def. of } F(x, k), \\
 &= \int_0^x \left(\int_0^1 \frac{k}{\sqrt{1-k^2 \sin^2 \theta}} dk \right) d\theta, \text{ by interchange of the order} \\
 &\quad \text{of integrals.}
 \end{aligned}$$

$$= \int_0^x \left[-\frac{1}{2} (2/\sin^2 \theta) \sqrt{1-k^2 \sin^2 \theta} \right]_{k=0}^1 d\theta$$

$$= - \int_0^x (\cos \theta / \sin^2 \theta - \operatorname{cosec}^2 \theta) d\theta$$

$$= - \left[- (1/\sin \theta) + \cot \theta \right]_{\theta=0}^x$$

$$= - \left[-\frac{1}{\sin x} + \frac{\cos x}{\sin x} \right] \quad \therefore \lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\sin \theta} = 0$$

$$= \frac{1 - \cos x}{\sin x} = \tan \frac{x}{2}.$$

Exercises 17

Express the following integrals in terms of standard elliptic integrals :

1. (a) $\int_0^1 \frac{dx}{\sqrt{(x^2+4)(x^2+1)}}$, (b) $\int_0^3 \frac{dx}{\sqrt{(x^2+4)(x^2+3)}}$,

(c) $\int_0^{\sqrt{3}} \frac{dx}{\sqrt{(x^2+16)(x^2+9)}}$, (d) $\int_0^\infty \frac{dx}{\sqrt{(x^2+3)(x^2+1)}}$,

(e) $\int_0^1 \cos^{-1} x^2 dx$, (f) $\int_0^1 x^2 \sin^{-1} x^6 dx$

(g) $\int_0^1 x^{n-1} \sin^{-1} x^{2n} dx$

[Hints : Use procedures (A) and (B) of sec. 17.4 for 1]

2. (i) Prove that

$$\int_0^1 \frac{dt}{\sqrt{t(1-t)(1-k^2t)}} = \frac{1}{2} \int_{1/k^2}^\infty \frac{dt}{\sqrt{t(t-1)(k^2t-1)}} = K$$

[Hints : Use $t = 1/(k^2x)$ in left hand integral to get middle one and $t = \sin^2 \theta$ to get right most one].

(ii) Prove that $\int_0^u \frac{dx}{\sqrt{x(1-x)(1-k^2x)}} = 2F(\sin^{-1} \sqrt{u}, k)$

(iii) Show that $\int_u^1 \frac{dx}{\sqrt{x(1-x)(x-k_c^2)}} = 2F\left(\sin^{-1} \frac{\sqrt{1-u}}{k}, k\right)$

3. (i) Prove that $\int_u^a \frac{dx}{\sqrt{(a-x)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}} F(\lambda, p)$, $a > u \geq b > c$

where $\lambda = \sin^{-1} \sqrt{\frac{a-u}{a-b}}$ and $p^2 = \frac{a-b}{a-c}$.

[Hints : Use $a-x = y^2$ and then $y = \sqrt{a-b} \sin \lambda$]

(ii) Prove that $I = \int_c^u \sqrt{\frac{a-x}{(b-x)(x-c)}} dx = 2\sqrt{a-c} E(\gamma, q)$, $a > b \geq u > c$

where $\gamma = \sin^{-1} \sqrt{\frac{u-c}{b-c}}$ and $q^2 = (b-c)/(a-c)$

$$\begin{aligned} \text{[Hints : } I &= \int_c^u \frac{(a-x) dx}{\sqrt{(a-x)(b-x)(x-c)}} \\ &= a \int_c^u \frac{dx}{\sqrt{(a-x)(b-x)(x-c)}} - \int_c^u \frac{x dx}{\sqrt{(a-x)(b-x)(x-c)}} \end{aligned}$$

Now use (iii) and (iv) of 17.4 (c)]

$$\begin{aligned} \text{(iii) Show that } \int_u^a \frac{x dx}{\sqrt{(a-x)(x-b)(x-c)}} &= \frac{2c}{\sqrt{a-c}} F(\lambda, p) \\ &+ 2\sqrt{a-c} E(\lambda, p), a > u \geq b > c, \end{aligned}$$

$$\text{where } \lambda = \sin^{-1} \sqrt{\frac{a-u}{a-b}} \text{ and } p^2 = \frac{a-b}{a-c} \quad \text{[Hints : as in 3 (i)]}$$

$$\begin{aligned} \text{(iv) Show that } \int_u^a \sqrt{\frac{a-x}{(x-b)(x-c)}} dx &= 2\sqrt{a-c} [F(\lambda, p) \\ &- E(\lambda, p)], a > u \geq b > c \text{ where } \lambda, p \text{ are as in (iii)} \end{aligned}$$

[Hints : Rationalise integrand as in 3 (ii) and then sum the results of 3 (i) and 3 (iii)]

4. Prove that

$$\text{(i) } \int_u^b \frac{dx}{\sqrt{(a-x)(b-x)(x-c)}} = \frac{2}{\sqrt{a-c}} F(\delta, p), a > b > u \geq c$$

$$\begin{aligned} \text{(ii) } \int_u^b \frac{x dx}{\sqrt{(a-x)(b-x)(x-c)}} &= \frac{2}{\sqrt{a-c}} [(b-a) \Pi(\delta, q^2, q) \\ &+ aF(\delta, q)], \end{aligned}$$

$$a > b > u \geq c \text{ where } \delta = \sin^{-1} \sqrt{\frac{(a-c)(b-u)}{(b-c)(a-u)}} \text{ and } q = \sqrt{\frac{b-c}{a-c}}$$

for both (i) and (ii).

[Hints : Type (v) of 17.4 (c)]

5. Prove that

$$\text{(i) } \int_u^\infty \frac{dx}{\sqrt{(x-a)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}} F(\gamma, q), u \geq a > b > c$$

$$\text{where } \gamma = \sin^{-1} \sqrt{\frac{a-c}{u-c}} \text{ and } q^2 = \frac{b-c}{a-c}.$$

[Hints : Use $x-c = 1/y^2$ and then $\sqrt{a-c} y = \sin \gamma$]

$$(ii) \int_c^u \frac{dx}{\sqrt{(x-a)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}} F(\theta, m), a > b > c,$$

where $\theta = \sin^{-1}(\sqrt{u-c}/\sqrt{b-c})$ and $m^2 = (b-c)/(a-c)$.

[Hints: Use $x-c=y^2$ and then $y=\sqrt{b-c}\sin\theta$].

6. Evaluate in terms of elliptic integral:

$$(i) \int_0^7 \frac{dx}{\sqrt{(x-1)(x-3)(x-4)(x-5)}}$$

$$(ii) \int_2^8 \frac{dx}{\sqrt{(x-3)(x-4)(x-5)(x-6)}}$$

$$(iii) \int_4^6 \frac{dx}{\sqrt{(x-1)(x-2)(x-3)}}$$

$$(iv) \int_2^3 \frac{dx}{\sqrt{(x-1)(x-2)(x-3)(x-4)}}$$

$$(v) \int_0^u \frac{dx}{\sqrt{x(1-x)(1-k^2x)}}, 0 < u < 1.$$

$$(vi) \int_u^1 \frac{dx}{\sqrt{x(1-x)(x-k_c^2)}}, 0 < u < 1.$$

$$(vii) \int_7^{10} \frac{dx}{\sqrt{(x-1)(x-3)(x-4)(x-5)}}$$

7. Evaluate in terms of elliptic integrals:

$$(i) \int_4^\infty \frac{dx}{\sqrt{(x^2-2x+10)(x^2+x+7)}},$$

$$(ii) \int_{1-3\sqrt{2}}^{1+3\sqrt{2}} \frac{dx}{\sqrt{(x^2-2x+10)(x^2-2x+7)}},$$

$$(iii) \int_1^\infty \frac{dx}{\sqrt{(x^2-4x+5)(x^2-4x+10)}},$$

$$(iv) \int_2^{10} \frac{dx}{\sqrt{(x^2-2x+4)(x^2-4x+8)}}.$$

8. Prove that

$$(i) \operatorname{sn} 2u = \frac{2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u}{\operatorname{cn}^2 u + \operatorname{sn}^2 u \operatorname{dn}^2 u}$$

$$(ii) \quad \operatorname{cn} 2u = \frac{\operatorname{cn}^2 u - \operatorname{sn}^2 u \operatorname{dn}^2 u}{\operatorname{cn}^2 u + \operatorname{sn}^2 u \operatorname{dn}^2 u};$$

$$(iii) \quad \operatorname{dn} 2u = \frac{\operatorname{dn}^2 u + \operatorname{cn}^2 u (\operatorname{dn}^2 u - 1)}{\operatorname{dn}^2 u - \operatorname{cn}^2 u (\operatorname{dn}^2 u - 1)};$$

$$(iv) \quad \operatorname{cn} (u + iK_c) = -i k_c / k \text{ from (19) of sec. 17.11;}$$

$$(v) \quad \text{equations (19), (20), (21) from equations (13), (14), (15) of sec. 17.11;}$$

$$(vi) \quad \operatorname{sn} (2K - u) = \operatorname{sn} u;$$

$$[\text{Hints: } \operatorname{sn} (2K - u) = \operatorname{sn} (4K - \overline{2K + u}) = \operatorname{sn} (2K + u) \text{ etc.}]$$

$$(vii) \quad \frac{d}{du} \operatorname{sn} (2K - u) = -\frac{d}{du} \operatorname{sn} u = -\frac{d}{du} \operatorname{sn} (u + 2K);$$

$$(viii) \quad \operatorname{sn} (2ik_c - u) = -\operatorname{sn} u;$$

9. Prove that

$$(i) \quad \int_0^x E(x, k) \sqrt{1 - k^2 \sin^2 x} dx = \frac{1}{2} [E(x, k)]^2, \quad 0 < x \leq \frac{\pi}{2};$$

$$(ii) \quad \int_0^x F(x, k) \sin x dx = -F(x, k) \cos x + \frac{1}{k} \sin^{-1}(k \sin x);$$

$$(iii) \quad \int_0^x E(x, k) \sin x dx.$$

$$= -E(x, k) \cos x + \frac{1}{2k} [k \sin x \sqrt{1 - k^2 \sin^2 x} + \sin^{-1}(k \sin x)];$$

$$(iv) \quad E(k, \phi + \pi) = 2E + E(k, \phi);$$

$$(v) \quad \operatorname{sn} K = (1 + k_c)^{-1/2}, \operatorname{cn} \frac{K}{2} = \sqrt{k_c} (1 + k_c)^{-1/2}, \operatorname{dn} \frac{K}{2} = \sqrt{k_c};$$

$$10. \text{ Show that if } u = \int_y^1 \frac{dt}{\sqrt{(1-t^2)(k_c^2 + k^2 t^2)}} \text{ then } y = \operatorname{cn} u.$$

[Hints: Use $t = \cos \theta$ and sec. 17.5]

11. Show that

$$(i) \quad \operatorname{dn} (u + v) \operatorname{dn} (u - v) = \frac{\operatorname{ds}^2 u \operatorname{ds}^2 v + k^2 k_c^2}{\operatorname{ns}^2 u \operatorname{ns}^2 v - k^2};$$

$$(ii) \quad \operatorname{sn} (u - v) \operatorname{dn} (u + v) = \frac{\operatorname{sn} u \operatorname{dn} u \operatorname{cn} v - \operatorname{sn} v \operatorname{dn} v \operatorname{cn} u}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v};$$

$$(iii) \quad 1 + \operatorname{cn} (u + v) \operatorname{cn} (u - v) = \frac{\operatorname{cn}^2 u + \operatorname{cn}^2 v}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}.$$

12. Show that

$$(i) \int_0^1 E(x, k) k dk = \frac{\sin^2 x + 1 - \cos x}{3 \sin x}$$

$$(ii) \int_0^1 \Pi(x, r^2, k) k dk \\ = \tan \frac{x}{2} - r \log \sqrt{\frac{1+r \sin x}{1-r \sin x}} - r^2 \Pi(x, r^2, 0).$$

Answers

$$1. (a) \frac{1}{2} F(\pi/4, \sqrt{3}/2), (b) \frac{1}{2} F(\pi/3, \frac{1}{2}), (c) \frac{1}{4} F(\pi/6, \sqrt{7}/3),$$

$$(d) (1/\sqrt{3}) F(\pi/3, 2/\sqrt{3}), (e) 2\sqrt{2} E(1/\sqrt{2}, \pi/2) - \sqrt{2} F(1/\sqrt{2}, \pi/2),$$

$$(f) (1/3) [\pi/2 - 2\sqrt{2} E(1/\sqrt{2}, \pi/2) + \sqrt{2} F(1/\sqrt{2}, \pi/2)],$$

$$(g) \frac{1}{n} \left[\frac{\pi}{2} - 2\sqrt{2} E\left(\frac{1}{\sqrt{2}}, \frac{\pi}{2}\right) + \sqrt{2} F\left(\frac{1}{\sqrt{2}}, \frac{\pi}{2}\right) \right].$$

$$6. (i) \frac{2}{\sqrt{6}} \left[F\left(\frac{\pi}{3}, \sqrt{\frac{2}{3}}\right) - F\left(\cos^{-1} \sqrt{\frac{5}{6}}, \sqrt{\frac{2}{3}}\right) \right],$$

$$(ii) \frac{2}{\sqrt{3}} \left[F\left(\cos^{-1} \frac{1}{\sqrt{6}}, \frac{\sqrt{3}}{2}\right) - F\left(\cos^{-1} \sqrt{\frac{2}{3}}, \frac{\sqrt{3}}{2}\right) \right],$$

$$(iii) \sqrt{2} \left[F\left(\frac{\pi}{3}, \frac{1}{\sqrt{2}}\right) - F\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right) \right], (iv) \frac{1}{2} K(\sqrt{3}/2),$$

$$(v) 2F(\sin^{-1} \sqrt{u}, k), \text{ Hints. Use } x = \sin^2 \theta, (vi) 2F\left(\sin^{-1} \frac{\sqrt{1-u}}{k}, k\right),$$

$$(vii) \frac{2}{\sqrt{6}} \left[F(\pi/3, \sqrt{2/3}) - F(\cos^{-1} \sqrt{5/14}, \sqrt{2/3}) \right].$$

$$7. (i) \frac{4}{3\sqrt{3}} \left[K\left(\frac{1}{\sqrt{3}}\right) - F(1/\sqrt{3}, \pi/3) \right] \text{ {Hints: } } \frac{x+2}{x-4} = t, \lambda = \frac{2}{3}, 2,$$

$$(ii) \lambda = 1, 3/2; Q_1 = (\sqrt{6})^2 + (x-1)^2, Q_2 = 3^2 + (x-1)^2,$$

$$\text{Put } x-1 = t \text{ and } t = \sqrt{6} \tan \theta. (2/3) F(1/\sqrt{3}, \pi/\sqrt{3}),$$

$$(iii) \frac{1}{\sqrt{6}} \left\{ F\left(\sqrt{\frac{5}{6}}, \frac{\pi}{4}\right) + K\left(\sqrt{\frac{5}{6}}\right) \right\},$$

$$(iv) \frac{1}{\sqrt{6}} \left\{ F\left(\sqrt{\frac{2}{3}}, \frac{\pi}{4}\right) + F\left(\sqrt{\frac{2}{3}}, \tan^{-1} \frac{3}{5}\right) \right\}.$$

PART IV

Differential Equation with Applications

[CH. 18—22 and Misc. (GRE) Guide]

DEFINITIONS AND FORMATION OF DIFFERENTIAL EQUATIONS

18.1. INTRODUCTION AND DEFINITIONS

Equation is a kind of relation either among the variables only or among the variables and their derivatives. The first relation mainly belongs to algebra and is called algebraic equations with which everyone is conversant from the very beginning of the study of algebra. The second relation is observed in the problems of Differential Calculus such as "If $y = Ae^x + Be^{-x}$ then prove that $\frac{d^2y}{dx^2} - y = 0$." This problem consists of two relations — the one contains only variables x and y while the second contains both the derivative $\frac{d^2y}{dx^2}$ and the variable y . But differential calculus gave no name of this equation. From now on this type of equations will be called **Differential Equations**.

Leibnitz in 1676 first used the term "Oequitio differentialis" or differential equation to denote the relationship between the differentials dx and dy of the variables x and y . Actually he published the equation $\int y dy = \frac{1}{2}y^2$ in 11 November, 1675. This equation expresses the symbol of both integration as well as simple differential equation. Newton also

called x and y as "fluxion" and x and y as "fluent" and classified "fluxional equation" (—today's first order differential equation) into three groups.

- (i) $\frac{\dot{y}}{x} = f(x)$, $\frac{\dot{y}}{x} = f(y)$ or to-day's symbol is $\frac{dy}{dx} = f(x)$, $\frac{dy}{dx} = f(y)$;
- (ii) $\frac{\dot{y}}{x} = f(x, y)$ with two fluxions and two fluents; (iii) contains more than two fluxions — today's partial differential equations.

The discussion may be summarised to give the **definition of Differential Equation** as "The equation containing derivative is a differential equation". It may or may not contain explicitly the variable or variables from which the derivatives were derived. But the presence of derivative is a must.

Differential equation may also be *defined as the equation involving rates of change*. Because mathematical modelling of the physical problems of the real world with some *rate change* gives rise to differential equation as geometrical and physical interpretations of the derivatives are some

sorts of rate change. This is why the physical laws involving rates change of Applied Mathematics, Physics, Chemistry, Biology, Economics and Engineering can be easily modelled as differential equations and their solutions are used in the qualitative and quantitative analysis of the laws.

For example Newton formed a system of differential equation from the laws of gravitation and solved them and showed that the earth moves round the sun in an elliptical path keeping the sun at one of its foci. Likewise Maxwell in 1865 formed and solved a system of partial differential equation from the laws electric current of the electro-magnetic field and forecast about radio waves.

18.2. KINDS OF DIFFERENTIAL EQUATIONS

Differential Calculus uses two types of derivatives $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$ etc. and $\frac{\partial y}{\partial x}$, $\frac{\partial^2 y}{\partial x^2}$, $\frac{\partial^3 y}{\partial x^3}$ etc. depending on the number of independent variables.

The first type of derivatives is known as ordinary derivatives and the second is partial derivatives of different order and accordingly the differential equations are classified into three – (i) Ordinary Differential Equations, (ii) Partial Differential Equations and (iii) Total Differential Equations. Their definitions and illustrations are given below one after another.

(i) Ordinary Differential Equations : *The differential equation containing only ordinary derivative is called an ordinary differential equation. It means the differential equation must have a single independent variable with a single dependent variable. Examples are*

$$\frac{dy}{dx} + xy = x^2 \quad (1)$$

$$\left(\frac{dy}{dx}\right)^3 + 2x \frac{dy}{dx} - y = 0 \quad (2)$$

$$\frac{d^2y}{dx^2} + 4 \frac{dy}{dx} - 3y = x \quad (3)$$

$$\left(\frac{d^3y}{dx^3}\right)^2 + x \frac{d^2y}{dx^2} + 4y = x^2 \quad (4)$$

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = 2 \frac{d^2y}{dx^2} \quad (5)$$

$$\frac{dy}{dx} = \frac{x}{3\sqrt{y}(1+3\sqrt{x})} \quad (6)$$

In all these differential equations x is the independent and y is the dependent variable.

(ii) Partial Differential Equations : *The differential equation containing partial derivative is a partial differential equation.* It means the differential contains more than one independent variables corresponding to a single independent variable. Examples are

$$\frac{\partial y}{\partial x} - \frac{\partial y}{\partial t} = 0 \quad (7)$$

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \quad (8)$$

$$\frac{\partial^3 u}{\partial x^3} + 2x \left(\frac{\partial u}{\partial y} \right) + (x - y) u = e^{x+2y} \quad (9)$$

$$\left(\frac{\partial^2 T}{\partial x^2} \right)^2 + \left(\frac{\partial T}{\partial t} \right) + z T = 3 \sin x \quad (10)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad (11)$$

In all these differential equations, there are more than one independent variables corresponding to each dependent variable.

(iii) Total Differential Equations : These are *ordinary differential equations with more than one dependent variable and a single independent variable*:

The dependent variable may be explicit or implicit. Examples are :

$$P(x, y, z) dx + Q(x, y, z) dy + R(x, y, z) dz = 0 \quad (12)$$

$$x^2 dx^2 + 2xy dx dy + y^2 dy^2 - z^2 dz^2 = 0 \quad (13)$$

$$P(x, y, z) + Q(x, y, z) \frac{dy}{dx} + R(x, y, z) \frac{dz}{dx} = 0 \quad (14)$$

$$P(x, y, z) \frac{dx}{dt} + Q(x, y, z) \frac{dy}{dt} + R(x, y, z) \frac{dz}{dt} = 0 \quad (15)$$

$$yz(y+z) dx + zx(x+z) dy + xy(x+y) dz = 0 \quad (16)$$

In all these equations there are more than one dependent variable with a single independent variable. Independent variable is implicit in eqns. (12), (13) and (16) and explicit in (14) and (15).

18.3. NOMENCLATURE

Algebraic equations are named on the basis of the highest power of the variable alone. But differential equations are named by stating their (i) order and (ii) degree simultaneously. *The order and the degree of a differen-*

tial equation are the order and the degree of the highest order derivative present in the equation. The degree is determined after rationalising the surd, if any, in the equation. The order and the degree of the equations in 18.2 are as follows : eqns. (1), (6), (7), (12), (14), (15) and (16) are each of order 1 and of degree 1; eqns. (3), (8) and (11) are each of order 2 and degree 1; eqns (5) and (10) are each of order 2 and degree 2; eqns. (2), (4), (9) and (13) are of order 1, 3, 3, 1 and of degree 3, 2, 1, 2 respectively.

N.B. This book only discusses ordinary differential equation and the word equation will mean ordinary differential equation (O D E) onward.

18.3.1. SOME DEFINITIONS ABOUT NOMENCLATURE

In addition to order and degree the following terms are also used in the nomenclature of all the differential equations.

(a) Co-efficient : The differential equation is of constant coefficients if the coefficients of all the derivatives and the dependent variable are constants. ; otherwise, of variable coefficients if some of the coefficients are functions of independent variables.

For example, the coeffs. of $\frac{d^2y}{dx^2}$, $\frac{dy}{dx}$ and y in the ODE

$$y \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + 3y = 5x \quad (1)$$

are 1, 2 and 3 respectively. Also, the order and the degree of the ODE are 2 and 1 respectively. So, the name of the ODE, is now "Second order first degree Ordinary differential equation with constant coefficients." On the other hand the ODEs,

$$\frac{d^2y}{dx^2} + 2x \frac{dy}{dx} + 3xy = 4 \cos x \quad (2)$$

and
$$r^2 \frac{d^2\theta}{dr^2} + r \frac{d\theta}{dr} + (k - a^2) \theta = 0 \quad (3)$$

are both "second order first degree ordinary differential equation with variable coefficients" due to the presence of the independent variable x and r as coefficients in second and third terms of first ODE and first and second terms of the second ODE respectively.

(b) Linearity : The differential equation with unit power of all the derivatives and the dependent variable is a linear differential equation otherwise non-linear if any of the derivatives or dependent variables has more than unit power. So, the linearity and the degree of differential equation are two

different entities. Both linearity and degree do not depend on the non-linearity or degree of the independent variable. Non-linearity appears due to the product of derivatives or dependent variables themselves and the product of derivative and dependent variable. For example, eqns. (1), (2), (3) above and

$$\frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - 3)y = x^3 \quad (4)$$

are all linear whereas the following equations

$$y' = 1 + xy^2 \quad (5)$$

$$\frac{d^2y}{dx^2} + 2 \left(\frac{dy}{dx} \right)^2 + y = 0 \quad (6)$$

$$\frac{d^2y}{dx^2} + 2x \frac{d^2y}{dx^2} + 2y \frac{dy}{dx} + 3y = \sin x \quad (7)$$

$$\frac{d^2y}{dx^2} + \sin y = 0 \quad (8)$$

are all non-linear due to y^2 in (5), $\left(\frac{dy}{dx} \right)^2$ in (6), product $y \frac{dy}{dx}$ in (7) and infinite power series of $\sin y$ in (8).

(c) Homogeneity: A differential equation is said to be homogeneous if each term of the equation contains at least a derivative or a dependent variable. Examples are

$$\left(\frac{d^2y}{dx^2} \right)^2 + \left(\frac{d^2y}{dx^2} \right) \left(\frac{dy}{dx} \right)^3 - y = 0 \quad (9)$$

and
$$\frac{d^2y}{dx^2} + 4 \frac{dy}{dx} + y = 0 \quad (10)$$

whereas
$$\frac{d^2y}{dx^2} + 4 \frac{dy}{dx} + y = x^2 + \sin x \quad (11)$$

is not homogeneous because of the absence any y or its derivative in the R. H. Side of (11). The complete name of (9) is "second order second degree non-linear homogeneous ordinary differential equation with constant coefficients." Similarly, the complete name of (5) is "first order first degree non-linear non-homogeneous ODE with variable coefficients."

18.4. SOLUTION AND ITS KINDS

It is observed that any ODE is a functional relation of x, y , and the derivatives of y i.e.

$$F(x, y, y', y'', \dots, y^n) = 0 \quad (1)$$

denotes an n th order ODE whose independent variable is x and dependent variable is y and primes (') and superscripts denote order of differentiations. Now the solution of (1) is some algebraic relation defined and differentiable in an interval, say, $a < x < b$ and satisfies (1).

In general differential equation is very difficult to solve. The equations like $(y')^2 = -1$ and $|y'| + |y| = 0$ do not have solutions at all. Certain special types of differential equations can be solved in closed form. This is the subject matter of the subsequent chapters of this book.

Let us consider a very simple ODE $y' = 2x$. This is a first order first degree linear non-homogeneous ODE with constant coefficients. This can be solved by *direct integration* as $y = x + c$. So, *integration* is the main tool for solving differential equation and the solution contains an arbitrary constant. In this way second order ODE $y'' = 2x$ has solution $y = \frac{1}{3}x^3 + C_1x + C_2$ with two arbitrary constant. So, these two examples

reveal two facts — (i) order of ODE equals the number of arbitrary constants in the solution, (ii) an ODE may have infinite number of solutions as the arbitrary constant is capable of assuming infinite number of particular values. So, solution with arbitrary constants represents a set of particular solutions. Thus solution may be classified into four categories. They are as follows :

(1) **General solution** : A solution of the differential equation having number of arbitrary constants equal to the order of the equation is called a *general solution*. Thus the n th order equation $F(x, y, \dots, y^n) = 0$ may be $f(x, y, c_1, c_2, \dots, c_n) = 0$ or $g(x, c_1, c_2, \dots, c_n) = 0$ with n arbitrary constants $c_1, c_2, \dots, c_n$. It is also called *complete integral* or *complete primitive*.

(2) **Particular Solution** : A solution obtained from the general solution giving particular values to the arbitrary constants is called a *particular solution*.

(3) **Parametric solution** : A solution represented by using some parameter is called a *parametric solution*.

(4) **Singular Solution** : A solution that cannot be derived from (1), (2) or (3) above is a *singular solution*.

18.4.1. INITIAL AND BOUNDARY CONDITIONS

Particular solution is obtained from general solution using some conditions in an interval. These conditions are called initial and boundary conditions and concerned problems are called *initial and boundary value problems*.

The *initial conditions* are those for which the dependent variable and its derivatives are assigned some initial values corresponding to some initial value of the independent variable for a given interval such as $y(x_0) = a, y'(x_0) = b$ when x_0 is an initial value of the independent variable x .

The *boundary conditions* are those for which the dependent variable and its derivatives are assigned some final values for some terminal value or values of the independent variable other than initial value such as $y(x_1) = c$ and $y'(x_1) = d$ when the independent variable x attains a terminal value x_1 in the given interval.

The initial value problems must have initial conditions only whereas boundary value problem have both initial and boundary conditions.

18.5. FORMATION OF DIFFERENTIAL EQUATIONS

ODE can be formed by two processes—(i) the elimination of arbitrary constants from a given general solution; (ii) the mathematical modelling of natural laws involving some kinds of rate change of continuous variables as indicated in 18.2 and of geometric problems.

(i) **Elimination of arbitrary constants** : As already indicated in 18.1 that differential calculus deals with many problems of forming ODEs by the elimination of arbitrary constants of the solution. Let

$$f(x, y, c_1) = 0 \quad (i)$$

be a solution with a single arbitrary constant, c_1 , and let $y(x)$ is differentiable in a given interval of x . So, differentiating (i) w.r.t. x , we have,

$$\phi\left(x, y, c_1, \frac{dy}{dx}\right) = 0 \quad (ii)$$

Now eliminating c_1 from (i) and (ii), we get a lowest order ODE $F\left(x, y, \frac{dy}{dx}\right) = 0$ corresponding to (i). Similarly, let

$$f(x, y, c_1, c_2) = 0 \quad (iii)$$

be a solution with two arbitrary constants c_1 and c_2 . To eliminate c_1, c_2 we need two more relations. So, we differentiate (iii) twice successively as

$$\phi(x, y, c_1, c_2, y') = 0 \quad (iv)$$

$$\text{and} \quad \psi(x, y, c_1, c_2, y', y'') = 0 \quad (v)$$

Now, eliminating c_1, c_2 from (iii), (iv) and (v), we get a second order ODE as $F(x, y, y', y'') = 0$ (vi)

In this way differentiating the solution $f(x, y, c_1, c_2, \dots, c_n) = 0$ with n arbitrary constants n times successively, we get $(n + 1)$ relations altogether and eliminating the n arbitrary constants from $(n + 1)$ relations we get the lowest order i.e., ODE $F(x, y, y', y'', \dots, y^n) = 0$. In conclusion it can be said that "independent" number of arbitrary constants in the solution determines the lowest order of the differential equation to be formed. The word independent means that the number of constants neither increases or decreases on simplification. For example, the three constants A, B, C in $A \cos^2 x + B \sin^2 x + C \cos 2x$ simplifies to $D \cos^2 x + E \sin^2 x$ where $D = A + C, E = B - C$ since $\cos 2x = \cos^2 x - \sin^2 x$.

(ii) Mathematical modelling of physical and geometrical problems :

Mathematical modelling of physical and geometrical problems often give rise to ODE. Geometrically $\frac{dy}{dx}$ represents the slope of the tangent at (x, y) to a curve and similarly, the derivatives of higher order can be interpreted. In the same way physical problems with certain rate change can be modelled into an ODE as $\frac{ds}{dt}$ signifies velocity, $\frac{d^2s}{dt^2}$ signifies acceleration etc. Different geometrical and physical problems of chemistry, physics, engineering, biology, economics etc. will appear in problem sets of the chapters next.

Examples 17

1. Discuss all the terms regarding nomenclature of the ODE $\left(\frac{d^2y}{dx^2}\right)^{1/3} = K \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{5/2}$ and give the complete name of the equation.

Let us raise 6th power of the eqn. to rationalise the surds. So, the eqn. becomes

$$\left(\frac{d^2y}{dx^2}\right)^2 = K^6 \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{15}$$

Now, the highest order derivative is $\frac{d^2y}{dx^2}$. It's

(i) order is 2 and its (ii) degree is 2. The eqn. is (iii) non-linear due to the presence of product and powers of derivatives, (iv) coefficients are constants. So the eqn. is of constants coeffs. (v) The eqn is non-homogeneous due to the presence of a term without y or its derivatives. Hence the complete name of the eqn is "Second order second degree non-linear non-homogeneous ODE with constant coefficients."

2. Form an ODE corresponding to $y = Ax^m + Bx^n$ (1)

The given relation contains two arbitrary constants A and B . So, differentiating twice successively, we get,

$$\frac{dy}{dx} = mAx^{m-1} + Bnx^{n-1} \quad (2)$$

$$\frac{d^2y}{dx^2} = m(m-1)Ax^{m-2} + Bn(n-1)x^{n-2} \quad (3)$$

$$\text{From (2), } \frac{1}{x} \frac{dy}{dx} = mAx^{m-2} + Bnx^{n-2} \quad (4)$$

$\therefore$ (3) - $(m-1) \times$ (4) gives

$$\frac{d^2y}{dx^2} - (m-1) \frac{1}{x} \frac{dy}{dx} = Bn(n-m)x^{n-2} \quad (5)$$

Also (2) - $\frac{m}{x} \times$ (1) gives

$$\frac{dy}{dx} - \frac{m}{x}y = B(n-m)x^{n-1} \quad (6)$$

From (5) and (6), we have

$$\frac{d^2y}{dx^2} - (m-1) \frac{1}{x} \frac{dy}{dx} = n \left[\frac{1}{x} \frac{dy}{dx} - \frac{my}{x^2} \right]$$

$$\text{or, } x^2 \frac{d^2y}{dx^2} - (m-n-1)x \frac{dy}{dx} + mny = 0$$

is the required equation.

3. Form an ODE corresponding to $x = 2t + c, y = ct + 3, t$ is a parameter.

The parametric relation has only one arbitrary constant c . Now eliminating t , we get,

$$y = \frac{c}{2}(x - c) + 3 \quad (1)$$

$$\text{Diff. w.r.t. } x, \quad \frac{dy}{dx} = y' = \frac{c}{2} \therefore c = 2y'$$

$\therefore$ from (1) the reqd. ODE is,

$$y = y'(x - 2y') + 3$$

α

$$2(y')^2 - xy' = 3 - y$$

is the reqd. ODE.

4. Derive an ODE corresponding to all circles lying in a plane.

From co-ordinate geometry, the eqn. of all circles lying in plane is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad (1)$$

which contains three arbitrary constants g, f & c . Diff. (1) thrice successively, we have,

$$x + y \frac{dy}{dx} + g + f \frac{dy}{dx} = 0 \quad (2)$$

$$1 + \left(\frac{dy}{dx}\right)^2 + (y + f) \frac{d^2y}{dx^2} = 0 \quad (3)$$

and
$$3 \frac{dy}{dx} \frac{d^2y}{dx^2} + (y + f) \frac{d^3y}{dx^3} = 0 \quad (4)$$

From (3) and (4)

$$\frac{1 + \left(\frac{dy}{dx}\right)^2}{3 \frac{dy}{dx} \frac{d^2y}{dx^2}} = \frac{\frac{d^2y}{dx^2}}{\frac{d^3y}{dx^3}}$$

or,

$$\frac{d^3y}{dx^3} \left[1 + \left(\frac{dy}{dx}\right)^2 \right] = 3 \frac{dy}{dx} \left(\frac{d^2y}{dx^2} \right)^2 = 0$$

is the required ODE.

Exercises 18

1. Determine order, degree, linearity, homogeneity of the following equations.

(i) $\frac{d^3y}{dx^3} + \frac{2y}{\left(\frac{d^3y}{dx^3}\right)} = 2x$, (ii) $x(y')^3 + y' = 7xy$,

(iii) $\frac{d^2y}{dx^2} = \frac{m}{H} \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2 \right]}$

(iv) $\log \left(\frac{dy}{dx}\right) + \log x^2 = 2y$, (v) $x^2 + yy' - (y')^{3/2} = 0$

(vi) $x^4 \frac{dy}{dx} - x^2 \frac{d^2y}{dx^2} = y^4 \frac{d^3y}{dx^3}$,

$$(vii) \frac{d^2 y}{dx^2} = 3x \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{3/2},$$

$$(viii) e^{y'''} + xy'' + y = 0, \quad (ix) \sqrt{y}' = 3(y'')^{2/3},$$

$$(x) \frac{d^2 y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx} \right)^4}, \quad (xi) \left(\frac{d^3 y}{dx^3} \right)^4 - \left(\frac{d^2 y}{dx^2} \right)^5 + x = 0$$

2. Form ODE by the elimination of arbitrary constants $A, B, C, D, a, b, c, d, c_1, c_2$, if any, from the following solutions :-

$$\checkmark (i) y = A e^{mx} + B e^{nx},$$

$$\checkmark (ii) y = c \sin(x + d),$$

$$\checkmark (iii) y = A \cos nx + B \sin nx,$$

$$\checkmark (iv) \sin x + \sin y = c,$$

$$\checkmark (v) y = e^{mx} (A \cos nx + B \sin nx), \quad (vi) y = e^{x/a} + e^{-x/a},$$

$$(vii) y = A \cosh \frac{x}{A},$$

$$(viii) y = ax^2 + bx + c,$$

$$(ix) y = x^n (A + B \log x),$$

$$(x) e^{\cos x} (1 - \cos y) = c,$$

$$(xi) y = e^{mx} (A + Bx),$$

$$\checkmark (xii) x^2 + y^2 = cx,$$

$$(xiii) y = (A + Bx) \cos nx + (C + Dx) \sin nx,$$

$$\checkmark (xiv) \log y = c_1 e^x + c_2 e^{-x},$$

$$(xv) y = e^{mx} \{ (A + Bx) \cos nx + (C + Dx) \sin nx \},$$

$$(xvi) x = -\log t, y = c_1 t + c_2 t^2, t \text{ is a parameter},$$

$$(xvii) y = Ax \cos \left(\frac{n}{x} + B \right),$$

$$(xviii) x = at + b, y = b^2 t^2, t \text{ is parameter},$$

$$(xix) y + a e^x + b x^2 = 0, \quad (xx) y = a \sin 2x + b \cos 2x,$$

3. Find the ODE of all circles with centre at the origin.

4. The equation of all confocal conics is $\frac{x^2}{a+\lambda} + \frac{y^2}{b+\lambda} = 1$ where λ

is an arbitrary constant and can assume any real value. Find the ODE whose solutions are the given conics.

5. Show that the ODE corresponding to the curves $(c^2 - a^2)x^2 + c^2 y^2 = c^2(c^2 - a)$, where c is an arbitrary constant, is

$$xy(y')^2 + (x^2 - y^2 - a^2)y' - xy = 0.$$

6. Find the ODE of all tangents to the circle $x^2 + y^2 = c^2$ where c is a given constant. [Hints : tangent at an arbitrary point (x_1, y_1) is $xx_1 + yy_1 = c^2$, (i) eliminate x_1, y_1 from (i) and $x_1^2 + y_1^2 = c^2$ and $x_1 + y_1' = 0$]

7. Find the ODE corresponding to all unit circles having centre on the line $x = y$.

8. Find the differential equation satisfied (i) by the equation of the family of tangents to $y = x^2$ and (ii) by the equation of the family of normals to $y = x^2$.

9. Prove that $2y' y''' = 3(y'')^2$ if $y = \frac{ax+b}{cx+d}$ and $(y-x)y'' = 2y'(1+y')$ if also $a+d=0$.

Answers

1. (i) 3, 2, non-linear, homogeneous (ii) 1, 3, non-linear homogeneous (iii) 2, 2, non-linear, non-homogeneous (iv) 1, 1, non-linear, homogeneous (v) 1, 3, non-linear, non-homogeneous (vi) 3, 1, non-linear, homogeneous (vii) 2, 2, non-linear, non-homogeneous (viii) 3, 1, non-linear, non-homogeneous (ix) 2, 4, non-linear, homogeneous (x) 2, 2, non-linear, non-homogeneous (xi) 3, 4, non-linear, non-homogeneous.

$$2. (i) \frac{d^2y}{dx^2} - (m+n) \frac{dy}{dx} + mny = 0. (ii) y'' + y = 0.$$

$$(iii) y'' + n^2y = 0. (iv) \cos x + y' \cos y = 0.$$

$$(v) \frac{d^2y}{dx^2} - 2m \frac{dy}{dx} + (m^2 + n^2)y = 0.$$

$$(vi) y = e^{xy\sqrt{y^2-4}} + e^{-xy\sqrt{y^2-4}}$$

$$(vii) \text{Hints: } \sinh^{-1} y' = \log y' + \sqrt{1 + (y')^2},$$

$$y \log \left\{ \frac{dy}{dx} + \sqrt{1 + \left(\frac{dy}{dx} \right)^2} \right\} = x \sqrt{1 + \left(\frac{dy}{dx} \right)^2}$$

$$(viii) y''' = 0. (ix) x^2 y'' - (2n-1)xy' + n^2y = 0.$$

$$(x) y' \sin y + \sin x \cos y = \sin x. (xi) \frac{d^2y}{dx^2} - 2m \frac{dy}{dx} + m^2y = 0.$$

$$(xii) 2xyy' = y^2 - x^2. (xiii) y^{2p} + 2n^2y'' + n^2y = 0.$$

$$(xiv) yy'' - (y')^2 = y^2 \log y.$$

$$(xv) y^{2m} - 2my''' + 2(3m^2 + n^2)y'' - 4m(n^2 + n^2)y' + (m^2 + n^2)y = 0.$$

$$(xvi) y'' + 3y' + 2y = 0. (xvii) x^4 y'' + n^2y = 0. (xviii) 2yy' = (y')^2.$$

$$(xix) y''(x^2 - 2x) + y'(2 - x^2) + 2y(x - 1) = 0. (xx) y'' + 4y = 0.$$

$$3.x + yy' = 0. 4. xy(p^2 - 1) + (x^2 - y^2 - a^2 + b^2)p = 0, p = \frac{dy}{dx}.$$

$$6. y = xy' \pm c \sqrt{(y')^2 + 1}. 7. (y-x)^2 \{1 + (y')^2\} = (1 + y')^2$$

$$8. (i) 4xy' = 4y + (y')^2; (ii) 4x(y')^3 + 2(1 - 2y)(y')^2 + 1 = 0.$$

FIRST ORDER AND FIRST DEGREE DIFFERENTIAL EQUATIONS

19.1 METHODS OF SOLUTIONS

The most general first order and first degree *ODE* is $\frac{dy}{dx} = f(x, y)$. The integration of this equation gives its general solution. Direct integration is **not** always possible. The ways or methods of identification to make this *ODE* integrable is the objective of the rest of this chapter. The following procedures will help in this respect to express the solution into elementary functions :

1. Separation of variables,
2. Homogeneous equations,
3. Reducible to homogeneous equations,
4. Exact equations,
5. Linear equations,
6. Reducible to linear.

19.2 EXISTENCE AND UNIQUENESS OF SOLUTIONS

The methods or techniques stated in 19.1 will be discussed and used in subsequent sections for finding solution of first order and first degree *ODE* assuming existence and uniqueness of the solution. But for practical utility of these techniques, some knowledge of existence and uniqueness is essential; otherwise countless effort may be spent for a fruitless result of a mathematical model of a real world problem. With this understanding in mind, only the meaning of existence and uniqueness, the statement and the use of the theorem concerned are cited below :

Existence gives the conditions for having *at least* one solution whereas **uniqueness** gives the condition for *only one* solution. These are true for both algebraic and differential equations. To understand the conditions for existence and uniqueness, let us define a more general condition for continuity, the Lipschitz continuity.

Lipschitz continuity : Let R denote the rectangular region in the xy -plane defined by

$$R : a \leq x \leq b, c \leq y \leq d \quad (1)$$

where $-\infty < a < b < +\infty$ and $-\infty < c < d < +\infty$ (fig. 131).

$$\text{If } |f(x, y_1) - f(x, y_2)| \leq k |y_1 - y_2| \quad (2)$$

for $0 < k < \infty$, then $f(x, y)$ is said to be Lipschitz continuous in R and k is called the Lipschitz constant.

It should be remembered that *every Lipschitz continuous is continuous in y for each fixed x . But not every continuous function is Lipschitz continuous.*

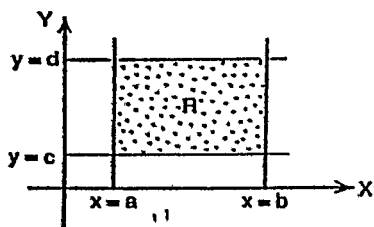


Fig. 131

Theorem: If $f(x, y)$ and $\frac{\partial f}{\partial y}$ be continuous in R then $f(x, y)$ is Lipschitz continuous in y over R .

Existence theorem :

If $f(x, y)$ is Lipschitz continuous and k is a Lipschitz constant in a region R satisfying the condition $|x - x_0| \leq a, |y - y_0| \leq b$ (fig. 132), then there exists *at least* a solution $y = y(x)$ of the initial value problem (IVP)

$$y' = f(x, y), \quad y(x_0) = y_0 \quad (2)$$

such that $|x - x_0| \leq \delta$ for some $\delta > 0$.

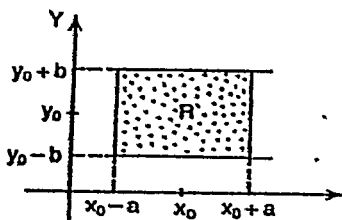


Fig. 132

The uniqueness of an “*at least*” solution governed by the existence theorem is established by using the following Gronwall’s inequality.

Gronwall’s inequality : Suppose $y(x)$ be a continuous non-negative function and $y(x) \leq A + B \int_{x_0}^x y(t) dt$ for positive constants A and B and for values $|x - x_0| \leq \delta$, then Gronwall’s inequality is $y(x) \leq Ae^{B|x-x_0|}$ for all x in the interval $|x - x_0| \leq \delta$.

Uniqueness theorem. : Let Picard’s iteration formula be

$$y_n(x) = y_0 + \int_{x_0}^x f(t, y_{n-1}(t)) dt \quad (3)$$

for the IVP (2) and let the conditions of existence theorem hold true, then $y(x) = \lim_{n \rightarrow \infty} y_n(x)$ is the *only* continuous solution of the IVP in $|x - x_0| \leq \delta$.

Theorem : Let $f(x, y)$ and $\frac{\partial f}{\partial y}$ be continuous in R . Then there exists a constant $\delta > 0$ such that the Picard's iteration $\{y_n(x)\}$ converge to a unique continuous solution of the IVP (2) in $|x - x_0| \leq \delta$.

These theorems are known as **local** results because of the guarantee of uniqueness near the initial point (x_0, y_0) . For proofs of these theorems, readers are advised to consult the books of ODEs by the authors E.L. Ince, R.P. Agnew, Polard and Tenebaum, W.R. Derrick etc.

19.3 SEPARATION OF VARIABLES

If P and Q are functions of two variables x and y , then the general first order and first degree ODE can be written as $P(x, y) + Q(x, y) \frac{dy}{dx} = 0$ (1)

or $P(x, y) dx + Q(x, y) dy = 0$ (2)

The equation (1) shows the independent and the dependent variables explicitly; while in equation (2) these are implicit and any one can be considered as dependent or independent. If these equations can be written as

$$\left. \begin{aligned} f_1(x) + f_2(y) \frac{dy}{dx} &= 0 \\ \text{or} \quad f_1(x) dx + f_2(y) dy &= 0 \end{aligned} \right\} \quad (3)$$

by some algebraic manipulation, then the variables are said to be **separated**. That is, variables of the same feather are grouped together. Apart from that if P and Q in (1) and (2) can be factored as $P(x, y) \equiv f_1(x) \phi(y)$ and $Q(x, y) \equiv f_2(y) \psi(x)$ then by division by $\phi(y) \psi(x)$ (1) and (2) becomes

$$\left. \begin{aligned} \frac{f_1(x)}{\psi(x)} + \frac{f_2(y)}{\phi(y)} \frac{dy}{dx} &= 0 \\ \text{or} \quad \frac{f_1(x)}{\psi(x)} dx + \frac{f_2(y)}{\phi(y)} dy &= 0 \end{aligned} \right\} \quad (4)$$

In (4) also variables are separated. Now (3) or (4) can be integrated and let from 2nd equation of (3)

$\int f_1(x) dx + \int f_2(y) dy = C$, where C is a constant of integration. Let the integrals be

$$F_1(x) + F_2(y) = C$$

This is the general solution of (1) or (2).

19.3.1 SEPARATION BY SUBSTITUTION

It will be found from 19.4 to 19.8 that there are specific techniques for separation of variables. Yet there are some ODEs which cannot be considered

by those techniques. But if a suitable substitution can be made then the new variables can be easily separated. This substitution completely depends on the choice but of no general procedures. Once the substitution is chosen, the following steps expedite the solution :

- (i) Selection of substitution;
- (ii) Differentiation of the substituting equation;
- (iii) Eliminate all the variables except those in (i) and (ii);
- (iv) Solve the *ODE* in (iii) by separation of variables;
- (v) Revive the original variables in (iv).

Examples

1. Solve the *ODE* $(x^2 + 1)(y^2 - 1) dx + xy dy = 0$.

Variables of the same kind are not grouped together in the given equation. To separate the variables, divide the equation by the LCM of the variables $(y^2 - 1)$ and x in their unwanted position. Therefore, the equation becomes,

$$\frac{x^2 + 1}{x} dx + \frac{y}{y^2 - 1} dy = 0$$

in which the variables are separated. Now integrating

$$\int \frac{x^2 + 1}{x} dx + \int \frac{y}{y^2 - 1} dy = C_1$$

$$\text{i.e.} \quad x^2 + \log x^2 + \log(y^2 - 1) = C$$

$$\text{or} \quad y^2 = 1 + \frac{1}{x^2} C e^{-x^2}.$$

This is the general solution of the given equation.

2. Solve the *ODE* $(x + y)^2 \frac{dy}{dx} = a^2$.

The variables cannot be separated at ease. So, let us substitute $x + y = u$. Therefore differentiating w.r.t. x ,

$$\frac{dy}{dx} = \frac{du}{dx} - 1$$

$$\therefore \text{ the given equation becomes } u^2 \left(\frac{du}{dx} - 1 \right) = a^2$$

$$\text{or} \quad \frac{du}{dx} = \frac{u^2 + a^2}{u^2}$$

Now, separating the variables and integrating, the general solution in terms of the original variables is

$$y = a \tan^{-1} \frac{x+y}{a} + C$$

3. Solve the ODE $\frac{x}{y}(dx+dy) + (x+y)(y dx - x dy) = 0$.

Let us substitute $x+y=u$ and $\frac{x}{y}=v$

$$\therefore dx+dy=du, \frac{y dx - x dy}{y^2} = dv \text{ and } y = \frac{u}{v+1}$$

Using in the given equation

$$v du + u \cdot \frac{u^2}{(v+1)^2} dv = 0.$$

Separating variables and integrating

$$-\frac{1}{2u^2} + \log \frac{v}{v+1} + \frac{1}{v+1} = C$$

$\therefore$ the required general solution is $-\frac{1}{2(x+y)^2} + \frac{y}{x+y} + \log \frac{x}{x+y} = C$.

19.3.2 APPLICATIONS

(1) **Geometric applications :** Many problems in geometry can be easily solved with the use of ODE. Some geometric problems related to separation of variables will be discussed. From tangents and normals of Differential Calculus, the established formulae according to figures 133 and 134 are enlisted below :

(i) $\frac{dy}{dx}$ = [Slope of the tangent to the curve $f(x, y) = 0$ at $P(x, y)$].

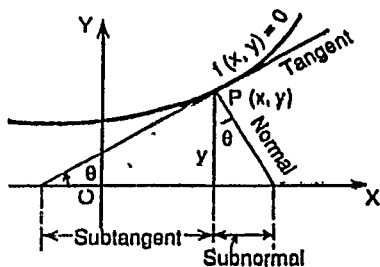


Fig. 133

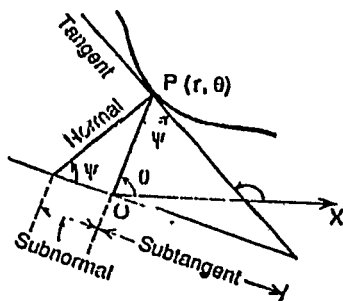


Fig. 134

(ii) $-\frac{dx}{dy}$ = [Slope of the normal to the curve $f(x, y) = 0$ at $P(x, y)$].

(iii) $-y \frac{dy}{dx}$ = Length of subnormal.

$$(iv) y / \left(\frac{dy}{dx} \right) = \text{Length of subtangent.}$$

$$(v) \tan \psi = r \frac{d\theta}{dr} = [\text{Slope of the tangent to the curve } r=f(\theta) \text{ at } P(r, \theta)].$$

$$(vi) r^2 \frac{d\theta}{dr} = \text{Length of polar sub tangent}$$

$$(vii) \frac{dr}{d\theta} = \text{Length of polar subnormal.}$$

(2) **Trajectories** : Dictionary meaning of trajectory is a curve described by a body in space *e.g.* path described by a throwing projectile, path of planets in heavens etc. Geometrically, when two family of curves intersect at a **fixed** angle, α then one is called the trajectory of the other. If $\alpha = \pi/2$, the curves are called **orthogonal trajectories**. The latter type occurs frequently in engineering, physics etc. *e.g.* hydrodynamics, heat conduction, electricity etc. Examples of orthogonal trajectories are (i) lines through origin and concentric circles having centre at the origin, *i.e.* $y = mx$ and $x^2 + y^2 = m^2$, (ii) lines of electric forces and equipotential lines, (iii) isothermal curves and lines of heat flow etc. Similarly $y = x + c$ and $y = c$ are isogonal trajectories.

Determination of trajectory of a given curve : Let the families of curves $F(x, y, c) = 0$ and $G(x, y, k) = 0$ be isogonal trajectories to each other. Let C_1 and C_2 be two curves of F and G families respectively and let they intersect at (x, y) at an angle α (fig. 135). Let the slopes of C_1 and C_2 be y'_g and y'_r . So, from analytic geometry, (cartesian system),

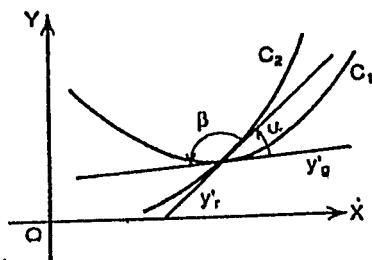


Fig. 135

$\tan \alpha = \frac{y'_g - y'_r}{1 + y'_g y'_r} \therefore \alpha = \tan^{-1} \left(\frac{y'_g - y'_r}{1 + y'_g y'_r} \right)$ if α is acute, for isogonal trajectory.

If $\alpha = \frac{\pi}{2}$, then $1 + y'_g y'_r = 0$ *i.e.* $y'_r = -\frac{1}{y'_g}$.

Similarly, for polar co-ordinates, $y'_g = \left(\frac{r d\theta}{dr} \right)_g$ and $y'_r = \left(\frac{r d\theta}{dr} \right)_r$.

$$\therefore \tan \alpha = \frac{\left(r \frac{d\theta}{dr}\right)_g - \left(r \frac{d\theta}{dr}\right)_r}{1 + \left(r \frac{d\theta}{dr}\right)_g \left(r \frac{d\theta}{dr}\right)_r} \text{ for isogonal trajectory.}$$

For orthogonal trajectory, $\alpha = \pi/2$ and so

$$\left(r \frac{d\theta}{dr}\right)_g = -\frac{1}{\left(r \frac{d\theta}{dr}\right)_r}.$$

Hence if α and a family of curve are given, then its isogonal trajectory can be determined by solving ODE arises from $\tan \alpha$ and for orthogonal trajectory, we have to solve the ODE corresponding to $y'_g y'_r = -1$ or $\left(r \frac{d\theta}{dr}\right)_g \left(r \frac{d\theta}{dr}\right)_r = -1$ of which one derivative is known. Thus many problems in physics and engineering can be solved using trajectories.

Examples

1. Find the equation of the family of curves which intersect the family of curves $r = a\theta$ at a constant angle α .

The problem is same as finding isogonal trajectories of the family of curves $r = a\theta$.

$\therefore r = a\theta \therefore$ diff. w.r.t. θ we have $\frac{dr}{d\theta} = a$. So, the given equation becomes $r = \theta \frac{dr}{d\theta}$. Therefore, the slope of the tangent to the given curve at (r, θ) is $\left(r \frac{d\theta}{dr}\right)_r = \theta$. Now

$$\pm \tan \alpha = \frac{\left(r \frac{d\theta}{dr}\right)_g - \left(r \frac{d\theta}{dr}\right)_r}{1 + \left(r \frac{d\theta}{dr}\right)_g \left(r \frac{d\theta}{dr}\right)_r}$$

[Note. Subscript g indicates given and r indicates required].

$$\text{or, } k = \frac{\theta - \left(r \frac{d\theta}{dr}\right)_r}{1 + \theta \left(r \frac{d\theta}{dr}\right)_r}, \text{ letting } k = \pm \tan \alpha$$

$$\text{or, } \left(r \frac{d\theta}{dr}\right)_r = \frac{\theta - k}{\theta k + 1}$$

or, $r \frac{d\theta}{dr} = \frac{\theta - k}{\theta k + 1}$, dropping subscript r , which is the ODE for the required curve.

$$\therefore \int \frac{dr}{r} = \int \frac{\theta k + 1}{\theta - k} d\theta + C_1$$

or, $r = C(\theta - k)^{k^2 + 1} e^{k\theta}$ where $k = \pm \tan \alpha$

This is the equation of the required curve.

2. Find the orthogonal trajectory of $y = cx^5$.

The ODE corresponding to the given curve is $\frac{dy}{dx} = \frac{5y}{x}$.

$\therefore$ the slope of the tangent to the given curve is $\left(\frac{dy}{dx}\right)_c = \frac{5y}{x}$.

For orthogonal trajectories $\left(\frac{dy}{dx}\right)_c \left(\frac{dy}{dx}\right)_r = -1$

or,
$$\left(\frac{dy}{dx}\right)_r = -\frac{1}{\left(\frac{dy}{dx}\right)_c} = -\frac{x}{5y}$$

$\therefore$ dropping the subscript r , we get $\frac{dy}{dx} = -\frac{x}{5y}$. Now separating the variables and integrating, the required orthogonal trajectory is $x^2 + 5y^2 = c$.

19.3.3 PHYSICAL APPLICATIONS

Proper mathematical modelling of most of the physical problems is really very difficult. But a model of an ideal situation satisfying important properties may be developed through experiments, observations and logic. In this section some simple physical problems will be discussed as an indication of the uses of ODE.

(i) **Motion in straight line—horizontal, vertical and inclined :** Let a body or particle be moving with velocity v , travelling a distance s in time t .

Therefore, from differential calculus velocity $v = \frac{ds}{dt}$ and acceleration

$$f = \frac{dv}{dt} = \frac{d^2s}{dt^2}. \text{ Also acceleration, } f = \frac{dv}{dt} = \frac{ds}{dt} \cdot \frac{dv}{ds} = v \frac{dv}{ds}. \text{ Now from Newton's}$$

second law of motion, the equation of motion is given by $P = mf$.

Let a body of mass m falling under gravity with a velocity v and resisted by a force proportional to v^n . Then the equation of motion is

$$P = W - K v^n, \text{ where } W \text{ is the weight and } K \text{ is a const. of proportionality.}$$

or, $mf = W - K v^n$

or, $m \frac{dv}{dt} = W - Kv^n$. This a first order and first degree ODE (non-linear).

Therefore, separating variables, and integrating, we get

$$\int \frac{dv}{g - \alpha v^n} = t + c, \quad \alpha = \frac{K}{m}, \quad g = \frac{W}{m}.$$

For different values of n , the left hand integral may be evaluated. The final velocity is v when $n \rightarrow \infty$.

Examples

1. A speedboat has weight 322 lbs. The engine drives with a force of 15 lbs. wt.; but the motion is resisted by twice its velocity. If it starts from rest, find its velocity.

Let v be the velocity at any time t (fig. 136).

Initial conditions (I.C.): $v = 0$ when $t = 0$. By hypothesis, $P = 15 - 2v$ and mass, $m = \frac{322}{32.2} = 10$ slugs.

$$\therefore P = m \frac{dv}{dt} = 15 - 2v \text{ is the equation of motion.}$$

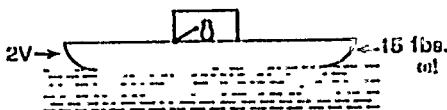


Fig. 136

$$\text{or, } 10 \frac{dv}{dt} = 15 - 2v$$

$$\text{Solving the ODE, } v = \frac{1}{2} (15 - C e^{-v/5})$$

Using I.C., $C = 15$. Therefore, the required velocity, $v = 7.5 (1 - e^{-2t})$.

2. E is a sphere of uniform density. Its weight and radius are respectively equal to those of the earth. It attracts a body of weight W at a distance of S feet from its centre. If the radius of earth is $R = 3960$ miles, then what will be the velocity of the body falling from rest to its surface from a distance of $4R$? What will be that velocity for infinite distance?

Let v be the velocity of the body at any time t (fig. 137).

I.C.: $v = 0$ when $s = 4R$. From Newton's law of gravitation,

$$f = \frac{dv}{dt} = \frac{k}{s^2}, \quad k \text{ is the const. of proportionality.}$$

But for $s = R$, $f = -g$.

$$\therefore -g = \frac{k}{R^2} \text{ or } k = -gR^2$$

$$\therefore f = -\frac{gR^2}{s^2}$$

Now the equation, from Newton's 2nd law of motion, is $P = mf$

$$\text{or, } mv \frac{dv}{ds} = -m \cdot \frac{gR^2}{s^2}$$

$$\text{or, } \frac{W}{g} v \frac{dv}{ds} = -W \frac{R^2}{s^2}$$

It's general solution, is $v^2 = \frac{2gR^2}{s} + C$

$$\text{From I.C., } C = -\frac{gR}{2}$$

$$\therefore v^2 = 2gR^2 \left(\frac{1}{s} - \frac{1}{4R} \right)$$

On the surface of the sphere, $s = R$. Therefore

$$v^2 = \frac{3}{2} gR = \frac{3}{2} \times .006093 \times 3960 \text{ m/sec. or } v = 6.02 \text{ m/sec}$$

Second part : When the body falls from infinity, $v = 0$, $s = \infty$.

$$\therefore C = 0 \text{ and } v^2 = 2g \frac{R^2}{s}. \text{ On the surface of the sphere}$$

$$s = R \therefore v^2 = 2gR = 2 \times .006093 \times 3960 \text{ m/sec.}$$

$$\therefore v = 6.95 \text{ m/sec.}$$

3. A car with its passengers weighs 1000 lbs. It is coming down an inclined road of inclination $5^\circ 15'$. Frictional force is 40.2 lbs. wt. and wind resistance in feet/sec. is twice the velocity of the car. What will be its velocity from rest after t seconds and after 10 seconds? What will be its final velocity?

Let v be the velocity of the car.

$$\text{I.C. : } v = 0 \text{ when } t = 0$$

$$\text{B.C. : (i) } v_{10} = ? \text{ when } t = 10 \text{ seconds}$$

$$\text{(ii) } v_{\infty} = ? \text{ when } t = \infty$$

Gravitational force draws down and this force is the weight of the car, $W = mg$. From fig. 138,

$$F_1 = W \sin \alpha = 1000 \sin 5^\circ 45' = 100.2 \text{ ft/sec.}$$

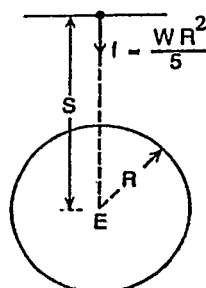


Fig. 137

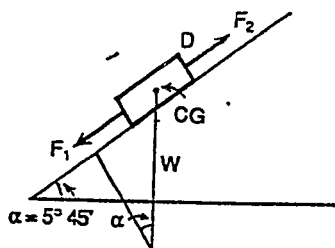


Fig. 138

Now from Newton's 2nd law of motion,

$$m \frac{dv}{dt} = 60 - 2v$$

It's solution is $\frac{1}{2} \log(30 - v) = -\frac{1}{m}t + C_1$

or, $\log(30 - v) = -\frac{2 \times 32.17}{1000}t + C, g = 32.17 \text{ ft/sec}^2$

Using I.C., $C = \log 30$.

$$\therefore \log(30 - v) = -.064t + \log 30$$

or, $v = 30(1 - e^{-.064t})$

When $t = 10$, $v_{10} = 30(1 - e^{-.064 \times 10}) = 14.24 \text{ ft/sec}$.

When $t = \infty$, $v_{\infty} = \text{final velocity} = 30 \text{ ft/sec}$.

(ii) Heat flow :

Heat flows from higher temperature to lower temperature. This property is common to heat, liquid or gaseous flows. The heat flow is said to be in **steady state** if every parallel surfaces of a wall attains the same temperature and each of those surfaces is called an **isothermal surface**. But temperature in each iso-surface is different at different position. So, temperature is a function of position i.e. distance. If the distance of a point from one end of a wall is x , then temperature, T is a function of x i.e. $T = T(x)$.

Let the width of a solid plane be $x_2 - x_1 = L$ (fig. 139).

Let its surface temperature be T_1 and T_2 . The law of heat flow in it is "Heat flow, Q is proportional to the negative of the product of the cross-sectional area, A and the temperature gradient, $\frac{dT}{dx}$ " i.e.

$$Q \propto -A \frac{dT}{dx}$$

$$\therefore Q = -kA \frac{dT}{dx}$$

The -ve sign indicates heat flow towards the decreasing temperature. Here k is constant of proportionality or **thermal conductivity** and the equation is called the **Fourier heat conduction equation**. Separating the variables and integrating,

$$\frac{Q}{kA} \int_{x_1}^{x_2} dx = - \int_{T_1}^{T_2} dT$$

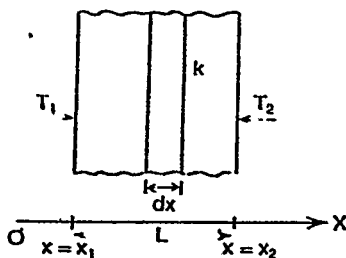


Fig. 139

or, $Q = \frac{kA}{L} (T_1 - T_2)$ where $L = x_2 - x_1$.

Similarly, for circular pipe the Fourier equation of heat conduction becomes (fig. 140)

$$Q = -kA \frac{dT}{dr}$$

where $A = 2\pi rL$, L is the length of the pipe and therefore

$$Q = -k(2\pi L) r \frac{dT}{dr}$$

Separating the variables and integrating,

$$\frac{Q}{2\pi kL} \int_{r_1}^{r_2} \frac{dr}{r} = - \int_{T_1}^{T_2} dT$$

$$\therefore Q = \frac{2\pi kL(T_1 - T_2)}{\log r_2 - \log r_1}$$

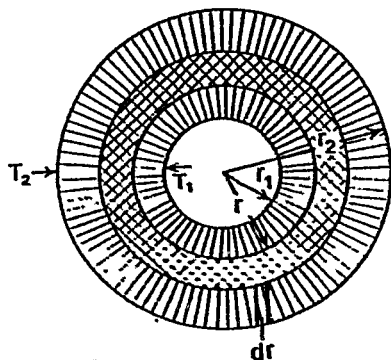


Fig. 140

Examples

1. The length, breadth and height of the walls of an warehouse are 10 meters, 25 cms, and 3 meters respectively. The walls are made of cement. It's inner and outer temperatures are 10°C and -10°C respectively. If the thermal conductivity of cement be .0022, then find one day's heat loss.

Let Q be the heat loss per day.

I.C.: $x = 0$, $T = 10^\circ\text{C}$. B.C.: $x = 25$ cm, $T = -10^\circ\text{C}$. Here

$A = 300 \times 1000$ sq. cms and $k = .0022$. Therefore,

$$Q = -kA \frac{dT}{dx}$$

or, $\frac{Q}{kA} \int_0^{25} dx = - \int_{10}^{-10} dT$

$$\therefore Q = \frac{.0022 \times 300 \times 1000 \times 20}{25} = 528 \text{ cal/sec.}$$

$$\therefore \text{heat loss per day} = 528 \times 3600 \times 24 \text{ cal.} = 46 \times 10^6 \text{ calories.}$$

(iii) Electric current :

A simple series circuit (fig. 141) consists of a constant *EMF*, E , a resistance R and an inductance L . Resistance R is a circuit parameter. It's symbol is $\text{---}\text{---}\text{---}$. It prevents flow of current and dissipate energy in the form of heat. If the switch of the circuit is closed, current i will flow through the resistor and this will cause a *voltage drop* i.e. electric potential is different at

the two ends of the resistor. This is governed by *Ohm's law* which states that *the voltage drop E_R across a resistor is proportional to the instantaneous current i i.e. $E_R = Ri$ where R is the resistance of the resistor, measured in ohms, i measured in amperes and E_R in volts.*

Inductance, L is a circuit parameter. It also resists the flow of current. Its symbol, is . Inductor differs from resistor in that when a current flows through an inductor, there is no voltage drop unless the current is changing.

When this is the case, the voltage drop E_L across the inductor is proportional to the rate of change of current i.e. $E_L = L \frac{di}{dt}$, L is measured in henry.

To establish ODE governing behaviour of currents **Kirchhoff's law**, are used. They are

(i) the algebraic sum of emfs in any loop of a network equals the algebraic sum of products Ri and $L \frac{di}{dt}$ i.e. $\Sigma E = \Sigma \left(Ri + L \frac{di}{dt} \right)$ (for series circuit).

(ii) the algebraic sum of the currents towards any branch point of a network is zero i.e. $\Sigma i = 0$.

The first law is applicable for the present simple series circuit. At time t if E is the electromotive force (e.m.f.) and i is the instantaneous current then by Kirchhoff's first law

$$E - Ri - L \frac{di}{dt} = 0$$

$$\text{or, } L \frac{di}{dt} + Ri = E \quad (1)$$

Let at $t = 0$, $E = \text{const.}$ (i.e. no previous current is assumed).

$$\therefore \text{ from (1), } L \frac{di}{dt} = (E - Ri)$$

$$\text{or, } \int \frac{di}{E - Ri} = \int \frac{1}{L} dt + C$$

$$\text{or, } \log (E - Ri) = -\frac{R}{L} t + C \quad (2)$$

But at $t = 0$, $i = 0$ as assumed. So, $C = \log E$.

$$\therefore (2) \text{ becomes, } i = \frac{E}{R} (1 - e^{-Rt/L}) \quad (3)$$

Equation (3) shows that as time increases, the current i also increases.

But at $t = \infty$, $i = \frac{E}{R}$ i.e. $E = Ri$ which is Ohm's law.

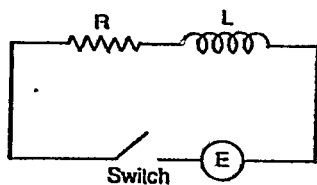


Fig. 141

Example

1. A series circuit consists of an e.m.f. of E volts, an inductance of 2 henrys and a resistor of 20 ohms. If $E = 100$ volts and initially no current flows, then what will be current after 0.01 secs.

Here I.C. : $i = 0$ when $t = 0$. The ODE is $L \frac{di}{dt} + Ri = E$

$$\text{or,} \quad \int \frac{di}{E - Ri} = \frac{1}{L} \int dt + C$$

$$\text{or,} \quad \log(E - Ri) = -\frac{R}{L}t + C$$

Using I.C., $C = \log E$

$$\therefore i = \frac{E}{R}(1 - e^{-Rt/L})$$

Here, $E = 100$, $R = 20$, $L = 2$ and $t = .01$ secs.

$$\therefore i = 5(1 - e^{-1}) = .4758 \text{ amperes.}$$

(iv) Miscellaneous applications :

(a) **Flow of water through an orifice :** Let a water tank containing water has a hole at the bottom. When the hole is open the water flows out of the tank. The rate of flow through the hole is $(a, \text{ area of the hole}) \times (v, \text{ velocity of flow})$ i.e. av . This velocity of flow v is given in hydraulics by Torricelli as $\sqrt{2gh}$ or by Borda as $(.6)\sqrt{2gh}$ for smaller hole where h is the height of water level above the hole. Now from the condition of continuity, the rate of change of volume, V of water in the tank for any time t must be equal to the amount of water flowing out through the hole or orifice i.e.

$$\frac{dV}{dt} = -av \quad \text{where } v = \sqrt{2gh} \text{ or } 0.6\sqrt{2gh}$$

according as the hole is small or too small. The $-ve$ sign is for flowing out.

(b) **Growth and decay :** Compound interest, growth and decay of bacteria, food supply, humidity, radio-active decay of radium, uranium or any metal etc. are governed by the same law – the rate of change is directly proportional to the amount present. Let Q be the amount present at any time t . Therefore $\frac{dQ}{dt} = \pm KQ$, K is the constant of proportionality and $-ve$ sign is for decay and $+ve$ sign for growth.

(c) **Dilution (Compartmental problem) :** Let a vessel of water contains V gallon of water and x lbs. of salt at any time t . Then the concentration, c of salt is $\frac{x}{V}$. Let ' a ' gallons of brine enter the vessel and after complete mixing leaves it keeping the original volume unchanged. Therefore

the rate of salt entering and leaving vessel are ac and $a \frac{x}{V}$ respectively. So, from continuity, the rate of change of salt is given by the ODE

$$\begin{aligned}\frac{dx}{dt} &= (\text{rate of entry}) - (\text{rate of exit}) \\ &= ca - a \frac{x}{V}.\end{aligned}$$

(d) **Cooling bodies** : Newton's law of cooling is "The rate of change of temperature difference between an object and its surrounding medium is proportional to the temperature difference." Let T be the temperature of the object and T_0 that of the surroundings at any time t , then

$$\frac{dT}{dt} = K(T - T_0), K \text{ constant of proportionality.}$$

Exercises 19 (1)

1. Solve the following ODEs by separation of variables :

Q(i) $(1+y^2) dx + (1+x^2) dy = 0$, Q(ii) $\frac{dy}{dx} = \frac{4-y^2}{4+x^2}$,

Q(iii) $(e^x + e^{-x}) dy + (e^y + e^{-y}) dx = 0$,

Q(iv) $x(1+y^2) dx + y(1+x^2) dy = 0$,

Q(v) $\sec^2 x \sec y dx + \sin y dy = 0$,

(vi) $x(1-y) dx + (1+y^2)(x-1) dy = 0$,

(vii) $e^x \tan y dx + (1-e^x) \sec^2 y dy = 0$,

Q(viii) $x^2(1+y^2) dx + y^2(1+x^2) dy = 0$,

(ix) $e^{\tan^{-1} x} \sqrt{1-y^2} dx + e^{\sin^{-1} y} (1+x^2) dy = 0$,

Q(x) $x \log x dy + \sqrt{1+y^2} dx = 0$,

(xi) $(y^2 + 4y + 8) \tan x dx + \cos x dy = 0$,

(xii) $\frac{dy}{dx} = \frac{e^x (\sin x + \cos x)}{e^y (\cos y - \sin y)}$, Q(xiii) $\log \left(\frac{dy}{dx} \right) = ax + by$,

(xiv) $xy(x dy + y dx) = 6y^3 dy$.

2. Solve using separation by substitutions :

(i) $(x+y)^2 dy = dx$, Q(ii) $(2x+y)^2 dx - 2dy = 0$,

(iii) $x + yy' = F(x^2 + y^2)$, (iv) $y(1 + e^{x/y}) dx + e^{x/y}(y-x) dy = 0$,

(v) $(1 + e^{y/x}) dy + (1 - y/x) e^{y/x} dx = 0$,

Q(vi) $x(x+y)(dx+dy) = \frac{y}{x}(x dy - y dx)$,

$$(vii) x^m y^n (x dy + y dx) = \phi(xy) dx,$$

$$(viii) dx + dy = (x+y) \left(1 + \frac{y}{x}\right)^2 (x dy - y dx),$$

$$\mathcal{Q}(ix) y^3 (x dx + y dy) = (x^2 + y^2)^3 dy,$$

$$(x) \sin^{-1} \left(\frac{dy}{dx} \right) = x + y,$$

$$\mathcal{Q}(xi) \frac{dy}{dx} = \sqrt{y-x},$$

$$(xii) x^m y^n (2x dy + y dx) = \phi(xy^2) dx, \quad (xiii) xy' = yF(xy).$$

3. Solve the following ODEs using given conditions :

$$\mathcal{Q}(i) x dx + y dy = 0 \text{ when } y=2, x=1,$$

$$\mathcal{Q}(ii) \theta \frac{dr}{d\theta} = -2r, \text{ given } r=9 \text{ when } \theta = -1/3,$$

$$(iii) e^x \sec y dx + (1 + e^x) \sec y \tan y dy = 0, \text{ given } y=60^\circ \text{ when } x=3,$$

$$(iv) a \left(x \frac{dy}{dx} + 2y \right) = 2y \left(\frac{dy}{dx} \right) \text{ given } y=a \text{ when } x=2a,$$

$$(v) 2y dx + x^2 dy = -dx, \text{ given } y=7/2 \text{ when } x=1/\log 2.$$

4. (i) Find the equation of the curve when the radius vector bisects the angle β between the tangent and the normal to the curve at any point.
 (ii) Find the equation of the curve for which subnormal is proportional to the abscissa of any point.
 (iii) Determine the equation of a curve if its slope at any point is equal to the square of the abscissa and it passes through $(-1, 1)$.
 (iv) If the polar subtangent to a curve is unity, find its equation.
 (v) The tangent at any point (x, y) to a curve intersects the x -axis at $(x-1, 0)$, find the equation of the curve.
[Hint : Write the equation of the tangent through $(x-1, 0)$]
 (vi) Find the equation to the curve when the portion of its tangent between the point of contact and the x -axis is bisected by the y -axis.
[Hints : Point of bisection is $(0, y/2)$. Write the equation of the tangent through $(0, y/2)$]
 (vii) Find the equation to the curve when the portion of its tangent between the point of contact and the y -axis is bisected by the x -axis.
 (viii) If the segment of each tangent between the co-ordinate axes is bisected by the point of contact, then show that the curve is a rectangular hyperbola.
[Hints : Let the points of intersection with axes be $(2x, 0)$ and $(0, 2y)$, so that point of contact will be (x, y)]
5. (i) Find the isogonal trajectories of the family of curves $y(x+c)=1$ making an angle 45°

- (ii) Find the orthogonal trajectories of the family of curves $y^2 = Cx^3$.
- (iii) Find both isogonal and orthogonal trajectories of the family of curves $r = a \sin \theta$.
- (iv) Determine the orthogonal trajectories of the curve $r = K \sin \theta$.
- (v) Determine the isogonal trajectories of the curve $4y = 3x + c$ intersecting at an angle $\tan^{-1} 4$.
6. (i) A steamer weighs 64,000 tons. It's propeller starts moving from rest with a force of 2,00,000 lbs. wt. If the resistance of water is 100,000V lbs. wt., then the velocity V in F.P.S. system is a function of time, t and the final velocity is miles-hours unit. ($g = 32 \text{ ft/sec}^2$)
- (ii) A car weighing 1000 lbs. coming down an 5° inclined plane is resisted by a frictional force of 40 lbs. wt. and wind resistance of 1.5 time the velocity of the car in FPS system. Determine its velocity at any time t , after 10 secs. and at infinite time.
7. (i) The inner and the outer radii of a hollow sphere are 4 cms and 9 cms respectively. It's thermal conductivity is .75. The inner and the outer faces are maintained at temperatures 100°C and 0°C respectively. If heat flows from inner to outer, find the heat loss and the temperature T at a distance of 5 cms from the centre.
- [Hints : $A = 4\pi r^2$, I.C. : $r_1 = 4$, $T_1 = 0^\circ\text{C}$, $r_2 = 9$, $T_2 = 100^\circ\text{C}$]
- (ii) The breadth of the wall of a furnace is 45.7 cms. It's plaster has a thermal conductivity of .0024. The two faces of the wall are maintained at temperatures 1000°C and 120°C respectively. Find the heat loss per square unit per hour.
8. (i) A simple series circuit consists of 150 volts (e.m.f.), an inductance of 3 henries and a resistance of 3 ohms. If $i = 0$ when $t = 0$ then determine the current flow after .01 secs.
- (ii) If $E = 10$ volts, $R = 5$ ohms, $L = t + 1$ henries ($0 \leq t \leq 1$), $L = 2$ henries ($t > 1 \text{ sec.}$) and $i(0) = 0$ then determine the current flow as a function of time t .
9. (i) A water tank of 4 ft. length, 3 ft. breadth and 9 ft. height has a hole of radius 1" at the bottom. It is full of water. If the hole remains open when the tank will be empty and when height of water level will be 5 feet?

[Hints : Vol = $12h$, area of hole = $\frac{\pi}{12^2}$ sq. ft., v , velocity of flow = $.6\sqrt{2gh}$, h is the instantaneous height of water level,

$$\frac{dv}{dt} = 12 \frac{dh}{dt} = -\frac{\pi}{12^2} \cdot (.6) \sqrt{2gh}$$

- (ii) A vessel contains 150 lbs. salt in 100 gallon water. A brine of one pound per gallon enters the vessel at the rate of 2 gallon/minute and leaves the vessel at the same rate after completely stirred. What will be the amount of salt in the vessel after one hour ?

[Hints : $ca = 2$, $\frac{a}{V}x = \frac{2}{100}x$, at $t = 0$, $x = 150$ lbs.]

- (iii) The half life of radium is 1600 yrs. Find the amount of 500 mg of radium present after 250 yrs.

[Hints : Half life means time in which 50% will disappear. I.C. : $t = 0$, $Q = 500$; B.C. : $t = 1600$, $Q = 125$]

- (iv) The lines of electric forces of two oppositely charged sources of the same strength at $(-1, 0)$ and $(1, 0)$ are circle. Find the equipotential lines (orthogonal trajectories).

- (v) A body heated at 100°C is kept in air of 20°C for cooling. In 10 minutes temperature of the body falls to 60°C . Express temperature as a function of time.

- (vi) A hemispherical tank of radius R is initially filled with water. At the bottom of the tank, there is a hole of radius r through which the water drains under the influence of gravity. Find the depth of the water tank at any time t , and determine how long it will take the tank to drain completely.

[Hints : Fig. 142, dh elementary height of the level of water from bottom. So, $r_1^2 = R^2 - (R-h)^2 = 2Rh - h^2$, Instantaneous vol. $dV = \pi r_1^2 dh = \pi (2Rh - h^2) dh$. Also vol. of water leaving $= dV = \pi r^2 \sqrt{2gh} dt$

$$\therefore \pi (2Rh - h^2) dh = -\pi r^2 \sqrt{2gh} dt$$

$$\text{or } (2R\sqrt{h} - h^{3/2}) dh = -r^2 \sqrt{2g} dt.$$

Integrate and use I.C. $h = R$

at $t = 0$ and find t when $h = 0$]

10. Evaluate $I(y) = \int_0^{\pi/2} \frac{\log(1+y \sin^2 x)}{\sin^2 x} dx$

by showing that

$$I'(y) = \frac{\pi}{2\sqrt{1+y}}$$

[Hints : Differentiate under the integral sign. Note that $I(0) = 0$ which gives $C = -\pi$]

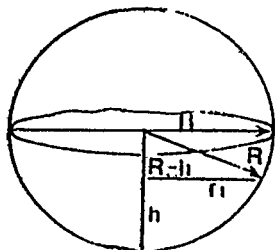


Fig. 142

11. (i) (**Pursuit curve or Tractrix**) : A heavy weight P located at $(a, 0)$ is attached to a man Q by a chain of length ' a '. Find the path that P will follow if Q starts at the origin and moves up the y -axis. This path is called pursuit curve or tractrix.

[Hints : Note that the line PQ (fig. 143) is tangent to the path of P . Hence the slope is given by

$$\frac{dy}{dx} = -\frac{\sqrt{a^2 - x^2}}{x}$$

$\therefore$ the length of the chain PQ is ' a '.]

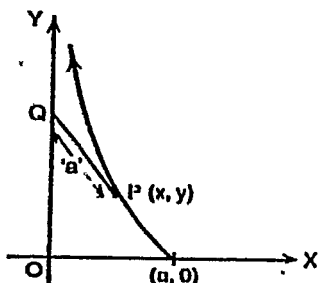


Fig. 143

- (ii) Strontium 90 (S_r^{90}) has a half life of twenty five years. If 10 grams of S_r^{90} are initially placed in a sealed container, how many grams will remain after ten years?
- (iii) Carbon 14 (C^{14}) has a half life of 5700 years and is uniformly distributed in the atmosphere in the form of carbon dioxide. Living plants absorb carbon dioxide and maintain a fixed ratio of C^{14} to the stable element C^{12} . At death, the disintegration of C^{14} changes this ratio. Compare the concentration of C^{14} in two identical pieces of wood, one of them freshly cut, the other 2000 years old.

[Hints : ODE for decay is $\frac{dQ}{dt} = -KQ$. I.C. : $t=0, Q=12$ for fixed maintenance of stable C^{12} and B.C. : $t=5700, Q=7$ = half life

of C^{14} . Required ratio = $\frac{Q}{14} = \frac{11.054}{14} \approx .789$]

- (iv) Water drains out of a cylindrical tank of radius 4 ft. through a hole of radius 1 inch in the bottom of the tank. How long will it take the tank to empty if water is initially 10 ft. deep?

[Hints : Vol. of water, $V = \pi 4^2 h$ at any height h , take velocity of flow, $v = \sqrt{2gh}$]

- (v) Find the equation of the family of curves which have the property that the tangent at any point P determines with co-ordinate axes a triangle whose area is equal to twice the area of the rectangle determined by P and the co-ordinate axes.

[Hints : Tangent intersects the axes at $A(x_1, 0)$ and $B(0, y_1)$. $\tan \psi = -\frac{y}{x}$. Draw PM and $PN \perp$ rs to axes.

$\tan NBP = \tan \left(\psi - \frac{\pi}{2} \right)$ gives $y_1 = 2y \cdot x_1 =$ subtangent. Given condition $x_1 y_1 = 4xy$

Answers

- $x+y=c(1-xy)$
 - $\frac{1}{2} \tan \frac{x}{2} = \frac{1}{4} \log \frac{2+y}{2-y} + c$
 - $e^x + e^y = c(1 - e^{x+y})$
 - $(1+x^2)(1+y^2) = c$
 - $4 \tan x - \cos 2y = c$
 - $2x + 2 \log(x-1)(1-y)^2 - y^2 - 2y = c$
 - $\tan y = c(1 - e^y)$
 - $x+y - \tan^{-1} x - \tan^{-1} y = c$
 - $e^{\tan^{-1} x} + e^{\tan^{-1} y} = c$
 - $\sec x + \frac{1}{2} \tan^{-1} \left(\frac{y+2}{2} \right) = c$
 - $(y + \sqrt{1+y^2}) \log x = c$
 - $e^x \sin x = e^y \cos y + c$
 - $ae^{-by} + be^{\alpha x} = c$
 - $(xy)^2 = 3y^4 + c$
- $y = \tan^{-1}(x+y) + c$
 - $2x+y = 2 \tan(x-c)$
 - $2x = \int \frac{dV}{F(V)} + c, v = x^2 + y^2$
 - $x + ye^y = c, v = x/y$
 - $y + xe^y = c, v = y/x$
 - $u = v - \log(1+v), u = x+y, v = y/x$
 - $x^{n-m+1}/(n-m+1) = \int u^n/\phi(u) du + c, xy = u$
 - $(2y+cx)(x+y)^2 + x = 0, u = x^2 + y^2, v = xy$
 - $2(x^2 + y^2)^2 - y^2 = cy^2(x^2 + y^2)^2$
 - $\tan(x+y) - \sec(x+y) = c + x$
 - $\sqrt{y-x} + \log(\sqrt{y-x}-1) = \frac{1}{2}x + c$
 - $(2x^{(n-2m+1)/2})/(n-2m+1) = \int \frac{v^{(n-1)/2}}{\phi(v)} dv + c, v = xy^2$
 - $\log x = \int \frac{du}{u(1+F(u))}, u = xy$
- $x^2 + y^2 = 5$
 - $r\theta^2 = 1$
 - $1 + e^x = 2(1 + e^3) \cos y$
 - $x^2 y = 4a^3 e^{-1} y/a$
 - $2y + 1 = 2e^{2/x}$
- $r = ce^{\theta}$ when $\theta = 45^\circ$
 - $3y^2 = 2kx^3 + c$
 - $3y = x^2 + 4$
 - $r(c-\theta) = 1$
 - $y = ce^x$
 - $y^2 = cx$
 - $y = cx^2$
- $y+1 = (y^2-1)ce^{x+y}, y = 2 \tan^{-1} y + x + c$
 - $2x^2 + 3y^2 = c$
 - $r = c \sin(\theta \pm \alpha), r = c \cos \theta$
 - $r \sin \theta = c$
 - $13x + 16y = c, 19x + 8y = c$
- $V = 20(1 - e^{-0.025t}), V_{\infty} = 20 \text{ ft/sec} = 13.63 \text{ m/hr}$
 - $V_1 = 31.5(1 - e^{-0.48t}), V_{10} = 12.1 \text{ ft/sec}, V_{\infty} = 31.5 \text{ ft/sec}$
- $2160 \pi \text{ cal/sec}, 64^\circ \text{C}$
 - $1.66 \times 10^6 \text{ cal}$
- 476 amps.
 - $i = 2[1 - (1+t)^{-5}], i = 2(1 - e^{-(5/2)t})$
- $\frac{36}{\pi} \text{ minutes; } 12(3 - \sqrt{5})/\pi \text{ minutes}$
 - 115.1 lbs.
 - 455 mg.
 - $(x+c_1)^2 + y^2 = c_1^2 - 1$
 - $T = 20 + 87e^{-.693t}$
 - $\text{Const. of int.} = \frac{14}{15} R^{5/2}, t = \frac{14}{15} \frac{R^{5/2}}{r^2 \sqrt{2g}} \text{ sec.}$
- $I(y) = \pi(\sqrt{1+y}-1)$
- $y = a \log \left(\frac{a + \sqrt{a^2 - x^2}}{x} \right) - \sqrt{a^2 - x^2}$
 - 7578 gm.
 - 789
 - 30.4 minutes
 - $y^2 = cx$

19.4 HOMOGENEOUS EQUATIONS

The general homogeneity defined in 17.3.1 (c) is **different** from that in this section. The definition that will be given here is completely confined for first order and first degree *ODE* of this section. This homogeneous is the analogue of the homogeneous algebraic functions dealt in algebra as well as in Euler's theorem in partial differentiation of differential calculus.

Definition : If $P(x, y)$ and $Q(x, y)$ are two homogeneous functions of the same degree then the most general first order and first degree *ODE*

$$P(x, y) dx + Q(x, y) dy = 0 \quad (1)$$

is called a homogeneous *ODE* of the first order and first degree.

Test for homogeneous function : There are several tests for homogeneous function. Of which one important test is the following :

Replace x and y by tx and ty and if $f(tx, ty) = t^n f(x, y)$, then $f(x, y)$ is a homogeneous function of degree n .

Substitution : Once the equation is identified as a homogeneous *ODE*, express one variable (call first variable) as a product of second variable and an arbitrary function of second variable. Normally, y is expressed as $v(x) x$ i.e. $y = v(x) x$ where v is a function of x . Here no guessing or searching is necessary for substitution as in 19.3.1 and the rest of the problem becomes a straight forward routine work as follows :

Procedure : Diff. the substitution, we have, $\frac{dy}{dx} = v + x \frac{dv}{dx}$. Since $P(x, y)$ and $Q(x, y)$ are homogeneous of the same degree, say n , so $P(x, y) = x^n P(1, v)$ and $Q(x, y) = x^n Q(1, v)$. Therefore, writing equation (1) as $\frac{dy}{dx} = -\frac{P(x, y)}{Q(x, y)}$ and using substitution and the new expression for $\frac{dy}{dx}$, we get

$$v + x \frac{dv}{dx} = -\frac{P(1, v)}{Q(1, v)} = F(v), \text{ (say)}$$

$$\text{or} \quad x \frac{dv}{dx} = F(v) - v$$

Now, separation of variables is very easy as

$$\frac{1}{x} dx = \frac{1}{F(v) - v} dv$$

$$\text{Integrating, } \log x = \int \frac{1}{F(v) - v} dv + c$$

which is the general solution.

N.B. : If the equation (1) is homogeneous of degree $n \neq -1$ then it can be more easily solved by the method of 19.5 next.

19.4.1 APPLICATIONS

ODE of homogeneous type may arise in many physical and geometrical applications. Formulae for tangent and normal, trajectory as discussed in 19.3.1 remain same. Here pursuit curves will be discussed only. These play an important role in the pursuance of any object like prey of any kind by the pursuer like hunter of any kind and this pursuance gives interesting *ODE* for study.

(i) Pursuit curves

in cartesian co-ordinates :

Pursuit curve is a curve or path described by an object in pursuance of another fixed or moving object. Let us consider the pursuit curve described by an aeroplane in moving from one port to another. Let the landing port be considered

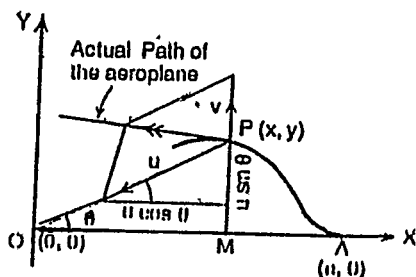


Fig. 144

as origin of reference, O and the departing port, A be at a distance in miles from O , both lying in the same plane. Let OA be considered as x -axis and y -axis be perpendicular to OA at O (fig. 144). Let wind blows towards south perpendicular to y -axis with a velocity of v m/hr and the velocity of the aeroplane be u m/hrs. Let the position of the aeroplane be $P(x, y)$ at time t in its path and $\vec{PO} = \vec{u}$ makes an angle θ with OA . Since u and v are the velocities of the aeroplane and the wind, so, their resultant velocity along the diagonal of the parallelogram contained by them is the actual path of the aeroplane and that path must be tangential to the curve traced by the aeroplane. Hence the slope of the curve $\frac{dy}{dx}$ at $P(x, y)$ expressed in terms of x, y will be the equation of motion. Now the components of u in the directions of x and y are

$$\frac{dx}{dt} = -u \cos \theta \quad \text{and} \quad \frac{dy}{dt} = -u \sin \theta \quad (1)$$

respectively. Considering the velocity of wind the effective component in the y -direction is

$$\frac{dy}{dt} = -u \sin \theta + v \quad (2)$$

Also from ΔPOM , $\sin \theta = \frac{y}{\sqrt{x^2 + y^2}}$ and $\cos \theta = \frac{x}{\sqrt{x^2 + y^2}}$

$$\therefore \frac{dx}{dt} = -\frac{ux}{\sqrt{x^2 + y^2}} \quad \text{and} \quad \frac{dy}{dt} = -\frac{uy}{\sqrt{x^2 + y^2}} + v \quad (3)$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{y - k\sqrt{x^2 + y^2}}{x}, \quad k = \frac{v}{u}, \quad (4)$$

Equation (4) is the ODE of homogeneous type for the equation of motion with initial condition (I.C.) as :

$x = a, y = 0$ at $t = 0$. Substituting $W = \frac{y}{x}$, the general solution of (4) is

$$\log(W + \sqrt{W^2 + 1}) = -k \log x + c$$

I.C. gives $c = k \log a$. Therefore

$$\log(W + \sqrt{W^2 + 1}) = -k \log x + k \log a$$

$$\text{or,} \quad W = \frac{1}{2} \left[\left(\frac{x}{a} \right)^{-k} - \left(\frac{x}{a} \right)^k \right]$$

$$\text{or} \quad y = \frac{a}{2} \left[\left(\frac{x}{a} \right)^{(1-v/u)} - \left(\frac{x}{a} \right)^{(1+v/u)} \right] \quad (5)$$

is the required pursuit curve of the aeroplane.

Cor. 1 : Let $u = v$ in (5), then

$$y = \frac{a}{2} \left(1 - \frac{x^2}{a^2} \right) \text{ or } x^2 = -2a \left(y - \frac{a}{2} \right)$$

The equation represents a parabola (fig. 145) with vertex at $(0, a/2)$. The curve intersects the x -axis at $(-a, 0)$ and $(a, 0)$. So, the aeroplane passes over the destination O and lands somewhere back of O . So, it will **never reach** the destination.

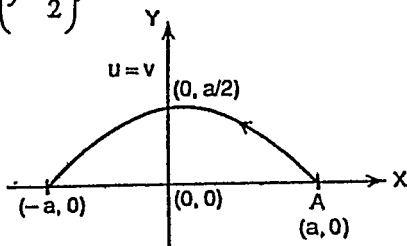


Fig. 145

Cor. 2 : If $v > u$ i.e. $\frac{v}{u} > 1$,

then $1 - \frac{v}{u} < 0$.

From (5), the expression

$$\left(\frac{x}{a} \right)^{1-(v/u)} = \left(\frac{a}{x} \right)^{(v/u-1)} \rightarrow \infty \text{ as } x \rightarrow 0$$

$\therefore y \rightarrow \infty$ as $x \rightarrow 0$ in (5).

So in such case the aeroplane will **not reach** the destination O but to some other place in infinity (fig. 146).

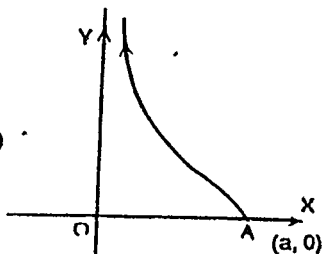


Fig. 146

Cor. 3 : If $v < u$, then $\frac{v}{u} < 1$.

So $\left(1 - \frac{v}{u}\right) > 0$. Therefore, from (5),

$y = 0$ when $x = 0$. Hence the aeroplane will *positively reach* the destination O (fig. 147).

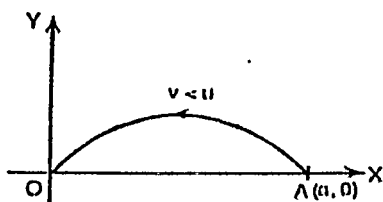


Fig. 147

(ii) Pursuit curves in polar co-ordinates :

Fig. 148, the radial and transverse components of the wind velocity v are

$$\frac{dr}{dt} = v \sin \theta \quad \text{and}$$

$$r \frac{d\theta}{dt} = v \cos \theta.$$

$\therefore$ the resultant velocity of the aeroplane in the radial direction is

$$\frac{dr}{dt} = -u + v \sin \theta$$

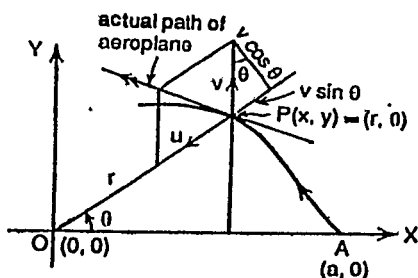


Fig. 148

$$\therefore \frac{1}{r} \frac{dr}{d\theta} = \frac{-u + v \sin \theta}{v \cos \theta} = k \sec \theta + \tan \theta, k = -\frac{u}{v}$$

Its general solution is $cr = (\sec \theta + \tan \theta)^k \sec \theta$

Using initial condition $r = a$, $\theta = 0$ at $t = 0$, we get $c = a^{-1}$.

$\therefore$ the required pursuit curve in polar co-ordinate is

$$r \cos \theta = a (\sec \theta + \tan \theta)^{-u/v}.$$

(iii) Pursuit curve when wind direction makes an angle α with the vertical :

The determination of pursuit curve for the aeroplane under this wind direction can be approached in two ways : (a) taking wind direction as y-axis, (b) normal to y-axis as perpendicular to the line joining the landing and the departing ports.

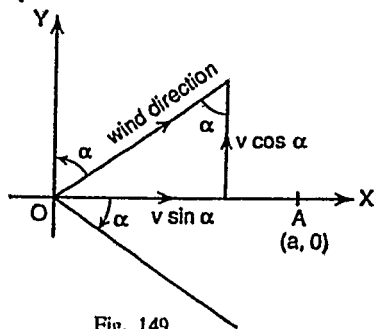


Fig. 149

Case (a) : If wind direction is considered as y-axis, then the corresponding x-axis becomes inclined to old x-axis joining the line of landing and departing ports O and A respectively at the same angle α of the wind with the vertical (fig. 149): Let the new axes be rotated through an angle α to coincide with the

old axes, so that the line OA remains inclined at an angle α with its previous position (fig. 150). The initial conditions now becomes $x = a \cos \alpha$, $y = a \sin \alpha$ when $t = 0$ instead of $x = a$, $y = 0$ in 19.4.1 (i). Now this case is same as in 19.4.1 (i) with the exception of the initial conditions. So, the ODE for the components of the velocities are, as before, $\frac{dx}{dt} = -u \cos \theta$, $\frac{dy}{dt} = -u \sin \theta + v$ and the ODE for equation of motion is, from equation (4) in 19.4.1 (i),

$$\frac{dy}{dx} = \frac{y - k \sqrt{x^2 + y^2}}{x}, \quad k = \frac{v}{u}$$

Its general solution is

$$\log(W + \sqrt{W^2 + 1}) = -k \log x + \log c \quad \text{Using}$$

$$I.C., c = (\tan \alpha + \sec \alpha) (a \cos \alpha)^k$$

$\therefore$ the required pursuit curve is

$$x^{(v/u-1)} (y + \sqrt{x^2 + y^2})$$

$$= (\tan \alpha + \sec \alpha) (a \cos \alpha)^{v/u}$$

Case (b) : In this case (fig. 151) no change is taken into consideration. Frame of reference remains as in 19.4.1 (i). So, the components of velocities in x and y directions are

$$\frac{dx}{dt} = -u \cos \theta + v \sin \alpha,$$

$$\frac{dy}{dt} = -u \sin \theta + v \cos \alpha$$

$$\therefore \frac{dy}{dx} = \frac{-u \sin \theta + v \cos \alpha}{-u \cos \theta + v \sin \alpha}$$

$$\text{But } \sin \theta = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\text{and } \cos \theta = \frac{x}{\sqrt{x^2 + y^2}}$$

$$\therefore \frac{dy}{dx} = \left(\frac{-uy}{\sqrt{x^2 + y^2}} + v \cos \alpha \right) / \left(\frac{-ux}{\sqrt{x^2 + y^2}} + v \sin \alpha \right)$$

The ODE is a homogeneous one. Therefore, substitution of $W = y/x$ makes the variables separable as

$$\frac{dx}{x} = \left[-\frac{u}{v} \cdot \frac{1}{\sqrt{(1+W^2)} (\cos \alpha - W \sin \alpha)} + \frac{\sin \alpha}{\cos \alpha - W \sin \alpha} \right] dW$$

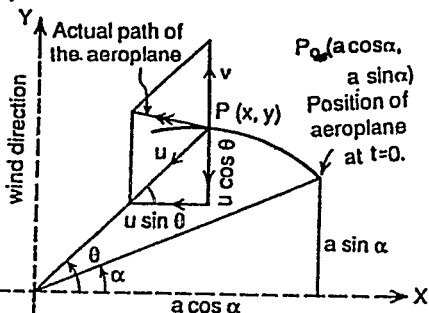


Fig. 150

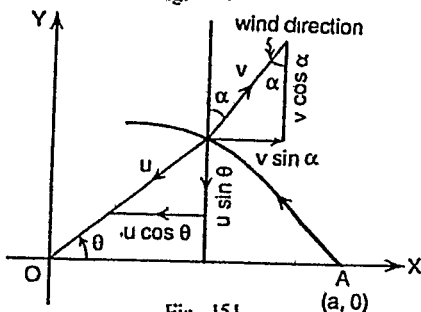


Fig. 151

Integrating,

$$x(\cos \alpha - W \sin \alpha) = c [\sec(\phi + \alpha) + \tan(\phi + \alpha)]^k \text{ where}$$

$$k = -\frac{v}{u} \text{ and } \tan \phi = W. \text{ Using I.C. } x = a, y = 0 \text{ i.e. } x = a,$$

$$W = \frac{v}{x} = 0, \text{ we get } c = (a \cos \alpha) / (\sec \alpha + \tan \alpha)^k. \text{ Therefore, the}$$

required pursuit curve is

$$(x - y \tan \alpha)^2 (\sec \alpha + \tan \alpha)^k = a(y + x \tan \alpha + \sqrt{x^2 + y^2}), k = -\frac{v}{u}.$$

19.4.2 RELATIVE PURSUIT CURVES

It is a curve described by a hunter pursuing its prey as seen by the prey. In this case origin is considered with respect to the prey. Let the hunter be a fighter and the prey be a bomber. Now the problem is to find the curve described by fighter as seen by bomber.

Let the origin on the earth be $(0, 0)$ and $(\bar{0}, \bar{0})$ be the moving origin of bomber (fig. 152). So, the position of the fighter after time t will be determined by (x, y) w.r.t. the origin $(0, 0)$ as well as $(\bar{x}, \bar{y})$ w.r.t. the origin $(\bar{0}, \bar{0})$.

The effective components of velocity of the fighter as seen from the ground are

$$\frac{dx}{dt} = v_F \cos(\pi - \theta) = -v_F \cos \theta \text{ and } \frac{dy}{dt} = -v_F \sin(\pi - \theta) = -v_F \sin \theta$$

respectively along x and y directions, where v_F is the velocity of fighter and θ is the inclination of the line joining the fighter and the bomber.

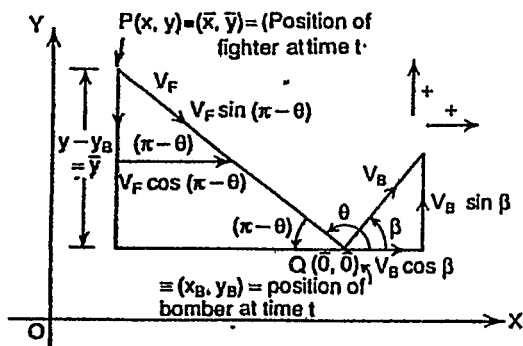


Fig. 152

Let the velocity of bomber at time t be v_B and it makes an angle β with the horizon.

The following statements are essential for the determination of the

relative velocities :

(a) The relative velocity is the sum of the velocities while both move towards each other in straight line;

(b) The relative velocity is the difference of the velocities while both move in straight but in same direction;

(c) The relative velocity is the sum of the velocities while each moves away from one another in the same straight line.

In the present case, let both fighter and bomber have horizontal components in the same direction. Let the velocity of fighter is greater than that of bomber and the fighter approaches towards bomber. So, the relative velocity is the difference of their velocities. Therefore, the rate of change of $\bar{x}$ relative to the bomber is

$$\frac{d\bar{x}}{dt} = -v_F \cos \theta - v_B \cos \beta$$

But the vertical component of their velocities are in opposite directions. That is, the velocity of the fighter w.r.t. bomber is in the -ve direction of y . So, relative velocity is the sum of their vertical components. Hence the rate of change of $\bar{y}$ relative to the bomber is

$$\frac{d\bar{y}}{dt} = -(v_F \sin \theta + v_B \sin \beta)$$

$$\therefore \frac{d\bar{y}}{d\bar{x}} = \frac{v_F \sin \theta + v_B \sin \beta}{v_F \cos \theta + v_B \cos \beta}$$

This is ODE for the equation of motion and its solution is the relative pursuit curve where $\sin \theta = \bar{y} / \sqrt{\bar{x}^2 + \bar{y}^2}$ and $\cos \theta = \bar{x} / \sqrt{\bar{x}^2 + \bar{y}^2}$ and v_F , $v_B \sin \beta$, $v_B \cos \beta$ are constants.

Similarly, in polar co-ordinates, radial and transverse components of the velocity of fighter relative to bomber are

$$\frac{dr}{dt} = -v_F - v_B (\cos \theta - \beta) \text{ and } r \frac{d\theta}{dt} = v_B \sin (\theta - \beta)$$

$$\therefore \frac{1}{r} \frac{dr}{d\theta} = -\frac{\cos (\theta - \beta)}{\sin (\theta - \beta)} - \frac{v_F}{v_B} \operatorname{cosec} (\theta - \beta)$$

Separating the variables, the general solution is

$$cr = \frac{1}{[\sin (\theta - \beta)^{(1-v_F/v_B)} [1 - \cos (\theta - \beta)]]} - \frac{v_F}{v_B}$$

with I.C. : $t = 0$, $r = r_0$ and $\theta = \theta_0$.

Examples

1. Solve the ODE $\frac{dy}{dx} = \frac{y}{x} + \tan \frac{y}{x}$.

The given equation is a homogeneous ODE. So, the substitution is $y = v(x)x$. Therefore,

$$\frac{dy}{dx} = v + x \frac{dv}{dx}, \text{ dropping the argument of } v.$$

$\therefore$ the given equation becomes

$$v + x \frac{dv}{dx} = v + \tan v \text{ or } x \frac{dv}{dx} = \tan v$$

Separating variables and integrating,

$$\int \frac{dx}{x} = \int \frac{dv}{\tan v} + c_1$$

$$\text{or, } \log x = \log \sin v + \log c, \text{ where } c_1 = \log c$$

$$\text{or, } x = c \sin v = c \sin \frac{y}{x}, \text{ reviving original variables.}$$

This is the general solution.

2. Find the equation of the orthogonal trajectories of the circle having centre on the y -axis and passing through the origin.

Let the co-ordinate of the centre be $(0, a)$. As it passes through the origin, the radius is evidently 'a'. So, the equation of the given circle is

$$x^2 + (y-a)^2 = a^2 \text{ or } x^2 + y^2 = 2ay \quad (1)$$

$$\text{The corresponding ODE is } \frac{dy}{dx} = \frac{2xy}{x^2 - y^2} \quad (2)$$

Now, the slope of the required orthogonal trajectory is

$$\left. \frac{dy}{dx} \right|_r = -\frac{x^2 - y^2}{2xy}$$

Dropping the subscript r , the ODE governing the required trajectory is

$$\frac{dy}{dx} = -\frac{x^2 - y^2}{2xy} \text{ which is a homogeneous one.}$$

It's general solution is $x^2 + y^2 = cx$ which is the required trajectory.

3. A river is 100 ft. wide. A man starts swimming with a velocity of 3 ft/sec. pointing to a tree on the opposite bank of the river. Find the equation of the path of the man if the river runs (a) 1 ft/sec., (b) 3 ft/sec. and (c) 4 ft/sec. and he wishes to cross the river directly.

Let the X axis be taken across the river and Y axis along the current. Let $P(x, y)$ be the position of the man on his way. Let the man starts swimming at an angle θ with the bank (fig. 153). So, the components of his velocity in the x and y -directions are respectively

$$\frac{dx}{dt} = -u \cos \theta, \quad \frac{dy}{dt} = u \sin \theta$$

But the resultant velocity of the man in y -direction is $\frac{dy}{dt} = -u \sin \theta + v$.

But $\sin \theta = \frac{y}{\sqrt{x^2 + y^2}}$ and $\cos \theta = \frac{x}{\sqrt{x^2 + y^2}}$. So, the ODE for the equation of motion is

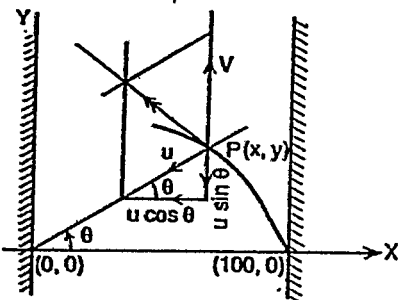


Fig. 153

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{1}{(dx/dt)} = \frac{y - k \sqrt{x^2 + y^2}}{x}, \quad k = \frac{v}{u}.$$

This is a homogeneous ODE. Its general solution is

$$\log \left[\frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}} \right] = -k \log x + c \quad (1)$$

I.C. : $y = 0$ and $x = 100$ when $t = 0$. Therefore, $c = k \log 100$. Hence from (1)

$$y = \frac{100}{2} \left[\left(\frac{x}{100} \right)^{(1-v/u)} - \left(\frac{x}{100} \right)^{(1+v/u)} \right]$$

$$= 50 \left[\left(\frac{x}{100} \right)^{(1-v/3)} - \left(\frac{x}{100} \right)^{(1+v/3)} \right] \quad \because u = 3 \text{ ft/sec.}$$

Now using $v = 1, 3, 4$, ft/sec separately, we get,

$$y = 50 \left[\left(\frac{x}{100} \right)^{2/3} - \left(\frac{x}{100} \right)^{4/3} \right] \quad (a)$$

$$x^2 = -200(y - 50) \quad (b)$$

$$y = 50 \left[\left(\frac{x}{100} \right)^{-1/3} - \left(\frac{x}{100} \right)^{7/3} \right] \quad (c)$$

These shows that the man only cross the river if the river flows at 1 ft/sec.

4. An aircraft, A at a slope of $\frac{3}{4}$ with a velocity of 200 m/hr. Another aircraft B pursued A from 50 miles north with a velocity of 300 m/hr. Find the equation of the path of B as observed from A. Use both cartesian and polar co-ordinates.

Figure as in 19.4.2, **Cartesian co-ordinates**. The component of velocity of B in x and y directions as seen from the ground are respectively

$$\frac{dx}{dt} = 300 \cos(\pi - \theta) = -300 \cos \theta$$

and $\frac{dy}{dt} = -300 \sin(\pi - \theta) = -300 \sin \theta$

where θ is the inclination of the line joining A and B .

Again the slope of A is $\tan \beta = \frac{3}{4}$. So, $\cos \beta = \frac{4}{5}$ and $\sin \beta = \frac{3}{5}$. Therefore,

at time t the horizontal and the vertical component of A are $200 \times \frac{4}{5}$ and $200 \times \frac{3}{5}$ miles/hr i.e. 160 and 120 m/hr.

Since the horizontal components of A and B are in straight line and in the same direction and that of vertical in opposite directions, so the rate of change of $\bar{x}$ and $\bar{y}$ w.r.t. A are

$$\frac{d\bar{x}}{dt} = -300 \cos \theta - 160 \quad \text{and} \quad \frac{d\bar{y}}{dt} = -(300 \sin \theta + 120)$$

$$\therefore \frac{d\bar{y}}{d\bar{x}} = \frac{300 \sin \theta + 120}{300 \cos \theta + 160} = \frac{15 \sin \theta + 6}{15 \cos \theta + 8}$$

$$\therefore \sin \theta = \bar{y} / \sqrt{\bar{x}^2 + \bar{y}^2} \quad \text{and} \quad \cos \theta = \bar{x} / \sqrt{\bar{x}^2 + \bar{y}^2}$$

$$\therefore \frac{d\bar{y}}{d\bar{x}} = \frac{15\bar{y} + 6\sqrt{\bar{x}^2 + \bar{y}^2}}{15\bar{x} + 8\sqrt{\bar{x}^2 + \bar{y}^2}} \text{ which is a homogeneous ODE.}$$

So, using $\bar{y} = v(\bar{x}) \bar{x}$ and separating the variables its general solution is

$$\log \bar{x} = \frac{15}{2} \int \frac{dv}{(3-4v)\sqrt{1+v^2}} - \log(3-4v) + c_1.$$

For the integral on the right using $v = \tan \theta$, the required general solution is

$$(3\bar{x} - 4\bar{y})^2 = c \tan^3 \left[\frac{1}{2} \tan^{-1} \frac{3\bar{y} - 4\bar{x}}{3\bar{x} + 4\bar{y}} \right].$$

I.C. : at $t=0$, $\bar{x}=0$, $\bar{y}=50$. Using this initial condition, the required equation of the path is

$$(3\bar{x} - 4\bar{y})^2 \tan^3 \left(\frac{1}{2} \cot^{-1} \frac{4}{3} \right) = 16 \times 50^2 \tan^3 \left[\frac{1}{2} \tan^{-1} \frac{3\bar{y} - 4\bar{x}}{3\bar{x} + 4\bar{y}} \right].$$

Polar co-ordinates : The radial and the transverse components of B relative to A are

$$\frac{dr}{dt} = -300 - 200 \cos(\theta - \beta), \quad r \frac{d\theta}{dt} = 200 \sin(\theta - \beta) \quad \text{where} \quad \tan \beta = \frac{3}{4}.$$

Therefore,

$$\frac{1}{r} \frac{dr}{d\theta} = -\cot(\theta - \beta) - \frac{3}{2} \operatorname{cosec}(\theta - \beta)$$

Separating variables and integrating,

$$r \sin(\theta - \beta) \left[\tan \frac{\theta - \beta}{2} \right]^{3/2} = c$$

or, $r [1 - \cos \theta \cos \beta - \sin \theta \sin \beta]^{3/2} = c [\sin \theta \cos \beta - \cos \theta \sin \beta]^{1/2}$

I.C. : at $t = 0$, $\theta = \pi/2$, $r = 50$ and $\tan \beta = 3/4$ i.e. $\cos \beta = 4/5$ and $\sin \beta = 3/5$. Using these $c = 10\sqrt{2}$. So, the required equation of the path is

$$r [5 - 4 \cos \theta - 3 \sin \theta]^{3/2} = 50\sqrt{2} [4 \sin \theta - 3 \cos \theta]^{1/2}$$

Exercises 19 (2)

1. Solve the following ODEs :

(i) $\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$,

(ii) $(y - \sqrt{x^2 + y^2}) dx = x dy$,

(iii) $y^2 dx + (x^2 + xy) dy = 0$, (iv) $x^2 y dx + (x^3 + y^3) dy = 0$,

(v) $(x + y) dx + (x - y) dy = 0$, (vi) $(5x + 6y) dx - (6x - 5y) dy = 0$,

(vii) $y(x^2 + xy - y^2) dx + x(y^2 - x^2) dy = 0$,

(viii) $x(x - 2y) dy + (x^2 - 3xy + 2y^2) dx = 0$,

(ix) $\left(x \sin \frac{y}{x} - y \cos \frac{y}{x} \right) dx + x \cos \frac{y}{x} dy = 0$,

(x) $y^2 + (3xy + x^2) \frac{dy}{dx} = 0$, (xi) $\frac{dy}{dx} = \frac{ax + by}{bx - cy}$,

(xii) $\frac{y}{x} \cos \frac{y}{x} dx - \left(\sin \frac{y}{x} + \cos \frac{y}{x} \right) dy = 0$,

(xiii) $y dx + x \log \frac{y}{x} dy - 2x dy = 0$,

(xiv) $(xe^{x/y} - y \sin(y/x)) dx + x \sin \frac{y}{x} dy = 0$,

(xv) $2ye^{x/y} dx + (y - 2xe^{x/y}) dy = 0$.

2. Solve the following initial value problems :

(i) $(x + y) dy - y dx = 0$, $y = 1$ when $x = 0$,

(ii) $(x^2 + 2xy - 2y^2) dx + (y^2 + 2xy - 2x^2) dy = 0$, $y = 3$ when $x = 0$.

3. Prove that the solution of $(ax - by) dx + (bx - ay) dy = 0$ is $(x + y)^{a+b} (x - y)^{a-b} = k$.

4. Find the equation to the curve whose slope at any point (x, y) is $(3ax^2 + 2bxy - cy^2)/(3fy^2 + 2exy - bx^2)$ and it passes through $(m, 0)$.

5. Find the isogonal and the orthogonal trajectories of the following curves (included angle α is measured from the given curve) :

(i) Hyperbola $xy = c$, (ii) Ellipse $x^2 + 4y^2 = c$.

6. Solve the following ODE :

$$(i) \quad [2(y-1)(x+y-1) + (y-1)^2] dx + [4(y-1)(x+y-1) + (x+y-1)^2] dy = 0$$

$$(ii) \quad \frac{dy}{dx} = \frac{y}{x - \sqrt{xy}}$$

7. Find the isogonal trajectory for the given angle measured towards the given curve from the trajectory :

$$(i) \quad y^2 = 4kx, \alpha = 45^\circ,$$

$$(ii) \quad x^2 + y^2 = k^2, \alpha = 45^\circ,$$

$$(iii) \quad y^2 + 2xy - x^2 = k, \alpha = 45^\circ,$$

$$(iv) \quad y = ax^2, \alpha = 45^\circ.$$

8. (i) From worked out example 3 show that (a) the man will reach the tree if the velocity of the river is 1 ft/sec., (b) he will reach 50 ft. away from the tree when the velocity of the river is 3 ft/sec., (c) he will never reach when the velocity of the river is 4 ft/sec.

(ii) A pilot is flying towards west 400 miles away. Air is blowing from south 20 m/hrs. Find the pursuit curve of the aircraft if its velocity is 300 m/hrs. Give the answer both in cartesian and polar co-ordinates.

(iii) A pilot is flying his aircraft towards west at a distance of 300 miles. Wind flows at the rate of 25 m/h with a slope of $4/3$. Find pursuit curve of the aircraft if its velocity is 200 m/hrs.

(iv) A boy is pulling a toy in the road by a rope of l ft. Suddenly the road turns into a right angle. Assume that he is always kept tight and neglecting the height of the boy, find the equation of the curve followed by the toy.

9. Solve the problem in Ex. 4 (worked out) :

(i) If aircraft A flies in the south-east direction and B pursues from 50 miles north-east.

[Hints : at $t=0$, $\theta = \pi/4$, $\beta = -\pi/4$, $r = 50$]

(ii) If A flies towards north-east direction and B pursues from 50 miles south-east.

10. Let t be the tangent to a curve C at a general point P , let F be the foot of the perpendicular from P to the x -axis and let T be the point in which t intersects the x -axis. Find the equation of the family of curves which have the property that the length TF is equal to the sum of the abscissa and ordinate of P .

[Hints : Subtangent $= x + y$]

Answers

1. (i) $x^2 - y^2 = cx$, (ii) $y + \sqrt{x^2 + y^2} = c$, (iii) $xy^2 = c(x + 2y)$,
 (iv) $x^3 + 3y^3 \log y = cy^3$, (v) $x^2 + 2xy - y^2 = c$, (vi) $\frac{1}{2} \log(x^2 + y^2) = \frac{6}{5} \tan^{-1} \frac{y}{x} + c$.

$$(vii) \log x = c - \left(\frac{y}{x}\right) - \left(\frac{x}{y}\right) \quad (viii) \log x + \frac{y}{x} = c, \quad (ix) x \sin \frac{y}{x} = c,$$

$$(x) x v = c \sqrt{4v+1}, v = y/x, \quad (xi) \frac{1}{2} \log (ax^2 + cy^2) = \frac{b}{\sqrt{ac}} \tan^{-1} \frac{y}{x} \sqrt{\frac{c}{a}} + k,$$

$$(xii) y \sin \left(\frac{y}{x}\right) = c,$$

$$(xiii) y = c(1 + \log(x/y)),$$

$$(xiv) \log x^2 - e^{-y/x} \left(\sin \frac{y}{x} + \cos \frac{y}{x} \right) = c,$$

$$(xv) 2e^{x/y} + \log y = c,$$

$$2. (i) y \log y = x,$$

$$(ii) (y^2 - xy + x^2) = 3(y+x),$$

$$4. ax^3 + bx^2y - cxy^2 - fy^3 = am^3,$$

$$5. (i) (y^2 - 2x^2) \tan \alpha + 2xy = c, y^2 - x^2 = c,$$

$$(ii) \log(4y^2 - 3xy \tan \alpha + x^2) + \frac{5}{2} \tan \alpha \tan^{-1} \left(\frac{8y - 3x \tan \alpha}{x \sqrt{16 - 9 \tan^2 \alpha}} \right) + c, y = cx^4,$$

$$6. (i) 9x(y-1)^2 + 3x^2(y-1) + 5(y-1)^3 = c,$$

$$(ii) ay = e^{-2\sqrt{x/y}}$$

$$7. (i) \log(2x^2 + xy + y^2) + \frac{6}{\sqrt{7}} \tan^{-1} \frac{x+2y}{x\sqrt{7}} = c,$$

$$(ii) \log c(x^2 + y^2) - 2 \tan^{-1} \frac{y}{x} = 0,$$

$$(iii) xy = c,$$

$$(iv) \log \sqrt{2y^2 - xy + x^2} + \frac{3}{\sqrt{7}} \tan^{-1} \frac{4y-x}{x\sqrt{7}} = c,$$

$$8. (ii) y = 25 \left[\left(\frac{x}{400} \right)^{15} - \left(\frac{x}{400} \right)^{16/15} \right], r \cos \theta = 400 (\sec \theta + \tan \theta)^{-15},$$

$$(iii) y + \sqrt{x^2 + y^2} = 2(240)^{1/8} x^{7/8},$$

$$\text{Second method : } (200y - 20\sqrt{x^2 + y^2}) dx - (200x - 15\sqrt{x^2 + y^2}) dy = 0$$

$$(iv) x = l \log \frac{l + \sqrt{l^2 - y^2}}{y} - \sqrt{l^2 - y^2},$$

$$9. (i) r = 50\sqrt{2} (\sin \theta + \cos \theta) / (\sqrt{2} - \cos \theta + \sin \theta)^{3/2},$$

$$(ii) 50\sqrt{2} (\cos \theta - \sin \theta) / (\sqrt{2} - \cos \theta - \sin \theta)^{2/3}$$

$$10. y = ce^{x/y}.$$

19.5 REDUCIBLE TO HOMOGENEOUS

The general equation of this category is $\frac{dy}{dx} = \frac{ax + by + c}{a_1x + b_1y + c_1}$

$$\text{or, } (ax + by + c) dx = (a_1x + b_1y + c_1) dy \quad (1)$$

Here the expressions $(ax + by + c)$ and $(a_1x + b_1y + c_1)$ are both linear in x and y in general. These expression when equated to zero i.e.

$$\text{and } \begin{cases} ax + by + c = 0 \\ a_1x + b_1y + c_1 = 0 \end{cases} \quad (2)$$

represent two straight lines and so there arise three possibilities —

(a) the lines are intersecting if $\frac{a}{a_1} \neq \frac{b}{b_1}$,

(b) the lines are parallel if $\frac{a}{a_1} = \frac{b}{b_1}$,

(c) the lines are coincident if $\frac{a}{a_1} = \frac{b}{b_1} = \frac{c}{c_1}$.

Case (a) : First method : If the equation (1) does not contain the constant c and c_1 , then it becomes a homogeneous ODE. Co-ordinate geometry assures that if the equations (2) satisfy (a) then they are intersecting and if the origin is transferred to the point of intersections of (2) then the constants c and c_1 will be removed.

Let (h, k) be the point of intersection of (2).

So, $ah + bk + c = 0$ and $a_1 h + b_1 k + c_1 = 0$ (3)

Let the new axes through (h, k) be X' and Y' . So relation between the new and old co-ordinates is, from co-ordinate geometry,

$$x = x' + h, y = y' + k \quad (4)$$

$$\therefore \frac{dy'}{dx'} = \frac{dy}{dx} \quad (5)$$

Using (3), (4), (5) in (1) we have

$$\frac{dy'}{dx'} = \frac{ax' + by'}{a_1 x' + b_1 y'} \quad (6)$$

which is homogeneous i.e. (6) is the reduced form of (1) if (a) is satisfied. Let the substitution for (6) be $y' = v(x') x'$. Therefore, $\frac{dy'}{dx'} = v + x' \frac{dv}{dx'}$. Using this substitution, the general solution of (6) is

$$\log x' = \int \frac{a_1 + b_1 v}{a + (b - a_1)v - b_1 v^2} dv + k, \text{ where}$$

$$v = \frac{y'}{x'} = \frac{x - h}{y - k} \text{ and } (h, k) \text{ is given by (3).}$$

Second method : Let $u = ax + by + c$ and $v = a_1 x + b_1 y + c_1$

$$\therefore du = a dx + b dy \text{ and } dv = a_1 dx + b_1 dy$$

$$\therefore dx = \frac{a_1 du - a dv}{a_1 b - ab_1} \text{ and } dy = \frac{b_1 du - b dv}{ab_1 - a_1 b}$$

$\therefore$ equation (1) becomes

$$u \left(\frac{a_1 du - a dv}{a_1 b - ab_1} \right) = v \left(\frac{b_1 du - b dv}{ab_1 - a_1 b} \right)$$

$$\begin{aligned} \text{or, } [b_1(a_1 b - ab_1) v - a_1(ab_1 - a_1 b) u] du \\ = [b(a_1 b - ab_1) v - a(ab_1 - a_1 b) u] dv \end{aligned}$$

This is also a homogeneous ODE. This can also be solved by 19.4.

Case (b) : Here separation by substitution in 19.3.1 gives the general solution. Let $\frac{a}{a_1} = \frac{b}{b_1} = m$ (say)

or, $a = a_1 m$ and $b = b_1 m$. Therefore equation (1) becomes

$$\frac{dy}{dx} = \frac{m(a_1 x + b_1 y) + c}{(a_1 x + b_1 y) + c_1} \quad (7)$$

Let $a_1 x + b_1 y = v$. So, the general solution of (7) can be easily obtained as

$$\begin{aligned} x = \frac{a_1 x + b_1 y}{a_1 + mb_1} - \frac{b_1 c - (a_1 + mb_1) c_1}{(a_1 + mb_1)^2} \\ \cdot \log [(a_1 + mb_1)(a_1 x + b_1 y) + b_1 c] + k. \end{aligned}$$

Case (c) : Let $\frac{a}{a_1} = \frac{b}{b_1} = \frac{c}{c_1} = m$ (say)

$\therefore$ (1) becomes $\frac{dy}{dx} = m$ or Integrating, $y = mx + c$ is the general solution.

Examples

1. Solve the ODE $(2x - 3y + 4) dx + (3x - 2y + 1) dy = 0$.

Here $\frac{2}{3} \neq \frac{-3}{-2}$. Therefore, the equation can be reduced to homogeneous form.

First method : Let (h, k) be the point of intersection of the lines $2x - 3y + 4 = 0$ and $3x - 2y + 1 = 0$.

$$\therefore 2h - 3k + 4 = 0 \text{ and } 3h - 2k + 1 = 0 \quad (1)$$

Solving (1), $h = 1, k = 2$. Let the origin be transferred to $(1, 2)$ and the new and old co-ordinate be related by $x = x' + 1$ and $y = y' + 2$. Therefore

$$\frac{dy}{dx} = \frac{dy'}{dx'}$$

$\therefore$ the given equation becomes

$$\frac{dy'}{dx'} = -\frac{2x' - 3y'}{3x' - 2y'} \quad (2)$$

which is a homogeneous ODE.

Let $y' = v(x') x'$ be the substitution. So equation (2) becomes

$$v + x' \frac{dv}{dx'} = -\frac{2 - 3v}{3 - 2v}$$

or,
$$x' \frac{dv}{dx'} = \frac{2(v^2 - 1)}{3 - 2v}$$

or, separating variables and integrating, we get,

$$(x')^4 (v + 1)^5 = (v - 1) c$$

Introducing original variables by $x' = x - 1$ and $y' = y - 2$,
 $v = \frac{y'}{x'} = \frac{y - 2}{x - 1}$, the required general solution is $(x + y - 3)^5 = c (y - x - 1)$.

Second method : Let $u = 2x - 3y + 4$ and $v = -(3x - 2y + 1)$.

$$\therefore du = 2dx - 3dy \text{ and } dv = -3dx + 2dy.$$

$$\therefore dx = -\frac{1}{5}(2du + 3dv) \text{ and } dy = -\frac{1}{5}(3du + 2dv).$$

$\therefore$ the given equation becomes,

$$u(2du + 3dv) = v(3du + 2dv)$$

or,
$$\frac{du}{dv} = \frac{2v - 3u}{2u - 3v} \text{ which is a homogeneous ODE.}$$

Using substitution $u = w(v)$, the general solution of the given equation becomes as before, $(x + y - 3)^5 = (y - x - 1)c$.

2. Solve the ODE $(2x + 3y - 1) dx + (4x + 6y + 2) dy = 0$.

Here $\frac{2}{4} = \frac{3}{6}$. Therefore, by case (b), let $2x + 3y = v$.

or,
$$2 + 3 \frac{dy}{dx} = \frac{dv}{dx} \text{ or } \frac{dy}{dx} = \frac{1}{3} \left(\frac{dv}{dx} - 2 \right).$$

Using the substitution in the given equation, we get

$$\frac{dv}{dx} = \frac{v + 7}{2(v + 1)} \text{ which by separation of variables and integration, the}$$

required general solution comes as $x + 2y - 4 \log(2x + 3y + 7) = c$.

3. Solve ODE $(2x + 3y + 2) dx + (4x + 6y + 4) dy = 0$.

Here $\frac{2}{4} = \frac{3}{6} = \frac{2}{4}$. Therefore by case (c), let $2x + 3y + 2 = m$. So, the given

equation becomes $dx + 2dy = 0$ if $m = 2x + 3y + 2 \neq 0$

or, $x + 2y = c$ is the required solution when $2x + 3y + 2 \neq 0$.

Exercises 19 (3)

Solve the following ODE :

1. $(2x + 3y - 6) dy - (6x - 2y - 7) dx = 0$.

2. $(6x - 2y - 7) dx - (3x - y + 4) dy = 0$.

3. $(3x - y + 2) dx - (6x - 2y) dy = 0$.

4. $(x - y + 1) dx + (2x + y - 2) dy = 0$.
5. $(2x - 5y + 3) dx - (5x - 12y + 8) dy = 0$.
6. $(x - 3y + 4) dy + (7y - 5x) dx = 0$.
7. $\frac{dy}{dx} = \left(\frac{2x - y}{4x - 2y - 1} \right)^2$.
8. $(x + y + 1) dx - (2x + 2y + 1) dy = 0$.
9. $(7y - 3) dx + (2x + 1) dy = 0$.
10. $(x + 2y) dx + (y - 1) dy = 0$.
11. $(x + y + 1) dx + (2x + 2y + 2) dy = 0$.
12. Solve by using the given conditions :
 - (a) $(x + y + 2) dx - (x - y - 4) dy = 0$, $y(1) = 0$,
 - (b) $(x + y) dx + (3x + 3y - 4) dy = 0$, $y(1) = 0$,
 - (c) $(x - 2y + 4) dx + (2x - y + 2) dy = 0$, $y(1) = 1$.
13. Show that the ODE $(4x + 3y + 1) dx + (3x + 2y + 1) dy = 0$ expresses such a parabola whose asymptotes are $x + y = 0$ and $2x + y + 1 = 0$.
14. Find the equation of the isogonal trajectories to curve $y^2 + (x - 2)^2 = c$ and intersecting them at a constant angle $\tan^{-1} 2$.
15. Find the solution of $\frac{dy}{dx} = \frac{x + y - 3}{x + y + 3}$ when $y = 1$ at $x = 0$.

Answers

1. $6x^2 - 4xy - 3y^2 - 14x + 12y = c$,
2. $2x - y - 15 \log(3x - y + 19) = c$,
3. $(2 - 15x + 5y)^4 = ce^{5(2y - x)}$.
4. $\log(3x^2 + 3xy + 3y^2 - 6x - 9y + 7) = c - \frac{6}{\sqrt{3}} \tan^{-1} \left[\frac{(3x + 3y - 5)/(x - y + 1)}{\sqrt{3}} \right]$.
5. $(x - 3y + 1)(x - y + 2) = c$.
6. $(3y - 5x + 10)^2 = c(y - x + 1)$.
7. $7x - 28y + 2 \log(28x^2 - 28xy + 7y^2 - 16x + 8y + 2) + \frac{9}{2\sqrt{2}} \log \frac{14x - 7y - 4 - \sqrt{2}}{14x - 7y - 4 + \sqrt{2}}$.
8. $6y - 3x = \log(3x + 3y + 2) + c$.
9. $7 \log(2x + 1) + 2 \log(7y - 3) = c$.
10. $[(x + 2)/(x + y + 1)] + \log(x + y + 1) = c$.
11. $x + 2y = c$.
12. (a) $\log[(x - 1)^2 + (y + 3)^2] + 2 \tan^{-1} \frac{(x - 1)/(y + 3)}{1} = 2 \log 3$,
 (b) $x + 3y + 2 \log(2 - x - y) = 1$,
 (c) $(x + y - 2)^{3/2} = 0$.
14. $y^2 - 4y(x - 2) - (x - 2)^2 = c$.
15. $3 \log(x + y) = x - y + 1$.

19.6 EXACT EQUATIONS

The most general first order and first degree ODE

$$M(x, y) dx + N(x, y) dy = 0 \quad (1)$$

will be exact if the expression on the left of (1) is exact differential i.e. it expresses the differential coefficient of some other expression.

The necessary and sufficient conditions for exactness and the procedure for solving (1) will be discussed now in the following paragraphs.

Theorem : If the functions $M(x, y)$, $N(x, y)$ with their first partial derivatives namely $\frac{\partial M}{\partial x}$, $\frac{\partial M}{\partial y}$, $\frac{\partial N}{\partial x}$ and $\frac{\partial N}{\partial y}$ are all continuous in a simply connected region R , then the necessary and sufficient conditions for

$$M(x, y) dx + N(x, y) dy = 0 \quad (1)$$

to be exact is

$$\text{is } \frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x} \quad (2)$$

Proof. Necessary conditions : Given that (1) is exact, we have to prove that (2) is true. From differential calculus, a function $f(x, y)$ is exact differential if

$$df(x, y) = \frac{\partial f(x, y)}{\partial x} dx + \frac{\partial f(x, y)}{\partial y} dy$$

Therefore, from the definition of exact ODE above,

$$M(x, y) dx + N(x, y) dy = \frac{\partial}{\partial x} f(x, y) dx + \frac{\partial}{\partial y} f(x, y) dy = df(x, y) = 0 \quad (3)$$

(3) and (1) can be identical only when $df(x, y) = 0$

i.e. integrating, $f(x, y) = c$ (3a)

$\therefore$ from (3), equating coeffs. of dx and dy , we get,

$$M(x, y) = \frac{\partial}{\partial x} f(x, y) \text{ and } N(x, y) = \frac{\partial}{\partial y} f(x, y) \quad (4)$$

$\therefore$ partial derivatives of $M(x, y)$ and $N(x, y)$ are continuous, so differentiating (4) partially,

$$\frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} \text{ and } \frac{\partial N}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ which is the condition (2).

Sufficient condition : Given (2), we have to prove that (1) is an exact differential equation. This proof and the method of solution (1) is same. Now integrating the first half of equation (4) above, we get,

$$f(x, y) = \int_{x_0}^x M(x, y) dx + g(y) \quad (5)$$

where the arbitrary function $g(y)$ is the constant of integration as equation (4) is a PDE and x_0 is a constant. Differentiating (5) and using the second half of (4), we have,

$$\frac{\partial}{\partial y} f(x, y) = \frac{\partial}{\partial y} \int_{x_0}^x M(x, y) dx + g'(y) = N(x, y)$$

$$\text{or, } \frac{\partial}{\partial y} \int_{x_0}^x M(x, y) dx + g'(y) = N(x, y) \quad (5a)$$

$$\text{or, } \int_{x_0}^x \frac{\partial}{\partial y} M(x, y) dx + g'(y) = N(x, y) \text{ by sec. 8.8, case (ii)}$$

$$\text{or, } \int_{x_0}^x \frac{\partial}{\partial x} N(x, y) dx + g'(y) = N(x, y) \text{ by condition (2)}$$

$$\text{or, } \left[N(x, y) \right]_{x=x_0}^x + g'(y) = N(x, y), \text{ by integration}$$

$$\text{or, } N(x, y) - N(x_0, y) + g'(y) = N(x, y)$$

$$\text{or, } g'(y) = N(x_0, y) \quad (5b)$$

$$\text{Integrating } g(y) = \int_{y_0}^y N(x_0, y) dy + k,$$

where k is const. of integration and y_0 is a const.

$$\therefore (5) \text{ becomes, } f(x, y) = \int_{x_0}^x M(x, y) dx + \int_{y_0}^y N(x_0, y) dy + k \quad (6)$$

Here it is assumed that the point (x_0, y_0) , the line segments joining (x_0, y_0) , (x, y_0) and (x_0, y_0) , (x_0, y) lie wholly within the region R .

19.6.1 EXACT-HOMOGENEOUS EQUATIONS

If the homogeneous equation $P(x, y) dx + Q(x, y) dy = 0$ in 19.4 has degree of homogeneity, n not equal to (-1) as well as the equation is exact, then the equation is called exact-homogeneous and such type can be more easily solved. Let the equation be exact i.e. $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ and homogeneous i.e. each of $P(x, y)$ and $Q(x, y)$ satisfy Euler's theorem on homogeneous function namely $x \frac{\partial P}{\partial x} + y \frac{\partial P}{\partial y} = nP$ etc. and $n \neq -1$.

$$\text{Let } U = x P(x, y) + y Q(x, y) \equiv x P + y Q$$

$$\therefore \frac{\partial U}{\partial x} = P + x \frac{\partial P}{\partial x} + y \frac{\partial Q}{\partial x} = P + x \frac{\partial P}{\partial x} + y \frac{\partial P}{\partial y}, \text{ for exactness.}$$

$$= P + nP, \text{ by Euler's theorem}$$

$$= (n+1) P$$

$$\text{Similarly, } \frac{\partial U}{\partial y} = (n+1) Q$$

$$\text{Now, } dU = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy = (n+1) P dx + (n+1) Q dy$$

$$= (n+1) (P dx + Q dy)$$

$$\begin{aligned}\therefore \text{integrating } \int (P dx + Q dy) &= \frac{1}{n+1} \int dU + c_1, n \neq -1 \\ &= \frac{1}{n+1} U + c_1, n \neq -1 \\ &= \frac{1}{n+1} (x^P + y^Q) + c, n \neq -1\end{aligned}$$

$\therefore$ the general solution of the exact-homogeneous ODE, $P(x, y) dx + Q(x, y) dy = 0$ is $\frac{1}{n+1} (x^P(x, y) + y^Q(x, y)) = c$ when the degree of homogeneity, $n \neq -1$.

Examples

1. Show that $(2x + y \cos x) dx + (2y + \sin x - \sin y) dy = 0$ is an exact ODE. Find its solution also.

Test : Let $M(x, y) = 2x + y \cos x$ and $N(x, y) = 2y + \sin x - \sin y$

$$\therefore \frac{\partial M}{\partial y} = \cos x \quad \text{and} \quad \frac{\partial N}{\partial x} = \cos x$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \cos x \quad \therefore \text{the given equation is an exact ODE.}$$

Solution : First method : Since $M(x, y)$ and $N(x, y)$ are defined and continuous for all x, y , so $x_0 = y_0 = 0$ can be chosen. Therefore, from equation (6) of 19.6 above, $N(x_0, y) = N(0, y) = 2y - \sin y$

$$\therefore f(x, y) = \int_0^x (2x + y \cos x) dx + \int_0^y (2y - \sin y) dy = c$$

$$\text{or} \quad x^2 + y \sin x + y^2 + \cos y = c$$

is the general solution of the given equation.

N.B. : $(0, 0)$ has been chosen as one of the values of (x_0, y_0) in R . Other than $(0, 0)$ may also be chosen. But that choice will incur more simplification and much labour. This procedure is sometimes expressed as "Integrate $M(x, y) dx$ keeping y const. and $N(x, y) dy$ neglecting x or putting $x = 0$ and then equate the sum of these two integrals to a constant."

Second method : Since the equation is exact. So, equation (4) can be used to get solutions. This procedure needs no choice of (x_0, y_0) and keeping or rejecting.

$$\therefore \frac{\partial}{\partial x} f(x, y) = M(x, y) = 2x + y \cos x$$

This is a partial differential equation. Yet this can be solved by direct integration i.e. def. of integration. But in this case constant of integration is some arbitrary function instead of arbitrary constant as in ODE.

∴ integrating w.r.t. 'x'

$$f(x, y) = x^2 + y \sin x + g(y)$$

Now differentiating it partially w.r.t. y, we get,

$$\frac{\partial}{\partial y} f(x, y) = N(x, y) = \sin x + g'(y)$$

or, $2y + \sin x - \sin y = \sin x + g'(y),$

using expression for $N(x, y)$ in the equation

or, $g'(y) = 2y - \sin y$ which is an ODE.

Separating variables and integrating or def. of integration, we get

$$g(y) = y^2 + \cos y + c_1$$

∴ the required solution is

$$f(x, y) = x^2 + y \sin x + y^2 + \cos y = c.$$

2. Solve the ODE, $x(x^2 + 3y^2) dx + y(y^2 + 3x^2) dy = 0$.

Here $M(x, y) = x(x^2 + 3y^2)$ and $N(x, y) = y(y^2 + 3x^2)$

$$\therefore \frac{\partial M}{\partial y} = 6xy \quad \text{and} \quad \frac{\partial N}{\partial x} = 6xy$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \therefore \text{the ODE is exact.}$$

Also, the ODE is homogeneous of degree $n = 3 \neq -1$

∴ the ODE is exact-homogeneous. Therefore, its general solution is

$$x^2(x^2 + 3y^2) + y^2(y^2 + 3x^2) = c$$

i.e. $x^4 + y^4 + 6x^2y^2 = c$ by 19.6.1.

19.6.2 SOLUTION OF EXACT ODE BY INSPECTION

This procedure requires easy identification of exact differential parts from the knowledge of differential calculus without going into the test for exactness and the solution procedures of 19.6 & 19.6.1. For example $(x dy + y dx)$, $(x dy - y dx)/x^2$ are respectively the exact differentials $d(xy)$, $d(y/x)$. Also re-arrangement of the terms sometimes makes the identification easy. A list of exact differentials is given below for ready reference, because quick identification saves time, energy and space.

Expressions

Exact differentials

1. $y dx + x dy$

$d(xy)$

2. $2xy dx + x^2 dy$

$d(x^2 y)$

3. $y^2 dx + 2xy dy$

$d(xy^2)$

4. $2xy^2 dx + 2x^2 y dy$

$d(x^2 y^2)$

5. $3x^2 y^3 dx + 3x^3 y^2 dy$

$d(x^3 y^3)$

- | | |
|---|---|
| 6. $y \cos x \, dx + \sin x \, dy$ | $d(y \sin x)$ |
| 7. $ye^{xy} \, dx + xe^{xy} \, dy$ | $d(e^{xy})$ |
| 8. $\frac{dx}{x} + \frac{dy}{y}$ | $d(\log(xy))$ |
| 9. $\frac{y \, dx - x \, dy}{y^2}$ | $d(x/y)$ |
| 10. $\frac{-y \, dx + x \, dy}{x^2}$ | $d(y/x)$ |
| 11. $\frac{2xy \, dy - y^2 \, dx}{x^2}$ | $d(y^2/x)$ |
| 12. $\frac{x \, dy - y \, dx}{x^2 + y^2}$ | $d\left(\tan^{-1} \frac{y}{x}\right)$ |
| 13. $2 \cdot \frac{x \, dy - y \, dx}{x^2 - y^2}$ | $d\left(\log \frac{x+y}{x-y}\right)$ |
| 14. $\frac{x \, dx + y \, dy}{\sqrt{x^2 + y^2}}$ | $d(\sqrt{x^2 + y^2})$ |
| 15. $\frac{x \, dy - y \, dx}{xy}$ | $d\left(\log \frac{y}{x}\right)$ |
| 16. $\frac{x \, dy - y \, dx}{(x-y)^2}$ | $d\left(\frac{1}{2} \cdot \frac{x+y}{x-y}\right)$ |
| 17. $\frac{x \, dy - y \, dx}{x\sqrt{x^2 - y^2}}$ | $d\left(\sin^{-1} \frac{y}{x}\right)$ |
| 18. $\frac{y \, dx - x \, dy}{(x+y)^2}$ | $d\left(\frac{1}{2} \cdot \frac{x-y}{(x+y)^2}\right)$ |
| 19. $\frac{x \, dy + y \, dx}{x^2 y^2}$ | $d\left(-\frac{1}{xy}\right)$ |
| 20. $\frac{dx + dy}{x+y}$ | $d(\log(x+y))$ |
| 21. $\frac{dx - dy}{x-y}$ | $d(\log(x-y))$ |
| 22. $\frac{y \, dx - x \, dy}{x\sqrt{x^2 - y^2}}$ | $d\left(\cos^{-1} \frac{y}{x}\right)$ |

Examples

1. Solve the ODE $(x - 2xy + e^y) \, dx + (y - x^2 + xe^y) \, dy = 0$.

By re-arrangement of terms the given ODE can be written as

$$x \, dx - (2xy \, dx + x^2 \, dy) + (e^y \, dx + xe^y \, dy) + y \, dy = 0.$$

$$\text{or, } d\left(\frac{1}{2}x^2\right) - d(x^2y) + d(xe^y) + d\left(\frac{1}{2}y^2\right) = 0$$

$$\text{or, } d\left(\frac{1}{2}x^2 - x^2y + xe^y + \frac{1}{2}y^2\right) = 0$$

$\therefore$ the given equation is an exact ODE. Integrating,

$$\frac{1}{2}x^2 - x^2y + xe^y + \frac{1}{2}y^2 = c$$

is the required solution.

19.6.3 INTEGRATING FACTORS

Integrating factor (I.F.) is a factor which makes an unexact expression exact on multiplication of the unexact expression by it and makes the product easily integrable.

I.F. of an expression is not its constant property as can be seen from the list in 19.6.2 and the following theorem. From 9, 10, 12, 15, 16, 17, 18 of 19.6.2

it is seen that $(y dx - x dy)$ has $\frac{1}{y^2}$, $\left(-\frac{1}{x^2}\right)$, $\left(-\frac{1}{x^2+y^2}\right)$, $\left(-\frac{1}{xy}\right)$, $\left\{-\frac{1}{(x-y)^2}\right\}$, $\left\{-\frac{1}{x\sqrt{x^2-y^2}}\right\}$ and $\frac{1}{(x+y)^2}$ as I.F.

Theorem : If the ODE $M(x, y) dx + N(x, y) dy = 0$ is not exact and $f(x, y) = c$ is its solution, then there exists one or more or even infinite number of I.F. for the ODE.

Proof : Differentiating $f(x, y) = c$ we get the ODE corresponding to it

$$\text{as } df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0$$

which must be identical to $M(x, y) dx + N(x, y) dy = 0$.

[as $f(x, y) = c$ is the assumed solution]

$\therefore$ the coeffs. of dx and dy are proportional.

$$\text{i.e. } \frac{\partial f}{\partial x} : \frac{\partial f}{\partial y} = M : N$$

$$\text{or, } \frac{(\partial f / \partial x)}{M} = \frac{(\partial f / \partial y)}{N} = F(x, y), \text{ const. of proportionality function.}$$

$$\therefore \frac{\partial f}{\partial x} = FM \text{ and } \frac{\partial f}{\partial y} = FN$$

$$\therefore df(x, y) = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = FM dx + FN dy$$

$$= F(x, y) [M(x, y) dx + N(x, y) dy] \quad (1)$$

which is exact. Therefore, $F(x, y)$ is an integrating factor which multiplies the unexact expression $[M(x, y) dx + N(x, y) dy]$.

Now multiply (1) by a constant function $G(f)$.

we get $G(f) df = GF [M dx + N dy]$ (2)

Here $G(f) df$ is exact differential and integrable, so (2) is also exact. Therefore, $M dx + N dy$ has infinite numbers of I.F. as $G(f)$ is an arbitrary function.

Note that $G(f)$ for $(y dx - x dy)$ in (12), (15), (16), (17), (18) of 19.6.2 are

$$-\frac{1}{\left\{\left(\frac{x}{y}\right)^2 + 1\right\}}, -\frac{1}{\left(\frac{x}{y}\right)}, -\frac{1}{\left(\frac{x}{y}-1\right)^2}, -\frac{1}{\frac{x}{y}\sqrt{\left(\frac{x}{y}\right)^2-1}}, -\frac{1}{\left(\frac{x}{y}+1\right)^2}$$

respectively for only I.F. $= \frac{1}{y^2}$.

This property can be studied for other cases also.

19.6.4 DETERMINATION OF INTEGRATING FACTOR

There is no general procedure for finding I.F. except that in 19.7 and those of the following special cases :

(A) Let $F(x, y)$ be an I.F. for $M(x, y) dx + N(x, y) dy = 0$.

By definition of I.F. the product

$$F(x, y) [M(x, y) dx + N(x, y) dy] = 0$$

is exact. Therefore, by condition of exactness

$$\frac{\partial}{\partial y} (FM) = \frac{\partial}{\partial x} (FN) \quad (1)$$

The following are the different possibilities of F :

(i) F is a function of x only i.e. $F(x, y) \equiv F(x)$

$$\therefore \text{from (1), } F(x) \frac{\partial M}{\partial y} = F(x) \frac{\partial N}{\partial x} + N \frac{d}{dx} F(x)$$

$$\therefore N \frac{dF(x)}{dx} = F(x) \left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right]$$

$$\text{or } \frac{dF(x)}{dx} = F(x) \left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right] / N$$

In this equation left hand side is clearly a function of x , so must be the right hand side. Let R.H.S. $= R(x)$.

$$\therefore \frac{dF(x)}{dx} = R(x)$$

Integrating, $F(x) = \int R(x) dx + \log c$

$$\therefore F(x) = ce^{\int R(x) dx}$$

$\therefore I.F. = F(x) = e^{\int R(x) dx}$, taking $c = 1$, is the required I.F.

(ii) F is a function of y only i.e. $F(x, y) \equiv F(y)$.

$$\therefore \text{from (1), } F(y) \frac{\partial M}{\partial y} + M \frac{dF(y)}{dy} = F(y) \frac{\partial N}{\partial x}$$

or,
$$\frac{dF(y)}{dy} = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$$

Arguing as in (i), $\frac{dF(y)}{dy} = R(y)$ (say)

$\therefore$ integrating, $F(y) = c e^{\int R(y) dy}$

$\therefore I.F. = F(y) = e^{\int R(y) dy}$ is the required I.F.

(iii) F is a function of the product xy i.e. $F = F(u)$, $u = xy$,

$$\therefore \text{from (1), } F(u) \frac{\partial M}{\partial y} + M \frac{\partial F(u)}{\partial y} = F(u) \frac{\partial N}{\partial x} + N \frac{\partial F(u)}{\partial x} \quad (a)$$

$$\because u = xy, \therefore \frac{\partial u}{\partial y} = x \text{ and } \frac{\partial F(u)}{\partial y} = F'(u) \frac{\partial u}{\partial y} = x \frac{dF(u)}{du}$$

$$\text{Similarly, } \frac{\partial u}{\partial x} = y \text{ and } \frac{\partial F(u)}{\partial x} = F'(u) \frac{\partial u}{\partial x} = y \frac{dF(u)}{du}$$

$\therefore$ (a) becomes

$$F(u) \frac{\partial M}{\partial y} + Mx \frac{dF(u)}{du} = F(u) \frac{\partial N}{\partial x} + Ny \frac{dF(u)}{du}$$

or,
$$\frac{dF(u)}{F(u)} = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} du.$$

Arguing as before right side must be a function of u i.e. xy . Let R.H.S. $= R(u) = R(xy)$. Therefore, integrating,

$$F(u) = c e^{\int R(u) du}$$

$\therefore I.F. = F(x, y) = e^{\int R(xy) d(xy)}$

(iv) F is a function of $\frac{x}{y}$ i.e. $F \equiv F(u)$, $u = \frac{x}{y}$.

$$\text{Here } u = \frac{x}{y}, \therefore \frac{\partial u}{\partial y} = -\frac{x}{y^2} \text{ and } \frac{\partial u}{\partial x} = \frac{1}{y}.$$

$$\therefore \frac{\partial F(u)}{\partial y} = -\frac{x}{y^2} \frac{dF(u)}{du} \text{ and } \frac{\partial F(u)}{\partial x} = \frac{1}{y} \frac{dF(u)}{du}$$

$$\therefore \text{As in (iii), } \frac{dF(u)}{F(u)} = \frac{y^2 \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right)}{xM + yN} du$$

As before, $F(u) = e^{\int R(u) du}$, $u = \frac{x}{y}$.

(v) F is a function of $\frac{y}{x}$ i.e. $F \equiv F(u)$, $u = \frac{y}{x}$.

As (vi), $F(u) = e^{\int R(u) du}$, $u = \frac{y}{x}$ and $R(u) = \frac{x^2 \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right]}{xM + yN}$.

Summary : The five results obtained in A are summarised below :

- (i) If $\frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = R(x)$, then $I.F. = e^{\int R(x) dx}$
- (ii) If $\frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) = R(y)$, then $I.F. = e^{\int R(y) dy}$
- (iii) If $\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) / (xM - yN) = R(xy) = R(u)$, then $I.F. = e^{\int R(u) du}$
- (iv) If $y^2 \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) / (xM + yN) = R(u)$, $u = \frac{x}{y}$, then $I.F. = e^{\int R(u) du}$
- (v) If $x^2 \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) / (xM + yN) = R(u)$, $u = \frac{y}{x}$, then $I.F. = e^{\int R(u) du}$

(B) If the ODE, $M(x, y) dx + N(x, y) dy = 0$ is homogeneous but not exact and $Mx + Ny \neq 0$, then its $I.F. = \frac{1}{Mx + Ny}$.

Proof : Since the ODE is homogeneous, so let $M = x^n \phi(y/x)$ and $N = x^n \psi(y/x)$.

Since $\frac{1}{Mx + Ny}$ is an I.F., so its multiplication with given ODE

$$\text{i.e.} \quad \frac{1}{Mx + Ny} (M dx + N dy) = 0 \quad (1)$$

is an exact ODE. Using the expressions for M and N and cancelling the common factor x^n , (1) takes form

$$\frac{1}{x \phi(y/x) + y \psi(y/x)} [\phi(y/x) dx + \psi(y/x) dy] = 0 \quad (2)$$

which remains exact. Therefore,

$$\begin{aligned} & \frac{\partial}{\partial y} \left[\frac{\phi(y/x)}{x \phi(y/x) + y \psi(y/x)} \right] \\ &= \frac{[x \phi(y/x) + y \psi(y/x)] \frac{1}{x} \phi'(y/x) - \phi(y/x) \left[\phi'(y/x) + \frac{y}{x} \psi'(y/x) + \psi(y/x) \right]}{[x \phi(y/x) + y \psi(y/x)]^2} \end{aligned}$$

$$= \frac{\frac{y}{x} [\psi(y/x) \phi'(y/x) - \psi'(y/x) \phi(y/x)] - \psi(y/x) \phi(y/x)}{[x \phi(y/x) + y \psi(y/x)]^2} \quad (3)$$

and

$$\begin{aligned} & \frac{\partial}{\partial x} \left[\frac{\psi(y/x)}{x \phi(y/x) + y \psi(y/x)} \right] \\ & [x \phi(y/x) + y \psi(y/x)] \left[-\frac{y}{x^2} \psi'(y/x) \right] \\ & - \psi(y/x) \left[\phi(y/x) - \frac{y}{x} \phi'(y/x) - \frac{y^2}{x^2} \psi(y/x) \right] \\ & = \frac{-\frac{y}{x} [\psi(y/x) \phi'(y/x) - \psi'(y/x) \phi(y/x)] - \psi(y/x) \phi(y/x)}{[x \phi(y/x) + y \psi(y/x)]^2} \\ & = \frac{\frac{y}{x} [\psi(y/x) \phi'(y/x) - \psi'(y/x) \phi(y/x)] - \psi(y/x) \phi(y/x)}{[x \phi(y/x) + y \psi(y/x)]^2} \quad (4) \end{aligned}$$

The right sides of equations (3) and (4) are equal. Therefore, equation (2) and hence equation (1) satisfies the condition for exactness. Therefore,

$\frac{1}{Mx+Ny}$ is an I.F. of $M(x, y) dx + N(x, y) dy = 0$ provided $Mx+Ny \neq 0$.

(C), If the ODE, $M(x, y) dx + N(x, y) dy = 0$ is of the form $y f_1(xy) dx + x f_2(xy) dy = 0$ and $Mx - Ny \neq 0$, then its I.F. = $\frac{1}{Mx - Ny}$.

Here $M \equiv y f_1(xy)$ and $N \equiv x f_2(xy)$. Therefore,

$\frac{1}{Mx - Ny} = \frac{1}{xy [f_1(xy) - f_2(xy)]}$. Let it be the I.F. of the given ODE. It's multiplication with the given ODE is

$$\begin{aligned} & \frac{1}{xy [f_1(xy) - f_2(xy)]} [y f_1(xy) dx + x f_2(xy) dy] = 0 \\ \therefore & \frac{\partial}{\partial y} \left[\frac{f_1(xy)}{x [f_1(xy) - f_2(xy)]} \right] = \frac{f_1 f_2' - f_1' f_2}{(f_1 - f_2)^2} \\ \text{and} & \frac{\partial}{\partial x} \left[\frac{f_2(xy)}{y [f_1(xy) - f_2(xy)]} \right] = \frac{f_1 f_2' - f_1' f_2}{(f_1 - f_2)^2} \end{aligned}$$

which satisfy the condition for exactness. Therefore, $\frac{1}{Mx - Ny}$ is an I.F. of the given form of ODE provided $Mx - Ny \neq 0$.

(D) The I.F. of the ODE $x^p y^q (Ay dx + Bx dy) + x^r y^s (Cy dx + Dx dy) = 0$ is $x^a y^b$ if $AD - BC \neq 0$ and $\frac{A}{B} = \frac{p+a+1}{q+b+1}$ and $\frac{C}{D} = \frac{r+a+1}{s+b+1}$.

Proof : Let, under the given conditions, $x^a y^b$ be the I.F. of the given ODE. Therefore,

$$x^{a+p} y^{q+b} (Ay dx + Bx dy) + x^{r+a} y^{s+b} (Cy dx + Dx dy) = 0$$

is an exact ODE. Let

$$M \equiv Ax^{p+a} y^{q+b+1} + Cx^{r+a} y^{s+b+1}$$

$$\text{and } N \equiv Bx^{p+a+1} y^{q+b} + Dx^{r+a+1} y^{s+b}$$

$$\therefore \frac{\partial M}{\partial y} = A(q+b+1)x^{p+a} y^{q+b} + C(s+b+1)x^{r+a} y^{s+b}$$

$$\text{and } \frac{\partial N}{\partial x} = B(p+a+1)x^{p+a} y^{q+b} + D(r+a+1)x^{r+a} y^{s+b}$$

But $\frac{A}{B} = \frac{p+a+1}{q+b+1}$ and $\frac{C}{D} = \frac{r+a+1}{s+b+1}$. Therefore $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ which is the condition for exactness.

Therefore, the I.F. of the given equation is $x^a y^b$,

$$\text{where } a = \frac{BC(p+1) - AD(r+1) + CA(s-q)}{AD - BC}$$

$$\text{and } b = \frac{BC(s+1) - AD(q+1) + BD(p-r)}{AD - BC}, \because AD - BC \neq 0.$$

Examples.

1. Find I.F. of the ODE, $y(2x+y^3) dx - x(2x-y^3) dy = 0$ and hence solve.

Let $M \equiv y(2x+y^3)$ and $N \equiv -x(2x-y^3)$. Therefore,

$$\frac{\partial M}{\partial y} = 2x + 4y^3 \text{ and } \frac{\partial N}{\partial x} = -4x + y^3. \text{ So,}$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 3(2x+y^3) = \frac{3M}{y}, \therefore \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = -\frac{3}{y} = \text{a function of } y$$

only.

$$\therefore \text{I.F.} = \int e^{\int -\frac{3}{y} dy} = e^{-\int \frac{3}{y} dy} = \frac{1}{y^3}.$$

Second part : Multiplying the given equation by I.F. $= \frac{1}{y^3}$, we get

$\frac{1}{y^2} (2x+y^3) dx - \frac{x}{y^3} (2x-y^3) dy = 0$ which is exact by def. Let us use the second method for solution. So, let $f(x, y) = c$ be the solution. Hence

$$\frac{\partial f(x, y)}{\partial x} = M(x, y) = \frac{1}{y^2} (2x+y^3), \text{ from exact equation}$$

or, integrating the PDE, $f(x, y) = \frac{x^2}{y^2} + xy + g(y)$ (a)

Diff. w.r.t. y partially, $\frac{\partial f(x, y)}{\partial y} = -\frac{2x^2}{y^3} + x + g'(y) = N(x, y)$

$\therefore -2\frac{x^2}{y^3} + x + g'(y) = -\frac{x}{y^3}(2x - y^3)$ from exact equation

or, $g'(y) = 0$, or Integrating, $g(y) = C_1$

$\therefore f(x, y) = \frac{x^2}{y^2} + xy = C - C_1 = k$ is the required solution.

2. Show that $y(2x^2y^3 + 3)dx + x(x^2y^3 - 1)dy = 0$ is not an exact ODE. Find its I.F.

Let $M \equiv y(2x^2y^3 + 3)$ and $N \equiv x(x^2y^3 - 1)$

$\therefore \frac{\partial M}{\partial y} = 8x^2y^3 + 3$ and $\frac{\partial N}{\partial x} = 3x^2y^3 - 1$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. Therefore, the ODE is not exact.

Second part : The given ODE can be written as

$$x^2y^3(2ydx + xdy) + (3ydx - xdy) = 0 \quad (1)$$

It is of type 19.6.4 (D). So, let $x^a y^b$ be the I.F.

$\therefore x^{a+2}y^{3+b}(2ydx + xdy) + x^a y^b(3ydx - xdy) = 0$ is an exact ODE. Therefore,

$$\frac{\partial}{\partial y} [2x^{a+2}y^{4+b} + 3x^a y^{b+1}] = \frac{\partial}{\partial x} [x^{a+3}y^{b+3} - x^{a+1}y^b]$$

or, $2(b+4)x^{a+2}y^{b+3} + (b+1) \cdot 3x^a y^b$
 $= (a+3)x^{a+2}y^{b+3} - (a+1)x^a y^b$

Equating the coeffs. of $x^{a+2}y^{b+3}$ and $x^a y^b$, we have,

$$2(b+4) = a+3 \text{ and } 3(b+1) = -(a+1)$$

$\therefore a = \frac{7}{5} \text{ and } b = -\frac{9}{5}$

$\therefore$ the required I.F. $= x^{7/5} y^{-9/5}$

3. Differentiating $\int \frac{f(xy) + F(xy)}{f(xy) - F(xy)} \cdot \frac{d(xy)}{xy} + \log \frac{x}{y} = c$,

prove that $\frac{1}{xy \{f(xy) - F(xy)\}}$ is I.F. of the ODE

$y f(xy) dx + x F(xy) dy = 0$. Hence solve the ODE

$$y(x^2 y^2 + xy + 1) dx - x(x^2 y^2 - xy + 1) dy = 0.$$

First part : Differentiating the integral equation, we get,

$$\frac{d}{d(xy)} \left[\int \left[\frac{f(xy) + F(xy)}{f(xy) - F(xy)} \cdot \frac{1}{xy} \right] d(xy) \right] d(xy) + d \left(\log \frac{x}{y} \right) = 0$$

$$\text{or, } \frac{f(xy) + F(xy)}{f(xy) - F(xy)} \cdot \frac{x dy + y dx}{xy} + \frac{y dx - x dy}{xy} = 0$$

$$\text{or, } \left[\frac{f(xy) + F(xy)}{f(xy) - F(xy)} \cdot \frac{1}{x} + \frac{1}{x} \right] dx + \left[\frac{f(xy) + F(xy)}{f(xy) - F(xy)} \cdot \frac{1}{y} - \frac{1}{y} \right] dy = 0$$

$$\text{or, } \frac{f(xy)}{x \{f(xy) - F(xy)\}} dx + \frac{F(xy)}{y \{f(xy) - F(xy)\}} dy = 0$$

$$\text{or, } \frac{1}{xy \{f(xy) - F(xy)\}} [y f(xy) dx + x F(xy) dy] = 0 \quad (1)$$

$$\text{Let } M = \frac{f(xy)}{x \{f(xy) - F(xy)\}} \text{ and } N = \frac{F(xy)}{y \{f(xy) - F(xy)\}}$$

$$\therefore \frac{\partial M}{\partial y} = \frac{f(xy) F'(xy) - f'(xy) F(xy)}{\{f(xy) - F(xy)\}^2} = \frac{\partial N}{\partial x}$$

$$\therefore \text{equation (1) is exact. Therefore, } \frac{1}{xy \{f(xy) - F(xy)\}}$$

is an I.F. of $y f(xy) dx + x F(xy) dy = 0$ and its solution is the first integral equation of the question.

$$\text{Second part : } f(xy) \equiv x^2 y^2 + xy + 1 \text{ and } F(xy) = -x^2 y^2 + xy - 1$$

$\therefore$ the required solution is

$$\int \frac{f(xy) + F(xy)}{f(xy) - F(xy)} \cdot \frac{d(xy)}{xy} + \log \frac{x}{y} = C$$

$$\text{i.e. } \int \frac{d(xy)}{(xy)^2 + 1} + \log \frac{x}{y} = C$$

$$\text{or, } \tan^{-1}(xy) + \log \frac{x}{y} = C.$$

Exercises 19 (4)

Solve the following ODE :

1. $(2xy + 1) dx + (x^2 + 4y) dy = 0.$

2. $(e^x + 1) dx + dy = 0.$

3. $x dy + y dx = 0.$

4. $(y \cos xy + 2x) dx + x \cos xy dy = 0.$

5. $(xy \cos xy + \sin xy) dx + x^2 \cos xy dy = 0.$

6. $(xy^2 + x) dx + (x^2 y - y) dy = 0$.
7. $x \cos \left(\frac{y}{x} \right) (y dx + x dy) - y \sin \left(\frac{y}{x} \right) (x dy - y dx) = 0$.
8. $x dy + y dx = x^2 y dx$.
9. $\sin x \frac{dy}{dx} - y \cos x + y^2 = 0$.
10. $(x^3 + 3xy^3) dx + (y^3 + 3x^2 y) dy = 0$.
11. $y(1 + xy) dx + x(1 - xy) dy = 0$.
12. $(x^2 - x + y^2) dx + (e^y - 2xy) dy = 0$.
13. $x \sqrt{x^2 + y^2} dx - \frac{x^2 y}{y - \sqrt{x^2 + y^2}} dy = 0$.
14. $(4x^3 - \sin x + y^3) dx - (y^2 + 1 - 3xy^2) dy = 0$.
15. $(1 + e^{x/y}) dx + e^{x/y} (1 - x/y) dy = 0$.
16. $(y^2 e^{xy^2} + 4x^3) dx + (2xy e^{xy^2} - 3y^2) dy = 0$.
17. $[2 \cos 2x + \sin (x + 2y)] dx + [\cos y + 2 \sin (x + 2y)] dy = 0$.
18. $\left[y \left(1 + \frac{1}{x} \right) \cos y \right] dx + [x + \log x - x \sin y] dy = 0$.
19. $(x + 2y - 3) dy - (2x - y + 1) dx = 0$.
20. $[\cos x \tan y + \cos (x + y)] dx + [\sin x \sec^2 y + \cos (x + y)] dy = 0$.
21. Find the equation of a cubic satisfying $(1 + x^2) y_1 + 2xy = 4x^2$ and passing through the origin.
22. Show that $x^h y^k$ is the I.F. of $py dx + qx dy + x^m y^n (ry dx + sx dy) = 0$ if $\frac{h+1}{p} = \frac{k+1}{q}$ and $\frac{h+m+1}{r} = \frac{k+n+1}{s}$. Hence solve $3y dx - 2x dy + x^2 y^{-1} (10y dx - 6x dy) = 0$.
23. Find the I.F. of the following ODE and solve :
 - (i) $(x^2 - y^2 - y) dx - (x^2 - y^2 - x) dy = 0$,
 - (ii) $y(y + 2x + 1) dx - x(2y + x - 1) dy = 0$,
 - (iii) $(x^4 y^2 - 1) dx + (x^2 y^4 - x) dy = 0$,
 - (iv) $(x^2 y^2 + xy + 1) y dx + (x^2 y^2 - xy + 1) x dy = 0$,
 - (v) $(x^2 y - 2xy^2) dx - (x^3 - 3x^2 y) dy = 0$.

$$(vi) \left(y + \frac{1}{3}y^3 + \frac{1}{2}x^2 \right) dx + \frac{1}{4}(x + xy^2) dy = 0,$$

$$(vii) (3x^2y^4 + 2xy) dx + (2x^3y^3 - x^2) dy = 0,$$

$$(viii) x(3y dx + 2x dy) + 8y^4(y dx + 3x dy) = 0.$$

24. Find the orthogonal trajectories of each of the following family of curves :

$$(i) x^2 + (y-k)^2 = k^2, \quad (ii) x^2 - y^2 = k^2, \quad (iii) (x-k)^2 + y^2 = k^2,$$

(iv) Circles having centre on the x -axis and passing through the origin.

Answers

$$1. x^2y + x + 2y^2 = C. \quad 2. e^x + x + y = C. \quad 3. xy = C. \quad 4. x^2 + \sin xy = C.$$

$$5. x \sin xy = C. \quad 6. x^2y^2 + x^2 - y^2 = C. \quad 7. xy \cos\left(\frac{y}{x}\right) = C.$$

$$8. Cy = e^{(-1/xy)}. \quad 9. \sin x = (x + C)y. \quad 10. x^4 + 6x^2y^2 + y^4 = C.$$

$$11. x = Cy e^{(1/xy)}. \quad 12. 2x^3 - 3x^2 + 6xy^2 - 6e^y = C. \quad 13. (x^2 + y^2)^{3/2} + y^3 = C.$$

$$14. 3x^4 + 3 \cos x + 3y^3x - y^3 - 3y = C. \quad 15. x + ye^{x/y} = C.$$

$$16. e^{xy^3} + x^4 - y^3 = C. \quad 17. \sin 2x - \cos(x + 2y) + \sin y = C.$$

$$18. y(x + \log x) + x \cos y = C. \quad 19. y^2 - x^2 + xy - x - 3y = C.$$

$$20. \sin x \tan y + \sin(x + y) = C. \quad 21. 4x^3 - 3y(1 + x^2) = 0.$$

$$22. h = 2, k = -3, x^3(y + 2x^2) = Cy^3.$$

$$23. (i) \frac{1}{x^2}, x - y + \frac{1}{2} \log \frac{x+y}{x-y} = C, \quad (ii) (xy)^{-4/3}, (y-x+1)^3 = Cxy,$$

$$(iii) \frac{1}{x^2y^2}, x^4y + xy^4 - Cxy = -3, \quad (iv) \frac{1}{2x^2y^2}, xy - \frac{1}{xy} + \log \frac{x}{y} = C,$$

$$(v) \frac{1}{x^2y^2}, \frac{x}{y} + \log \frac{y^3}{x^2} = C, \quad (vi) x^3, x^4y(3+y^2) + x^6 = C, \quad (vii) \frac{1}{y^2}, x^3y^3 + x^2 = Cy,$$

$$(viii) xy, x^2y^2(x + 4y^4) = C.$$

$$24. (i) x^2 + y^2 = Cx, \quad (ii) xy = C, \quad (iii) x^2 + y^2 = Cy, \quad (iv) x^2 + y^2 = Cy.$$

19.7 LINEAR EQUATIONS

This is a very important class of first order and first degree *ODE*. It contains no non-linearity like power, products or combination of the dependent variable and its derivative. In both these senses – first degree and linearity, it is linear. So, only by linear equation, a general equation of the following form is understood always :

$$A(x) \frac{dy}{dx} + B(x)y = C(x), \quad A(x) \neq 0 \quad (1)$$

called linear in y , the name of the dependent variable. Equation (1) can be written in normal form as

$$\frac{dy}{dx} + P(x)y = Q(x) \quad (2)$$

where $P(x) = B(x)/A(x)$ and $Q(x) = C(x)/A(x)$. A linear equation in x is also written as

$$\frac{dx}{dy} + P(y)x = Q(y) \quad (3)$$

Characteristics : The general characteristics for identification – the equation contains altogether **three** terms – one with first derivative, one with one simple dependent variable and the other with pure independent variable. So, it is a non-homogeneous *ODE*. It will be found next that the solution procedure is most simple and routinely formulated. The equation possesses a formulated *I.F.*

Solution procedures : The procedure for solving equation (2) will be discussed. If $Q(x) = 0$, the equation becomes homogeneous and can be easily solved by separation of variables as

$$\frac{dy}{y} = -P(x) dx \text{ or integrating, } y = c e^{-\int P(x) dx}.$$

If $Q(x) \neq 0$, the equation (2) is non-homogeneous. It's solution is of more interest. The equation (2) is not exact. Let $F(x)$ be the *I.F.* of (2). Therefore multiplying (2) by $I.F. = F(x)$ the equation becomes

$$F(x) \frac{dy}{dx} + P(x) F(x)y = F(x) Q(x)$$

which by definition is exact. So, it must satisfy the condition for exactness, namely,

$$\frac{\partial}{\partial y} (P(x) F(x)y - F(x) Q(x)) = \frac{\partial}{\partial x} F(x)$$

or,
$$P(x) F(x) = \frac{d}{dx} F(x) \text{ which is an ODE.}$$

Now, separating the variables F and x , and integrating,

$$F(x) = C e^{\int P(x) dx}$$

Dropping the multiplicative constant C i.e. considering its unit value, the *I.F.* for equation (2) is the formula

$$I.F. = F(x) = e^{\int P(x) dx} \quad (3)$$

Now, multiplying the equation (2) by (3), we get,

$$e^{\int P(x) dx} \frac{dy}{dx} + P(x)y e^{\int P(x) dx} = Q(x) e^{\int P(x) dx}$$

or
$$\frac{d}{dx} [y e^{\int P(x) dx}] = Q(x) e^{\int P(x) dx} \quad (4)$$

Note that after multiplication of (2) by (3), L.H.S. becomes the differential coefficient of the product of dependent variable, y and the I.F. of (2) and R.H.S. is a function of the independent variable x alone. So, both sides are integrable. Then integrating, the general solution of (2) is

$$y e^{\int P(x) dx} = \int Q(x) e^{\int P(x) dx} + C \quad (5)$$

Steps : (i) Compute I.F. using (3) or its equivalent.

(ii) Multiply the linear equation by I.F.

(iii) L.H.S. must be the differential coefficient of the product of dependent variable and I.F.

(iv) Step (iii) is a check point. If step (iii) fails, check the computation of I.F. for correctness.

(v) Integrate step (iii).

(vi) Compute the R.H. integral of (v) and hence the general solution.

Method of variation of parameter, an alternative to I.F. method :

This method is due to J.L. Lagrange (1736–1813) of French. The general solution of the associated homogeneous equation (2) i.e. of

$$\frac{dy}{dx} + P(x)y = 0 \quad (6)$$

is $y(x) = c e^{-\int P(x) dx} \equiv c v(x)$. In this method the constant c is considered as a function of the independent variable, x . So, the method is variation of parameter or constant. Let the solution of (2) be

$$y = c(x) v(x) \quad (7)$$

$$\therefore \frac{dy}{dx} = c(x) v'(x) + c'(x) v(x) \quad (8)$$

Using (7) and (8) in (2), we get,

$$c(x) v'(x) + c'(x) v(x) + P(x) c(x) v(x) = Q(x)$$

$$\text{or, } c'(x) v(x) + [v'(x) + P(x) v(x)] c(x) = Q(x)$$

$$\text{or, } c'(x) v(x) = Q(x) \quad \because \text{by (6) } v'(x) + P(x) v(x) = 0$$

$$\text{or, } c'(x) = \frac{Q(x)}{v(x)}$$

$$\text{Integrating, } c(x) = \int \frac{Q(x)}{v(x)} dx + c_1$$

$\therefore$ by (7), the general solution of (2) is

$$y = v(x) \left[\int \frac{Q(x)}{v(x)} dx + c_1 \right]$$

$$\text{or, } y e^{\int P(x) dx} = \int [Q(x) e^{\int P(x) dx}] dx + c_1 \quad \because v(x) = e^{-\int P(x) dx}$$

which is the same solution (5).

19.7.1 APPLICATION : ELECTRIC CURRENT

In 19.3.3 (iii) electromotive force (*e.m.f.*) was considered constant and the circuit was free of capacitor, *C*. If the *e.m.f.* is a function of time, *t* and the circuit contains capacitor *C* then 19.3.3 (iii) is not applicable.

Let in the circuit of 19.3.3 (iii), the *e.m.f.* is a function of time, *t*. Let $E(t) = E_0 \sin \omega t$ be periodic energy. Therefore according Kirchhoff's law, ODE for the series circuit is

$$L \frac{di}{dt} + Ri = E(t)$$

$$\text{or, } \frac{di}{dt} + \frac{R}{L} i = \frac{E_0}{L} \sin \omega t \quad (1)$$

This is a linear equation in *i*. So,

$$I.F. = e^{\int R/L dt} = e^{Rt/L}$$

Multiplying (1) by *I.F.*, we get

$$\frac{d}{dt} (i e^{Rt/L}) = e^{Rt/L} \frac{E_0}{L} \sin \omega t$$

Integrating,

$$\begin{aligned} i(t) &= C e^{-Rt/L} + \frac{E_0}{R^2 + \omega^2 L^2} (R \sin \omega t - \omega L \cos \omega t) \\ &= C e^{-Rt/L} + \frac{E_0}{\sqrt{R^2 + \omega^2 L^2}} \sin(\omega t - \delta), \quad \delta = \tan^{-1} \frac{\omega L}{R}, \end{aligned} \quad (2)$$

$$\text{When } t \rightarrow \infty, C e^{-Rt/L} \rightarrow 0 \text{ and } i(t) = \frac{E_0}{\sqrt{R^2 + \omega^2 L^2}} \sin(\omega t - \delta)$$

i.e. after sufficient time has been elapsed the electric flow will be harmonic in nature.

Electric flow in *R - C* circuit :

Capacitor, *C* stores energy. It's potential difference or voltage, E_C is directly proportional to the electric charge, *Q*

$$\text{i.e. } E_C \propto Q \quad \text{i.e. } E_C = \frac{1}{C} Q \quad (3)$$

where *C* is called capacitor. The unit of measurement of *C* and *Q* are farad and coulomb respectively. Since

$$i(t) = \frac{dQ}{dt}, \quad \therefore E_C = \frac{1}{C} \int i(t) dt \quad (4)$$

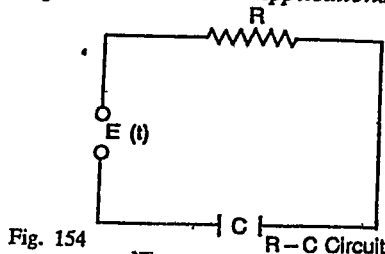
Using Kirchhoff's law in an *R - C* circuit (fig. 153), we get,

$$Ri + \frac{1}{C} \int i dt = E(t)$$

Differentiating it w.r.t. 't',

$$R \frac{di}{dt} + \frac{1}{C} i = \frac{dE}{dt} \quad (5)$$

Let us consider the following cases :



Case (a) : E or $e.m.f.$ is constant. Therefore $\frac{dE}{dt} = 0$. So, (5) becomes,

$R \frac{di}{dt} + \frac{1}{C} i = 0$. It's solution is $i(t) = C_1 e^{-t/(CR)}$. This shows that after sufficiently long time flow of current will stop.

Case (b) : $e.m.f.$ is sinusoidal. If $E(t) = E_0 \sin \omega t$, then

$\frac{dE}{dt} = \omega E_0 \cos \omega t$. Therefore, equation (5) becomes,

$$R \frac{di}{dt} + \frac{1}{C} i = \omega E_0 \cos \omega t$$

$$\text{or, } \frac{di}{dt} + \frac{1}{RC} i = \frac{\omega E_0}{R} \cos \omega t \quad (6)$$

This is a linear equation in i . So, it's I.F. = $e^{\int \frac{1}{RC} dt} = e^{t/RC}$. Multiplying (6) by I.F., we get,

$$\frac{d}{dt} (i e^{t/RC}) = e^{t/RC} \frac{\omega E_0}{R} \cos \omega t$$

$$\text{Integrating, } i(t) = K e^{-t/RC} + \frac{\omega E_0 C}{1 + (\omega RC)^2} (\cos \omega t + \omega RC \sin \omega t)$$

$$= K e^{-t/RC} + \frac{\omega E_0 C}{\sqrt{1 + (\omega RC)^2}} \sin(\omega t - \delta), \quad (7)$$

$$\text{where } \delta = \tan^{-1} \left(-\frac{1}{\omega RC} \right)$$

This shows that after sufficiently long time, the flow of current will be in harmonic in nature.

Examples

1. Solve the ODE, $x \cos x \frac{dy}{dx} + y(x \sin x + \cos x) = 1$.

The ODE can be written as

$$\frac{dy}{dx} + \left(\tan x + \frac{1}{x} \right) y = \frac{1}{x} \sec x \quad (1)$$

and is linear in y . Here $P(x) \equiv \tan x + \frac{1}{x}$ and $Q(x) \equiv \frac{1}{x} \sec x$. Therefore,

$$I.F. = e^{\int P(x) dx} = e^{\int \left(\tan x + \frac{1}{x} \right) dx} = e^{\log \sec x + \log x} = x \sec x$$

Multiplying (1) by I.F.

$$x \sec x \frac{dy}{dx} + x \sec x \left(\tan x + \frac{1}{x} \right) y = \sec^2 x$$

$$\text{or, } \frac{d}{dx} (y \cdot x \sec x) = \sec^2 x$$

Integrating, $xy \sec x = x \tan x + C$.

Alter (Variation of parameter)

The associated homogeneous equation of (1) is

$$\frac{dy}{dx} + \left(\tan x + \frac{1}{x} \right) y = 0 \quad (2)$$

$$\therefore \text{ sol. is } y = \frac{C}{x} \cos x \quad (3)$$

Let C be a function of x i.e. let $C = C(x)$ and let $y = \frac{C(x)}{x} \cos x$ be the solution of (1). Now diff. w.r.t.,

$$\frac{dy}{dx} = \frac{C'(x)}{x} \cos x + C(x) \left[-\frac{1}{x^2} \cos x - \frac{1}{x} \sin x \right]$$

$\therefore$ equation (1) becomes,

$$\begin{aligned} \frac{C'(x)}{x} \cos x + C(x) \left[-\frac{1}{x^2} \cos x - \frac{1}{x} \sin x \right] + \left(\tan x + \frac{1}{x} \right) \frac{C(x)}{x} \cos x \\ = \frac{1}{x} \sec x \end{aligned}$$

$$\text{or, } \frac{C'(x)}{x} \cos x = \frac{1}{x} \sec x$$

$$\text{or, } C'(x) = \sec^2 x$$

$$\text{or, integrating, } C(x) = \tan x + C$$

$$\therefore \text{ the general solution is } y = \frac{\tan x + C}{x} \cos x$$

$$\text{or, } xy \sec x = \tan x + C$$

$$2. \text{ Solve the ODE, } (x+y+1) \frac{dy}{dx} = 1$$

None of the methods is applicable to the given equation. Let us invert the equation, as

$$\frac{dx}{dy} = x + y + 1$$

or, $\frac{dx}{dy} - x = (y+1)$ which is linear in x .

Now $I.F. = e^{\int -dy} = e^{-y}$

Multiplying the linear equation in x by $I.F.$ and integrating, we have,

$$\int \frac{d}{dy}(xe^{-y}) dy = \int (y+1) e^{-y} dy + C$$

$$\text{or, } xe^{-y} = -(y+1)e^{-y} - e^{-y} + C$$

$$\text{or, } x + y + 2 = Ce^y \text{ as the required general solution.}$$

3. A series circuit consists of a constant inductance of 1 henry, a variable resistance R , and a constant e.m.f. E . If $R = \frac{1}{5+t}$ ohm at any time t and $i = 0$ at time $t = 0$, then find E when $i = 30$ amperes at $t = 5$ secs.

$$\text{The governing ODE is } L \frac{di}{dt} + Ri = E \quad (1)$$

Using given data $L = 1$ and $R = \frac{1}{5+t}$ and $E = \text{const.}$ in equation (1), we get,

$$\frac{di}{dt} + \frac{1}{5+t} i = E \quad (2)$$

which is linear in i . So, $I.F. = e^{\int \frac{1}{5+t} dt} = (5+t)$

$\therefore$ multiplying (2) by $I.F.$ and integrating, we get,

$$i(5+t) = E \left(5t + \frac{t^2}{2} \right) + C \quad (3)$$

Using $I.C.$: $i = 0$ at $t = 0$ in (3), we get $C = 0$, so (3) becomes

$$i(5+t) = E \left(5t + \frac{t^2}{2} \right) \quad (4)$$

Using $B.C.$: $i = 30$ at $t = 5$ in (4) we get the required result, $E = 8$ volts.

Exercises 19 (5)

Solve the following ODE :

$$1. y' - y = e^{2x}. \quad 2. xy' + y + 4 = 0. \quad 3. xy' + y = \sin x.$$

$$4. (1+x^2) \frac{dy}{dx} + (1-x^2)y = xe^{-x}. \quad 5. x \frac{dy}{dx} - y = (x-1)e^x.$$

$$6. x^2 \frac{dy}{dx} = (2x-1)y + x^2. \quad 7. \frac{dy}{dx} - y \cot x = e^x (1 - \cot x).$$

$$8. \frac{dy}{dx} + \frac{y}{x} = \sin x^2. \quad 9. (1+y^2) dx - (\tan^{-1} y - x) dy = 0.$$

First Order and First Degree Differential Equations

10. $y^2 + \left(x - \frac{1}{y}\right) \frac{dy}{dx} = 0.$

11. $ye^y dx = (y^3 + 2xe^y) dy.$

[Hints : arrange 9, 10, 11 as linear in x]

12. $(x+a) \frac{dy}{dx} - 3y = (x+a)^5.$

13. Solve the following initial value problems :

(i) $x^2 y' + 2xy - x + 1 = 0, y(1) = 0$

(ii) $y' + y \tan x = \sin 2x, y(0) = 1$

14. Slope of a curve at any point (x, y) is $(y + 2x)$. It passes through the origin. Show that its equation is $y = 2(e^x - x - 1)$.

15. Find the equation of the curves for which the sum of the reciprocals of the polar subtangent and the radius vector is constant.

16. An *e.m.f.* of $E_0 \sin 10t$ volts, an inductance of 1 henry and a resistance of 10 ohms are connected in series. $i = 0$ at $t = 0$. Find the value of E_0 if $i = 5$ amperes at $t = .1$ secs.

17. A piston is moving with a force $F_1 = v \cos 2s$ in a oil filled cylinder. The oil resists the piston with a force $F_2 = -v^2$. That is, equation of motion is

$$v \frac{dv}{ds} = v \cos 2s - v^2$$

where v is the velocity of the piston and s is the distance traversed in time t . Express v as a functions of s .

18. A mass of m slugs falls from a height of 1000 ft. Find its velocity and distance traversed in time t when the air resistance is linearly proportional to the velocity of the falling mass and the constant of proportionality is

$$\frac{m}{40}.$$

19. An *e.m.f.* of Ee^{-t} volts, inductance of 1 henry and a resistance of 2 ohms are connected in series. (i) If there is no current flow initially, find E when $i = 10$ amp. at $t = 1$ sec., (ii) at what time current flow will 5 amps when $E = 50$ volts.

20. An infectious disease is introduced to a large population. The proportion of people who have exposed to the disease increases with time. Suppose that $P(t)$ is the proportion of people who have been exposed to the disease with t years of its introduction. If $P'(t) = \frac{1}{3}(1 - P(t))$ and $P(0) = 0$, after how many years will the proportion have increased to 90%?

[Hints : $P = 90\% = 9/10$]

Answers

1. $y = e^{2x} + Ce^x$. 2. $xy + 4x = C$. 3. $xy + \cos x = C$. 4. $2ye^x + 1 = C(1+x^2)$.

5. $y = e^x + Cx$. 6. $y = (1 + Ce^{1/x})x^2$. 7. $y = e^x + C \sin x$

8. $xy + \frac{1}{2} \cos x^2 = C$. 9. $x + 1 = Ce^{-\tan^{-1} y} + \tan^{-1} y$. 10. $x - 1 = y^{-1} + Ce^{1/y}$.

11. $x = y^2 (C - e^{-y})$. 12. $2y = (x+a)^5 \left\{ 1 + \frac{C}{(x+a)^2} \right\}$.

13. (i) $xy = x - \log x - 1$.

(ii) $y \sec x + 2 \cos x = 3$.

15. $\theta = Cr - \log r + k$.

16. 149 volts.

17. $5v = \cos 2s + 2 \sin 2s + Ce^{-s}$.

18. $y = 40g(t + 40e^{-t/40} - 40)$ and $v = 40g(1 - e^{-t/40})$.

19. (i) 40 volts, (ii) 0.120, 2.18 secs.

20. $t = 3 \log 10 \approx 6.9$ years.

19.8 REDUCIBLE TO LINEAR

There are certain non-linear equations that can be reduced to linear equations by some manipulation. In general non-linear equations cannot be solved by finger rule. Here two types of these equations will be discussed – (i) Bernoulli's and (ii) Riccati's.

(i) **Bernoulli's equation** : The general form of this equation is

$$\frac{dy}{dx} + P(x)y = Q(x)y^n, n \neq 1 \quad (1)$$

This equation is named after the Swiss mathematician James Bernoulli (1654–1706) although it was solved by Liebtinz in 1696. The non-linearity occurs on the right side as a coefficient of $Q(x)$ of linear equation in 19.7. The equation (1) can be reduced to linear form in **three** steps – (a) division by non-linear part, (b) substitution for the non-linear part arises in the second term after (a), (c) removal of the coefficients of the derivative. Thus, dividing (1) by y^n , we have,

$$\frac{1}{y^n} \frac{dy}{dx} + P(x) \cdot \frac{1}{y^{n-1}} = Q(x) \quad (2)$$

Substituting, $\frac{1}{y^{n-1}} = u$, we get $-(n-1) \frac{1}{y^n} \frac{dy}{dx} = \frac{du}{dx}$.

$\therefore$ (2) becomes,

$$-\frac{1}{n-1} \frac{du}{dx} + P(x)u = Q(x), n \neq 1$$

or, $\frac{du}{dx} - (n-1)P(x)u = -(n-1)Q(x) \quad (3)$

This is the reduced linear equation of the non-linear equation (1). Therefore, the

$$I.F. = e^{-\int (n-1)P(x) dx} \quad (4)$$

Multiplying (3) by $I.F.$ and integrating, we have,

$$ue^{-\int (n-1)P(x) dx} = \int [-(n-1)Q(x)e^{-\int (n-1)P(x) dx}] dx + c \quad (5)$$

This is the general solution of (1) where $u = y^{-(n-1)}$

(ii) **Riccati's equation**: A first order and first degree ODE of the form

$$\frac{dy}{dx} + a_2(x)y^2 + a_1(x)y + a_0(x) = 0 \quad (6)$$

where $a_0(x)$, $a_1(x)$, $a_2(x)$ are continuous in x and $a_2(x) \neq 0$, is called a *Riccati equation*. The equation is non-linear because of the presence of y^2 . This equation arises in many physical applications. This equation can be reduced to linear equation by the substitution $y = y_1 + \frac{1}{u}$ where y_1 is known before hand.

Let us substitute $y = y_1 + \frac{1}{u}$ where y_1 is a known particular solution.

$$\therefore \text{differentiating,} \quad \frac{dy}{dx} = \frac{dy_1}{dx} - \frac{1}{u^2} \frac{du}{dx}$$

Putting in (6) we have,

$$\frac{dy_1}{dx} - \frac{1}{u^2} \frac{du}{dx} + a_2(x) \left[y_1 + \frac{1}{u} \right]^2 + a_1(x) \left[y_1 + \frac{1}{u} \right] + a_0(x) = 0$$

$$\text{or,} \quad \left[\frac{dy_1}{dx} + a_2(x)y_1^2 + a_1(x)y_1 + a_0(x) \right] - \frac{1}{u^2} \frac{du}{dx} + 2a_2(x)y_1 \frac{1}{u} + \frac{a_2(x)}{u^2} + \frac{a_1(x)}{u} = 0$$

$$\text{or,} \quad \frac{du}{dx} - \left(2a_2(x)y_1 + a_1(x) \right) u - a_2(x) = 0 \quad (7)$$

$\therefore y_1$ is a solution (6), so squared bracket part vanishes.

The equation (7) is now linear in u and y_1 , the known solution, is some function of x . Therefore,

$$I.F. \text{ of (7)} = e^{\int [a_1(x) + 2a_2(x)y_1] dx} = v(x) \quad (\text{say}) \quad (8)$$

Multiplying (7) by $I.F.$ and integrating, the general solution of (6) is

$$u v(x) = \int a_2(x) v(x) dx + C$$

where $u = \frac{1}{y_1 - y}$ and $v(x)$ is given by (8).

Examples

1. Solve the non-linear ODE $\frac{dy}{dx} + \frac{y}{x} \log y = \frac{y}{x^2} (\log y)^2$.

The equation is of Bernoulli's type because of presence of non-linear part on the right side. So, dividing by $y (\log y)^2$, we get,

$$\frac{1}{y (\log y)^2} \frac{dy}{dx} + \frac{1}{x} \cdot \frac{1}{(\log y)} = \frac{1}{x^2} \quad (1)$$

Now substituting, $u = \frac{1}{\log y}$ or $\frac{du}{dx} = -\frac{1}{y (\log y)^2} \frac{dy}{dx}$

Using these in (1), we get,

$$-\frac{du}{dx} + \frac{1}{x} u = \frac{1}{x^2} \quad \text{or} \quad \frac{du}{dx} - \frac{1}{x} u = -\frac{1}{x^2} \quad (2)$$

Equation (2) is now linear in u . Therefore,

$$I.F. = e^{-\int \frac{1}{x} dx} = \frac{1}{x}$$

$\therefore$ multiplying equation (2) by I.F. and integrating,

$$\frac{u}{x} = \frac{1}{2x^2} + C$$

or, $\frac{1}{x \log y} = \frac{1}{2x^2} + C$ is the required solution.

2. Find a general solution of the equation

$y' = y^2 + (1-2x)y + (x^2 - x + 1)$, given that $y = x$ is a particular solution.

The given ODE is non-linear and of Riccati's type. With the given particular solution, let $y = \tilde{x} + \frac{1}{u}$ be the substitution.

$\therefore y' = 1 - \frac{1}{u^2} u'$. Using all these in the given equation, we have,

$$1 - \frac{1}{u^2} u' = \left(x + \frac{1}{u}\right)^2 + (1-2x) \left(x + \frac{1}{u}\right) + x^2 - x + 1$$

or, $u^2 - u' = u^2 x^2 + 2ux + 1 + u^2 x + u - 2u^2 x^2 - 2ux + u^2 x^2 - u^2 x + u^2$

or, $u' + u + 1 = 0$ or $u' + u = -1$ which is linear in u .

$\therefore I.F. = e^{\int dx} = e^x$

Multiplying $u' + u = -1$ by I.F. we get

$$u' e^x + u e^x = -e^x$$

or, $\frac{d}{dx} (u e^x) = -e^x$

Integrating, $u e^x = -e^x + C$

or, $\frac{1}{y-x} e^x = -e^x + C$ or $y = x + \frac{1}{(Ce^{-x} - 1)}$ is the required general solution.

Exercises 19 (6)

Solve the following non-linear ODE :

- $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x) e^x \sec y.$
- $nx \frac{dy}{dx} + 2y = xy^{n+1}.$
- $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y.$
- $\frac{dy}{dx} + \frac{x}{1-x^2} y = x\sqrt{y}.$
- $e^y \frac{dy}{dx} + e^y = e^x.$
- $x \frac{dy}{dx} + \frac{y^2}{x} = y.$
- $3xy' + y + x^2 y^4 = 0.$
- Show that the substitution $y = v^{1/(1-n)}$, $n \neq 0, 1$, reduces the Bernoulli's equation to linear and hence solve
(i) $xy' + y = y^3$, (ii) $(1-x^2)y' - 2(1+x)y = \sqrt{y^5}$, (iii) $xy' - y = y^2 e^{-x}$.
- Solve the initial value problem :

$$y' + \frac{y}{x} = (\log x)y^2, y(1) = 1.$$

10. Find a general solution of each of the following non-linear equations, using the given particular solution : [Hint. Follow Sec. 19.8 (ii)]

- $y' = xy^2 + (1-2x)y + x - 1, y_1 = 1,$
- $y' = (x+y)(x+y-2), y_1 = 1-x,$
- $y' = x^3(y-x)^2 + y/x, y_1 = x,$
- $2y' - (y/x)^2 - 1 = 0, y_1 = x,$
- $y' - (\sin^2 x)y^2 + y \sec x \operatorname{cosec} x + \cos^2 x = 0, y_1 = \cos x \operatorname{cosec} x,$

Answers

- $\sin y = (C + e^x)(1+x).$
- $xe^{-1/(xy^2)} = C.$
- $\tan y = x^2 - 1 + Ce^{-x^2}.$
- $\sqrt{y} + \frac{1}{3}(1-x^2) = C(1-x^2)^{1/4}.$
- $e^{x+y} = \frac{e^{2x}}{2} + C.$
- $y = \frac{x}{C + \log x}.$
- $y^3 = \frac{1}{x^2 + Cx}.$
- (i) $\frac{1}{y^2} = Cx^2 + 1,$
- (ii) $\frac{1}{y^{3/2}(1-x)^3} = -\frac{3}{2} \left[\frac{1}{16} \log(1-x^2) + \frac{1}{8(1-x)} + \frac{3}{16(1-x)^2} + \frac{1}{6(1-x)^3} \right] + C,$
- (iii) $\frac{x}{y} = e^{-x} + C.$
- $\frac{1}{xy} = -\frac{(\log x)^2}{2} + 1.$
- (i) $y = 1 + \frac{1}{1-x + Ce^{-x^2}},$
- (ii) $y = 1 - x + \frac{1}{C-x},$
- (iii) $y = x + 5x/(C-x^5),$
- (iv) $y = x + 2x/(C - \log x),$
- (v) $y = \cos x \operatorname{cosec} x \left[1 + \left(Ce^{-\sin^2 x} - \frac{1}{2} \right)^{-1} \right].$

FIRST ORDER BUT HIGHER DEGREE ODE

20.1 DEFINITION AND INTRODUCTION TO METHODS OF SOLUTION

This is a first order *ODE* of degree more than one. This means, by the definitions of order and degree of an *ODE*, that the equation must have first order derivative and the degree of the first derivative must be more than one. So, the equation is non-linear by nature. A most general n th degree first order *ODE* is

$$\left(\frac{dy}{dx}\right)^n + P_1(x, y) \left(\frac{dy}{dx}\right)^{n-1} + P_2(x, y) \left(\frac{dy}{dx}\right)^{n-2} + \dots + P_{n-1}(x, y) \frac{dy}{dx} + P_n y = 0$$

The equation is homogeneous by definition of homogeneity and non-linearity is mainly due to the higher powers of $\frac{dy}{dx}$. But the non-linear equations studied in chapter 19 were due to the higher powers of the dependent variable, y but not for derivatives. For convenience, the symbol p for $\frac{dy}{dx}$ is normally used. So, the above *ODE* resembles an algebraic polynomial of p as

$$p^n + P_1(x, y) p^{n-1} + P_2(x, y) p^{n-2} + \dots + P_{n-1}(x, y) p + P_n y = 0 \quad (1)$$

This equation can be reduced to first order and first degree equations of chapter 19 and the reduced equations are solved by using the methods of chapter 19. This reduction is carried out in the following five ways : (i) Factorization into linear factors in p ; (ii) Solvable for x ; (iii) Solvable for y ; (iv) Lagrange's equation and (v) Clairaut's equation. These will be discussed in the following section one after another.

20.2 METHODS OF SOLUTION

The solution procedures named in the previous section will be discussed in details in the section.

(i) Factorization into linear factors in p : This is applicable when the equation (1) in 20.1 can either be factored into linear factors in p or solved for p . Since equation (1) contains variables p , x and y , the form of linear factor in p must be $[p - f(x, y)]$. Also the degree of the equation is n , by algebra, there must be n such linear factors. Let factored form of equation (1) be

$$[p - f_1(x, y)][p - f_2(x, y)][p - f_3(x, y)] \dots [p - f_n(x, y)] = 0 \quad (2)$$

$$\therefore p - f_1(x, y) = 0 \quad (a_1)$$

$$p - f_2(x, y) = 0 \quad (a_2)$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$p - f_n(x, y) = 0 \quad (a_n)$$

$\therefore$ the equation (1) breaks up into n first order and first degree equations of chapter 19. Let the solutions of those equations be

$$\phi_1(x, y, c_1) = 0, \phi(x, y, c_2) = 0, \dots, \phi_n(x, y, c_n) = 0.$$

$\therefore$ the solution of the original equation must be product of these solutions i.e.

$$[\phi_1(x, y, c_1)][\phi_2(x, y, c_2)] \dots [\phi_n(x, y, c_n)] = 0 \quad (3)$$

which contains n arbitrary constants contrary to the hypothesis of first order ODE i.e. it must contain a single arbitrary constant. To meet this hypothesis, let $c_1 = c_2 = \dots = c_n = c$ and the solution (3) becomes

$$[\phi_1(x, y, c)][\phi_2(x, y, c)] \dots [\phi_n(x, y, c)] = 0$$

This is the form of the required solution.

(ii) Solvable for x : When (i) above fails, then the equation (1) may be solved for x i.e. x can be expressed in terms of p and y . The general form is $x = f(p, y)$ (4)

Differentiating equation (4) w.r.t. y ,

$$\frac{dx}{dy} = f_p(p, y) \frac{dp}{dy} + f_y(p, y)$$

$$\text{or, } \frac{1}{p} = f_p(p, y) \frac{dp}{dy} + f_y(p, y), \left(\because \frac{dy}{dx} = p \right) \quad (5)$$

This is the reduced first order first degree equation of (4). This can be solved by any one of the six procedures of chapter 19. Let the solution of (5) be of the form

$$\phi(p, y, c) = 0 \quad (6)$$

as equation (5) contains the variables p and y .

Now the solution of the given equation (4) is either the eliminant of p between (4) and (6) or the parametric form

$$\left. \begin{array}{l} x = f(p, y) \\ \phi(p, y, c) = 0 \end{array} \right\} \text{ where } p \text{ is a parameter,} \quad (7)$$

if the elimination is not possible.

(iii) Solvable for y : If (i) and (ii) fails or only (i) fails, then equation (1) can be solved for y i.e. y can be expressed in terms of x and p . The general form is $y = f(x, p)$ (8)

The following two cases (iv) and (v) are the special cases of this case. So, it is better to solve for y , if an equation is solvable for both x and y , to investigate the other two particular cases.

Now as in (ii), differentiating (8) w.r.t. x

$$\frac{dy}{dx} = f_x(x, p) + f_p(x, p) \frac{dp}{dx}$$

$$\text{or, } p = f_x(x, p) + f_p(x, p) \frac{dp}{dx} \quad (9)$$

This is the reduced first order first degree equation in p and x of the equation (8). Let its solution be of the form

$$\psi(p, x, c) = 0 \quad (10)$$

Now the required solution of (8) is either the eliminant of p between (8) and (10) or the parametric form

$$\left. \begin{aligned} y &= f(x, p) \\ \psi(p, x, c) &= 0 \end{aligned} \right\} \text{ where } p \text{ is a parameter,}$$

if the elimination is impossible.

(iv) Lagrange's equation : When the given equation is solved for y , it takes form $y = x f(p) + g(p)$ (11)

This particular equation (11) is known as Lagrange's equation. If this form can be identified the reduced equation of (11) will linear in x as will be visible within a while.

Differentiating (11) w.r.t. x , we have,

$$\frac{dy}{dx} = f(p) + [x f'(p) + g'(p)] \frac{dp}{dx}$$

$$\text{or, } p - f(p) = [x f'(p) + g'(p)] \frac{dp}{dx}$$

$$\text{or, } \frac{dx}{dp} = \frac{x f'(p) + g'(p)}{p - f(p)}$$

$$\text{or, } \frac{dx}{dp} - \frac{f'(p)}{p - f(p)} \cdot x = \frac{g'(p)}{p - f(p)} \quad (12)$$

This reduced equation (12) of (11) is linear in x . It has a routine procedure for solving by computing I.F. Its general solution is

$$x e^{\int P dp} = \int (Q e^{\int P dp}) dp + c \quad (13)$$

$$\text{where } P = \frac{-f'(p)}{p - f(p)} \text{ and } Q = \frac{g'(p)}{p - f(p)}$$

$\therefore$ the required solution of the given equation is either the eliminant of p between (11) and (13) or their parametric form with p as parameter.

(v) **Clairaut's equation** : It is a particular form of (iv). Its general form is $y = xp + g(p)$ (14)

It is named after the name of French mathematician and astrologer Alexis Caude Clairaut (1713-1765).

Differentiating (14) w.r.t. x , we get,

$$p = p + [x + g'(p)] \frac{dp}{dx} \text{ or } [x + g'(p)] \frac{dp}{dx} = 0 \quad (15)$$

Since the product is zero in (15), either one must vanish. So there arises two cases :

(a) If $\frac{dp}{dx} = 0$, then $p = c$. Substituting this value in (14), produces the **general solution** $y = cx + g(c)$ (16) which is a family of straight lines.

(b) If $x + g'(p) = 0$, then $x = -g'(p)$. This is an algebraic relation. Elimination of p between this equation and (14) or their parametric representation

$$\left. \begin{aligned} y &= px + g(p) \\ x &= -g'(p) \end{aligned} \right\}, p \text{ is a parameter,} \quad (17)$$

produces another solution of (14) which can be verified as follows :

Differentiating 1st. equation of (17) w.r.t. x

$$\frac{dy}{dx} = p + [x + g'(p)] \frac{dp}{dx} \quad (18)$$

If $\frac{dp}{dx} \neq \infty$, then use of the 2nd equation of (17) into (18) gives $\frac{dy}{dx} = p$ again. This shows that (17) is also a solution of (14). But this solution cannot be derived by giving any particular value to c in (16), the general solution of (14) and it contains no arbitrary constant. Hence this solution is a **singular solution** of (14).

$\therefore$ Clairaut's equation has **two** solutions – **general** and **singular** solutions.

Geometrical interpretation of singular solution :

The slope of the singular solution (17) is seen to be $\frac{dy}{dx} = p = c$. So, the equation of the tangent to (17) is

$$Y - y = m(X - x)$$

Changing X and Y into current variables x and y and replacing x, y by (17), we obtain,

$$\text{or, } y - \{-cg'(c) + g(c)\} = c(x + g'(c)) \quad \because p = c, \quad m = \frac{dy}{dx} = p$$

$$\text{or, } y = cx + g(c) \text{ which is the general solution (16).}$$

$\therefore$ singular solution is geometrically the envelope of the general solution of Clairaut's equation.

20.3 SINGULAR SOLUTIONS IN GENERAL

Singular solutions for equations other than Clairaut's equation is the main theme of this section. It is seen that singular solutions are the envelopes of the general solutions for Clairaut's equation. This is true for other equations also. This section consists of two parts – (i) definitions related to the determination of singular solution and (ii) determination of singular solution.

(i) DEFINITIONS : Discriminant : The term discriminant means the expression $b^2 - 4ac$ for the quadratic equation $ax^2 + bx + c = 0$ in algebra. This is obtained by eliminating x between $f(x) \equiv ax^2 + bx + c = 0$ and $f'(x) \equiv 2ax + b = 0$. This procedure is true for any function. For more than one variable, partial derivative is used instead of ordinary derivative.

Now, if the first order but higher degree ODE, $f(x, y, p) = 0$ has a solution $F(x, y, c) = 0$, then the eliminant of p between $f(x, y, p) = 0$ and $\frac{\partial}{\partial p} f(x, y, p) = 0$ is called the p -discriminant while the eliminant of c between $F(x, y, c) = 0$ and $\frac{\partial}{\partial c} F(x, y, c) = 0$ is called the c -discriminant.

Nodal-point or node : If two distinct tangents can be drawn to a curve at its double point then that double point is called the **nodal-point** or **node** only and the locus of that point is called the **nodal locus** or **node locus** (fig. 155).

Cusp : If the tangents at a double point coincides instead of being distinct i.e. the two parts of the curve become tangents to each other, then the double point is called **cusp** and the locus of these cusps is called **cuspidal** or **cusp locus** (fig. 155).

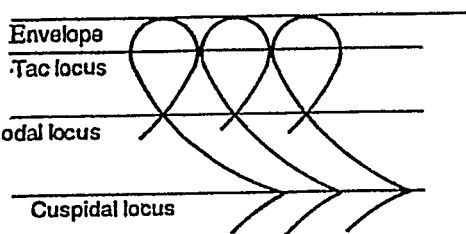


Fig. 155

Tac-locus : If the two parts of a family of curves touches each other, then the locus of the points of contact is called **tac-locus** (fig. 155).

Envelope : It is a family of curve tangent common to some other family of curve.

(ii) Determination of singular solutions : Since singular solutions are envelopes of general solutions and envelopes with tac, nodal and cuspidal loci can be computed from both p - and c -discriminants, so singular solutions can be determined from those discriminants. It is established by geometrical drawings that c -discriminant $= EN^2 C^3$

$$= (\text{Envelope}) (\text{Nodal locus})^2 (\text{Cuspidal locus})^3$$

$$\text{and } p\text{-discriminant} = ET^2 C \\ = (\text{Envelope}) (\text{Tac-locus})^2 (\text{Cuspidal locus})$$

where power indicates the number of loci.

$\therefore$ the singular solution or the envelope of the general solution of a first order higher degree ODE is obtained by equating the common linear factor of c and p -discriminants to zero. The same factor which is cubic in c -discriminant and linear in p -discriminant, equated to zero gives the cuspidal locus. The remaining factor which is quadratic in c -discriminant and p -discriminant respectively equated to zero gives the nodal and the tac loci.

Examples

1. Solve the ODE $p^3 (x+2y) + 3p^2 (x+y) + (y+2x)p = 0$.

Note that the given equation can be factorized into linear factors in p i.e.

$$p(p+1)[p(x+2y) + (y+2x)] = 0$$

which breaks up into three 1st order and 1st degree ODEs as

$$p = 0, p+1 = 0, p(x+2y) + (y+2x) = 0$$

or, $\frac{dy}{dx} = 0, \frac{dy}{dx} = -1, (x+2y)\frac{dy}{dx} + (y+2x) = 0$

The solutions of these equations are $y = c_1$, $y = -x + c_2$ and $xy + x^2 + y^2 = c_3$ respectively. Since the given equation is of 1st order, so let $c_1 = c_2 = c_3 = -c$, a single arbitrary constant. Therefore, the required general solution is $(y+c)(y+x+c)(xy+x^2+y^2+c) = 0$.

2. Solve the ODE $y + px = x^4 p^2$.

The given equation is easy to solve for y as

$$y = -px + x^4 p^2 \quad (1)$$

Differentiating (1) w.r.t. 'x' and using $\frac{dy}{dx} = p$, we get

$$2p + x \frac{dp}{dx} = 2px^3 \left(2p + x \frac{dp}{dx} \right)$$

or, $\left(2p + x \frac{dp}{dx} \right) (1 - 2px^3) = 0 \quad (2)$

$\therefore$ the reduced ODE in p is $2p + x \frac{dp}{dx} = 0 \quad (3)$

The general solution of (3) is $px^2 = c \quad (4)$

Eliminating p between (1) and (4), the required general solution is

$$xy = c^2 x - c \quad (5)$$

From eqn. (2), we have another algebraic relation, $1 - 2px^3 = 0 \quad (6)$

Now elimination of p between (1) and (6) gives a solution $4yx^2 = -1$ called singular solution of (1).

N.B.: Verify that the envelope of (5) is the above singular solution.

3. Solve the ODE

$$p^3 - 4xyp + 8y^2 = 0$$

This equation can easily be solved for x as

$$x = \frac{p^2}{4y} + \frac{2y}{p} \quad (1)$$

Differentiating (1) w.r.t. y and using $\frac{dy}{dx} = p$

$$\frac{1}{p} = -\frac{p^2}{4y^2} + \frac{p}{2y} \frac{dp}{dy} + \frac{2}{p} - \frac{2y}{p^2} \frac{dp}{dy}$$

$\therefore$ the reduced ODE in p is $\frac{dp}{dy} = \frac{p}{2y}$. Its general solution is

$$p = c\sqrt{y} \quad (2)$$

Eliminating p between (1) and (2), we get the general solution of the given equation as $c^3 + 8\sqrt{y} - 4cx = 0$.

4. Solve the ODE $y = (2+p)x + p^2$ (1)

The given equation (1) is a Lagrange's equation. Therefore, differentiating (1) w.r.t. x and using $\frac{dy}{dx} = p$, we get, the reduced equation in p as

$$\frac{dp}{dx} = -\frac{2}{x+2p} \quad (2)$$

The general solution of (2) is $x = 2(2-p) + ce^{-\frac{1}{2}p}$ (3)

$\therefore$ the required solution of (1) is the parametric representation

$$\left. \begin{aligned} y &= (2+p)x + p^2 \\ x &= 2(2-p) + ce^{-\frac{1}{2}p} \end{aligned} \right\} p \text{ is a parameter.}$$

5. Prove that the ODE $p^2y + p(x-y) - x = 0$ has no singular solution.

The p -discriminant of the given equation is $(x-y)^2 + 4xy = 0$ or $(x+y)^2 = 0$ which gives the tac-locus of the curves represented by the ODE but not an envelope i.e. not a singular solution. But the equation can be factorized into linear factors in p as $(p-1)(py+x) = 0$ and its general solution is $(y-x+c)(x^2+y^2+c) = 0$.

$\therefore$ this problem shows that equation expressible as linear factors in p does not have singular solution.

6. Solve the ODE $4p^2 x(x-a)(x-b) = \{3x^2 - 2x(a+b) + ab\}^2$. Find its singular solution and other possible loci.

The given equation is solvable for p as

$$p = \pm \frac{3x^2 - 2x(a+b) + ab}{2\sqrt{x(x-a)(x-b)}} \quad (1)$$

Separating the variables, the general solution is

$$y + c = \pm \sqrt{x(x-a)(x-b)}$$

or, $(y+c)^2 = x(x-a)(x-b) \quad (2)$

Again the p - and c -discriminants are respectively

$$[3x^2 - 2x(a+b) + ab]^2 = 0 \quad (3)$$

and $x(x-a)(x-b) = 0 \quad (4)$

There is no singular solution of the ODE as (3) and (4) have no common linear factor. The quadratic factor in (3) gives tac-locus

$$3x = a + b \pm \sqrt{a^2 - ab + b^2}.$$

Cases when (i) $a=b$ and (ii) $a=b=0$

- (i) $a=b$, the given equation simplifies to $4xp^2 = (3x-a)^2$ and its general solution becomes $(y+c)^2 = x(x-a)^2$. Therefore, the p - and c -discriminants are $(3x-a)^2 = 0$ and $x(x-a)^2 = 0$ respectively.

$\therefore$ the equation has no singular solution. But quadratic factor p - and c -discriminant respectively give the tac-locus $x = \frac{1}{3}a$ and nodal locus $x = a$.

- (ii) $a=b=0$, the given equation simplifies to $4p^3 = 9x$. Its general solution is $(y+c)^2 = x^3$. The p - and c -discriminants are respectively $x=0$ and $x^3=0$. This equation also does not have any singular solution but has cuspidal locus $x=0$ as the factor x occurs 3 times in c -discriminant and linearly in p -discriminant.

7. Solve the ODE $y = px + 2p^2$.

The given equation is a Clairaut's equation. Differentiating it w.r.t. x and using $\frac{dy}{dx} = p$, we get,

$$(x+4p) \frac{dp}{dx} = 0$$

As the product is zero, there arise two cases —

Case (i) : $\frac{dp}{dx} = 0$ is the reduced ODE in p . Its general solution is,

$p = c$. Eliminating p between the given equation and this solution, the general solution of Clairaut's equation is $y = cx + 2c^2$.

Case (ii): $x + 4p = 0$, an algebraic equation. Elimination of p between the given equation and this equation gives the singular solution of Clairaut's equation as $y = -\frac{1}{8}x^2$.

8. Solve the ODE $x^2(y - px) = p^2 y$.

Let $x^2 = u$ and $y^2 = v$. Therefore, $2x dx = du$ and $2y dy = dv$. Hence, $p = \frac{x}{y} \frac{dy}{du}$. So, the given equation becomes,

$$v = u \frac{dv}{du} + \left(\frac{dv}{du} \right)^2$$

$$\left[\because \text{the given equation can be written as } x^2 \left(y^2 - p \frac{y}{x} y^2 \right) = p^2 y^2 \right]$$

or, $v = uP + P^2$ when $P = \frac{dv}{du}$

which is of type Clairaut's. Therefore its general sol. is $v = cu + c^2$

or, $y^2 = cx^2 + c^2$ which is the general solution of the given equation.

Now the c - and p - discriminants are $x^4 + 4y^2 = 0$ and $x^2(x^4 + 4y^2) = 0$. Therefore the singular solution is $x^4 + 4y^2 = 0$ and the tac-locus is

$$x^2 = 0 \text{ i.e. } x = 0.$$

9. Find the equation of curve with the property that light rays emanating from the origin will all be reflected into rays which are parallel to the x -axis.

Let the equation of the curve be $y = f(x)$. Let i = incident angle and r = reflected angle (fig. 156). Let y' = slope of the tangent at $P(x, y)$ to the curve $y = f(x)$, $m_1 \left(= -\frac{1}{y'} \right)$ = slope of the normal at $P(x, y)$

and $m_2 \left(= \frac{y}{x} \right)$ = slope of OP .

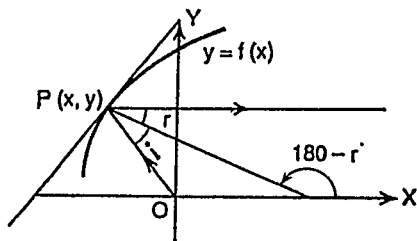


Fig. 156

From Physics, $\tan i = \tan r$ and from co-ordinate geometry,

$$\tan i = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{-\frac{1}{y'} - \frac{y}{x}}{1 - \frac{1}{y'} \cdot \frac{y}{x}} = \frac{-x - y y'}{x y' - y}$$

Fig. (156), $\tan (180^\circ - r) = -\frac{1}{y'} \therefore \tan r = \frac{1}{y'}$ i.e. $\tan i = \frac{1}{y'}$

$$\therefore \frac{1}{y'} = \frac{-x - y y'}{x y' - y} \text{ or } y = 2x y' + y (y')^2.$$

This is a first order second degree non-linear equation. Let $p = y'$. So, the equation becomes

$$y = 2xp + yp^2$$

or,

$$p = \frac{-x \pm \sqrt{x^2 + y^2}}{y}$$

i.e.

$$\frac{dy}{dx} = \frac{-x \pm \sqrt{x^2 + y^2}}{y} \text{ which is homogeneous equation.}$$

$\therefore$ its general solution is $\pm \sqrt{x^2 + y^2} = x + c$

$\therefore$ the required curve is $y^2 = 2cx + c^2$ which is a parabola with axis as x -axis, vertex at $\left(-\frac{c}{2}, 0\right)$ and focus at $(0, 0)$.

10. A fighter finds its enemy bomber 'a' miles from it at the same height and moving at a rate of k miles per hour. The fighter pursues the bomber at a rate of nk m/h. Find the path of pursuit.

Let the position of fighter and bomber be $F(x, y)$ and $B(a, kt)$ after t hours of pursuance (fig. 157). Therefore the direction and the velocity of the fighter are

$$y' = \frac{kt - y}{a - x} \quad (1)$$

$$\text{and } \frac{ds}{dt} = nk \quad (2)$$

To know the path of pursuit, t is not necessary. So to eliminate t from (1) and (2), differentiating (1) w.r.t. x , we have

$$y''(a - x) - y' = k \frac{dt}{dx} - y'$$

$$y''(a - x) = k \frac{(dt/dx)}{ds} \cdot \frac{ds}{dx}$$

or,

$$= k \cdot \frac{1}{nk} \cdot \sqrt{1 + (y')^2}$$

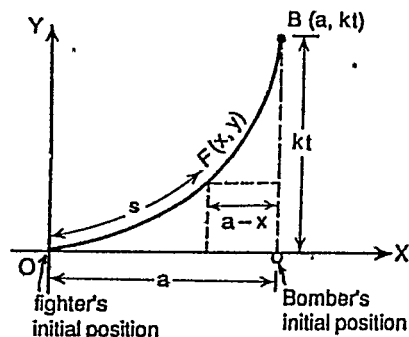


Fig. 157

or, $n y'' (a-x) = \sqrt{1+(y')^2}$. This is the equation of motion of the fighter.
Let $y' = p$. Therefore,

$$n(a-x) \frac{dp}{dx} = \sqrt{1+p^2}.$$

Its general solution is, $n \log(p + \sqrt{1+p^2}) = \log \frac{c}{a-x}$.

I.C. : $p = y' = 0$ when $x = 0$. So, $c = a$. Therefore, the required solution is $n \log(p + \sqrt{1+p^2}) = \log \frac{a}{a-x}$

$$\text{or, } 2p \left(\frac{a}{a-x} \right)^{\frac{1}{n}} = \left(\frac{a}{a-x} \right)^{\frac{2}{n}} - 1$$

$$\text{or, } 2 \int dy = \int \left[\left(\frac{a}{a-x} \right)^{\frac{1}{n}} - \left(\frac{a}{a-x} \right)^{-\frac{1}{n}} \right] dx + c$$

$$\text{or, } 2y = -\frac{a^{1/n} (a-x)^{(1-1/n)}}{(1-1/n)} + \frac{a^{-1/n} (a-x)^{(1+1/n)}}{(1+1/n)} + c, n > 1$$

Again, $y = 0$ when $x = 0$. Therefore $c = \frac{-2na}{1-n^2}$

$\therefore$ the required pursuit curve is

$$2y = -\frac{a^{1/n} (a-x)^{(1-1/n)}}{(1-1/n)} + \frac{a^{-1/n} (a-x)^{(1+1/n)}}{(1+1/n)} - \frac{2na}{1-n^2}.$$

Exercises 20

- $p^2 + p - 6 = 0$.
- $p^2 y - p(xy + 1) + x = 0$.
- $x^2 p^2 - 3x^2 y^3 p + 2p - 6y^3 = 0$.
- $p^3 - (x^2 + xy^2 + y^4) p^2 + (x^3 y^2 + x^2 y^4 + xy^6) p - x^3 y^6 = 0$.
- $p(p^2 + xy) = p^2(x + y)$.
- $(a^2 - x^2) p^3 + bx(a^2 - x^2) p^2 - p - bx = 0$.
- $3p^2 + 2p - y = 0$.
- $x^2 \left(\frac{dy}{dx} \right)^2 + 3xy \left(\frac{dy}{dx} \right) + 2y^2 = 0$.
- $y - px = x \sqrt{x^2 + p^2}$.
- $\left(p^2 - \frac{1}{a^2 - x^2} \right) \left(p - \sqrt{\frac{y}{x}} \right) = 0$.
- $2yp^2 - (4x - 5)p + 2y = 0$.
- $x^2 \left(\frac{dy}{dx} \right)^2 + xy \frac{dy}{dx} - 6y^2 = 0$.
- $2px - y + 2 \log p - \log 4 = 0$.

$$14. 2p^3 - (2x + 4 \sin x - \cos x) p^2 - (x \cos x - 4x \sin 2x + \sin 2x) p + x \sin 2x = 0.$$

$$15. p^3 x - p^2 y - 1 = 0. \quad 16. x^2 \left(\frac{dy}{dx} \right)^2 - 2xy \left(\frac{dy}{dx} \right) + 2y^2 - x^2 = 0.$$

$$17. (x-a)p^2 + (x-y)p - y = 0.$$

$$18. (1-y^2+x^{-2}y^4)p^2 - 2x^{-1}yp + x^{-2}y^2 = 0. \quad 19. 16y^2p^3 + 2px - y = 0.$$

$$20. y^2 \log y = xyp + p^2 \quad 21. p^2 - 2xp + 1 = 0. \quad 22. y = 2px + y^2 p^3.$$

$$23. 4p^3 + 3px = y. \quad 24. x + yp = ap^2. \quad 25. x + \frac{p}{\sqrt{1+p^2}} = 1.$$

$$26. y = px + \sin^{-1} p. \quad 27. p = \tan \left(x - \frac{p}{1+p^2} \right)$$

$$28. e^{(y-px)} = p^2. \quad 29. y = px + \frac{1}{p}. \quad 30. y = xy' + (y')^2.$$

$$31. x - yp = ap^2. \quad 32. y = (1+p)x + ap^2.$$

$$33. p^3 - p(y+3) + x = 0. \quad 34. \sin px \cos y = \cos px \sin y + p.$$

$$35. \text{Use } u = x^2, v = y^2 \text{ and solve the ODE } (px - y)(py + x) = h^2 p.$$

$$36. \text{Substituting } u = e^x, v = e^y \text{ solve the ODE } e^{3x}(p-1) + p^3 e^{2y} = 0.$$

37. A boat A moves along the y -axis with constant speed ' a '. Find the path of a second boat B which moves in the left half of the xy -plane with constant speed ' b ' and always points directly at A.

[Hints: W.O. 10 for ref. Take A $(0, at)$, B (x, y) at time t . Slope of AB is $\frac{y-at}{x} = \frac{dy}{dx}$ and velocity of B is $\frac{ds}{dt} = b$, Eliminate t . Solve the resulting

$$\text{equation } xy'' = -\frac{a}{b} \sqrt{1+y'^2}.$$

38. Find the path of a small boat in a wide river with uniform current if the boat has constant speed relative to the water and always heads towards a fixed point on the bank.

39. In calculus, the radius of curvature of a curve is defined to be $R \doteq [1 + (y')^2]^{3/2} / y''$. Solve this ODE when R is constant and show that the family of solution curves is the set of all circles in the xy -plane.

40. Find general solution, singular solution and additional loci, if any, of the following :

$$(i) \quad 4x(x-1)(x-2)p^2 - (3x^2 - 6x + 2)^2 = 0,$$

$$(ii) \quad 4xp^2 - (3x-1)^2 = 0.$$

Answers

1. $(3x+y+c)(2x-y+c)=0$.

2. $(x^2-2y+c)(2x+y^2+c)=0$.

3. $(y-(2/x)+c)(3x+(1/2y^2)+c)=0$.

4. $(x^3-3y+c)\left(\frac{x^2}{2}+\frac{1}{y}+c\right)\left(x+\frac{1}{3y^3}+c\right)=0$.

5. $(y+c)(x^2-2y+c)(x-\log y+c)=0$.

6. $(2y+bx^2+c)\left(y-\sin^{-1}\frac{x}{a}+c\right)\left(y-\cos^{-1}\frac{x}{a}+c\right)=0$.

7. $x=6p+2\log p+c$, $y=3p^2+2p$.

8. $(xy+c)(x^2y+c)=0$.

9. $n^2x^2+y^2+2cx=0$.

10. $\left(y-\sin^{-1}\frac{x}{a}+c\right)\left(y-\cos^{-1}\frac{x}{a}+c\right)(\sqrt{y}-\sqrt{x}+c)=0$.

11. $y(p^2+p+1)=0$, $2yp^2-(4x-5)p+2y=0$.

12. $(x^3y+c)(cx^2+y)=0$.

13. $xp^2+2p=c$, $y=2px+2\log p-\log 4$.

14. $(2y-x^2+c)(y+\sin x+c)(y+2\cos x+c)=0$.

15. $y=cx-\frac{1}{c^2}$, $4y^3+27x^2=0 \rightarrow S.S.$

16. $\left(\sin^{-1}\frac{y}{x}-\log cx\right)\left(\sin^{-1}\frac{y}{x}+\log cx\right)=0$.

17. $(x-a)c^2+(x-y)c-y=0$, $(x+y)^2=4ay$.

18. $\left(\log \frac{x+\sqrt{x^2-y^2}}{y}+y+c\right)\left(\log \frac{x+\sqrt{x^2-y^2}}{y}-y+c\right)=0$.

19. $16c^3+2cx-y^2=0$.

20. $\log y=kx+x^2$.

21. $x=\frac{1}{2}(p+p^{-1})$, $y=\frac{1}{4}p^2-\frac{1}{2}\log p+c$.

22. $y=2cx+c^3$.

23. $y=-(8p^3/7)+cp^{-1/2}$, $x=-(12p^2/7)+\frac{c}{3}p^{-3/2}$.

24. $x+yp=ap^2$, $x\sqrt{1+p^2}=p[c+a\log\{p+\sqrt{1+p^2}\}]$.

25. $(y+c)^2+(x-a)^2=1$.

26. $y=cx+\sin^{-1}c$, $y=\pm\sqrt{x^2-1}+\sin^{-1}\left(\pm\frac{\sqrt{x^2-1}}{x}\right)$.

27. $y=c-(1+p^2)^{-1}$, $x=\tan^{-1}p+p(1+p^2)^{-1}$.

28. $e^{(y-cx)}=c^2$, $x^2e^{y+2}=4$.

29. $y=cx+\frac{1}{c}$, $y^2=4x$.

30. $y=cx+c^2$, $x^2+4y=0$.

31. $x-yp=ap^2$, $y\sqrt{1-p^2}=a[\sin^{-1}p-p\sqrt{1-p^2}]+c$.

32. $x=2a(1-p)+ce^{-p}$, $y=(1+p)x+ap^2$.

33. $y\sqrt{1-p^2}+(1-p^2)^{3/2}=c$, $p^3-p(y+3)+x=0$.

34. $y=cx-\sin^{-1}c$, $y=\pm\sqrt{x^2-1}+\sin^{-1}\left(\pm\frac{\sqrt{x^2-1}}{x}\right)$.

35. $y^2=cx^2-c(c+1)^{-1}h^2$.

36. $e^y=ce^x+c^3$.

37. $\frac{a}{2b}\left[\frac{c^{a/b}}{x^{(a/b)+1}}-\frac{x^{(a/b)-1}}{c^{a/b}}\right]+k$.

38. $y^{v+y_0}=c(x+\sqrt{x^2+y^2})^v$.

40. (i) $y+c=\pm\sqrt{(x-1)(x-2)}$, tac-locus $3x=3\pm\sqrt{3}$,

(ii) $y=cx=(x-1)^2$, tac-locus $3x=1$; nodal locus, $x=1$.

HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS

21.1 DEFINITION AND THEOREMS RELATING TO SOLUTIONS

The most general n th order linear ODE is

$$a_0(x) \frac{d^n y}{dx^n} + a_1(x) \frac{d^{n-1} y}{dx^{n-1}} + a_2(x) \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_{n-1}(x) \frac{dy}{dx} + a_n(x) y = Q(x), \quad a_0(x) \neq 0 \quad (1)$$

where $a_0(x), a_1(x), a_2(x), \dots, a_n(x)$ and $Q(x)$ are all single-valued continuous functions of x in the interval $a < x < b$. The ODE (1) is by definition **non-homogeneous**, because of the presence of $Q(x)$ on the right. If $Q(x) = 0$ identically, then the equation (1) becomes

$$a_0(x) \frac{d^n y}{dx^n} + a_1(x) \frac{d^{n-1} y}{dx^{n-1}} + a_2(x) \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_{n-1}(x) \frac{dy}{dx} + a_n(x) y = 0 \quad (2)$$

and ODE (2) is called, by definition, **homogeneous**. This will also be called **related** or **associated homogeneous** equation of (1) in sec. 21.4 while general solution of (1) will be sought. This homogeneity has no identity with that defined in 19.4.

The solutions of equation (1) and (2) are based on the following **theorems 1, 2, 3** and **1, 2** respectively. Before going into the theorems let us recall the definitions of linear dependence and independence of functions from Algebra.

Definition. Linear dependence : The n functions $f_1, f_2, \dots, f_n$ are said to be linearly dependent if there exists n constants $c_1, c_2, \dots, c_n$ not all zero but $c_1 f_1 + c_2 f_2 + \dots + c_n f_n = 0$.

Linear independence : The n functions $f_1, f_2, \dots, f_n$ are said to be linearly independent if there exists n constants $c_1, c_2, \dots, c_n$ all equal to zero making $c_1 f_1 + c_2 f_2 + \dots + c_n f_n = 0$.

Necessary condition for linear dependence : If the functions $f_1, f_2, \dots, f_n$ are linearly dependent over an interval I and each function

possesses derivatives upto $(n-1)$ th order then the Wronskian $W(f_1, f_2, \dots, f_n)$ vanishes at all points of I .

The Wronskian $W(f_1, f_2, \dots, f_n)$ is a determinant of the functions $f_1, f_2, \dots, f_n$ and their $(n-1)$ th derivatives as

$$\begin{vmatrix} f_1 & f_2 & f_3 & \dots & f_n \\ f_1' & f_2' & f_3' & \dots & f_n' \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & f_3^{(n-1)} & \dots & f_n^{(n-1)} \end{vmatrix}$$

named after Polish mathematician H. Wronsky (1778-1853).

Proof (of above condition) : By definition, there exists n constants $c_1, c_2, \dots, c_n$ not all zero but

$$c_1 f_1 + c_2 f_2 + \dots + c_n f_n = 0 \quad (i)$$

Also, by hypothesis each of the functions can be differentiated successively upto $(n-1)$ times. Therefore,

$$c_1 f_1' + c_2 f_2' + \dots + c_n f_n' = 0 \quad (ii)$$

$$c_1 f_1'' + c_2 f_2'' + \dots + c_n f_n'' = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$c_1 f_1^{(n-1)} + c_2 f_2^{(n-1)} + \dots + c_n f_n^{(n-1)} = 0 \quad (n)$$

Now, eliminating n constants $c_1, c_2, \dots, c_n$ from n equations (i), (ii); ... (n), we have, the required

$$W(f_1, f_2, \dots, f_n) = 0.$$

Cor. : Non-vanishing of Wronskian $W(f_1, f_2, \dots, f_n)$ assures linear independence of the n functions.

Theorem 1 : If the homogeneous equation 2 possesses n linearly independent solutions $y_1, y_2, \dots, y_n$ then, for n constants, $y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$ is the general solution of (2).

More specifically, the linear combination of the n linearly independent solutions is the general solution of equation (1).

Proof : Since

$$\frac{d^k}{dx^k} (c_1 y_1 + c_2 y_2 + \dots + c_n y_n) = c_1 \frac{d^k y_1}{dx^k} + c_2 \frac{d^k y_2}{dx^k} + \dots + c_n \frac{d^k y_n}{dx^k},$$

$$k = 1, 2, \dots, (n-1), n.$$

Therefore, L.H.S. of equation (2)

$$\begin{aligned}
&= c_1 \left[a_0(x) \frac{d^n y_1}{dx^n} + a_1(x) \frac{d^{n-1} y_1}{dx^{n-1}} + a_2(x) \frac{d^{n-2} y_1}{dx^{n-2}} + \dots \right. \\
&\quad \left. + a_{n-1}(x) \frac{dy_1}{dx} + a_n(x) y_1 \right] \\
&\quad + c_2 \left[a_0(x) \frac{d^n y_2}{dx^n} + a_1(x) \frac{d^{n-1} y_2}{dx^{n-1}} + \dots + a_{n-1}(x) \frac{dy_2}{dx} + a_n(x) y_2 \right] \\
&\quad \dots \dots \dots \\
&\quad + c_n \left[a_0(x) \frac{d^n y_n}{dx^n} + a_1(x) \frac{d^{n-1} y_n}{dx^{n-1}} + \dots + a_{n-1}(x) \frac{dy_n}{dx} + a_n(x) y_n \right] \\
&= c_1 [0] + c_2 [0] + \dots + c_n [0] = 0
\end{aligned}$$

$\therefore$ each of $y_1, y_2, \dots, y_n$ satisfies equation (2).

$\therefore c_1 y_1 + c_2 y_2 + \dots + c_n y_n$ satisfies equation (2). So the general solution of (2) is

$$y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n \quad (3)$$

containing n constants equal to the order n of the equation (2).

N.B. : The number of arbitrary constants to be present in the solution is supported by theorem 2 next.

Theorem 2 : If the homogeneous equation (2) possesses n linearly independent solutions $y_1, y_2, \dots, y_n$ then there exists n constants $c_1, c_2, \dots, c_n$ such that $y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$ is a solution of (2).

Proof : The theorem is established by induction procedure, using Abel's identity. The proof starts with a second order ODE

$$a_0(x) \frac{d^2 y}{dx^2} + a_1(x) \frac{dy}{dx} + a_2(x) y = 0 \quad (4)$$

If equation (4) has two linearly independent solutions y_1 and y_2 then it is to be proved that for two constants c_1 and c_2 there is a third solution $y_3 = c_1 y_1 + c_2 y_2$.

$\therefore a_0(x) \neq 0$, therefore from (4)

$$\frac{d^2 y}{dx^2} + P_1(x) \frac{dy}{dx} + P_2(x) y = 0 \quad (5)$$

where $P_1(x) = \frac{a_1(x)}{a_0(x)}$ and $P_2(x) = \frac{a_2(x)}{a_0(x)}$. Since y_1 and y_2 are two solutions of (4), so they are solutions of (5) too.

$$\therefore \left. \begin{aligned} \frac{d^2 y_1}{dx^2} + P_1(x) \frac{dy_1}{dx} + P_2(x) y_1 &= 0 \\ \text{and } \frac{d^2 y_2}{dx^2} + P_1(x) \frac{dy_2}{dx} + P_2(x) y_2 &= 0 \end{aligned} \right\} \quad (6)$$

Now eliminating $P_2(x)$ from (6), we get

$$\left(y_2 \frac{d^2 y_1}{dx^2} - y_1 \frac{d^2 y_2}{dx^2} \right) + \left(y_2 \frac{dy_1}{dx} - y_1 \frac{dy_2}{dx} \right) P_1(x) = 0$$

$$\text{or, } \frac{d}{dx} \left[y_2 \frac{dy_1}{dx} - y_1 \frac{dy_2}{dx} \right] + \left[y_2 \frac{dy_1}{dx} - y_1 \frac{dy_2}{dx} \right] P_1(x) = 0$$

$$\text{or, } \frac{d}{dx} W + W P_1(x) = 0 \text{ where } W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \text{ prime indicates derivative.} \\ = W(y_1, y_2)$$

$$\text{or, } W(y_1, y_2) = c_{12} e^{-\int P_1(x) dx}$$

The subscripts of integrating constant c_{12} shows correspondence with subscripts of y_1 and y_2 .

$$\therefore W(y_1, y_2) = c_{12} e^{-\int \frac{a_1(x)}{a_0(x)} dx} \quad (7)$$

The equation (7) is known as **Abel's identity** after the name of the Norwegian mathematician Niels Abel (1802–1829).

Let equation (5) possesses a third solution y_3 . Therefore, using Abel's identity for the pair (y_3, y_1) and (y_3, y_2) we have

$$W(y_3, y_1) = c_{31} e^{-\int P_1(x) dx} \text{ and } W(y_3, y_2) = c_{32} e^{-\int P_1(x) dx}$$

$$\text{i.e. } y_3 y_1' - y_1 y_3' = c_{31} e^{-\int P_1(x) dx}$$

$$\text{and } y_3 y_2' - y_2 y_3' = c_{32} e^{-\int P_1(x) dx}$$

$$\therefore y_3 = \frac{(y_1 c_{32} - y_2 c_{31}) e^{-\int P_1(x) dx}}{y_1 y_2' - y_2 y_1'} = \frac{(y_1 c_{32} - y_2 c_{31}) e^{-\int P_1(x) dx}}{c_{12} e^{-\int P_1(x) dx}}, \text{ by (7)} \\ = \frac{c_{32}}{c_{12}} y_1 - \frac{c_{31}}{c_{12}} y_2 \quad (8)$$

$\therefore y_3$ is a linear combination of the two linearly independent solutions y_1 and y_2 i.e. y_3 is linearly dependent upon y_1 and y_2 by definition of linear dependence.

$$\therefore \left. \begin{aligned} \frac{d^2 y_1}{dx^2} + P_1(x) \frac{dy_1}{dx} + P_2(x) y_1 &= 0 \\ \text{and } \frac{d^2 y_2}{dx^2} + P_1(x) \frac{dy_2}{dx} + P_2(x) y_2 &= 0 \end{aligned} \right\} \quad (6)$$

Now eliminating $P_2(x)$ from (6), we get

$$\left(y_2 \frac{d^2 y_1}{dx^2} - y_1 \frac{d^2 y_2}{dx^2} \right) + \left(y_2 \frac{dy_1}{dx} - y_1 \frac{dy_2}{dx} \right) P_1(x) = 0$$

$$\text{or, } \frac{d}{dx} \left[y_2 \frac{dy_1}{dx} - y_1 \frac{dy_2}{dx} \right] + \left[y_2 \frac{dy_1}{dx} - y_1 \frac{dy_2}{dx} \right] P_1(x) = 0$$

$$\text{or, } \frac{d}{dx} W + W P_1(x) = 0 \text{ where } W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \text{ prime indicates derivative.}$$

$$= W(y_1, y_2)$$

$$\text{or, } W(y_1, y_2) = c_{12} e^{-\int P_1(x) dx}$$

The subscripts of integrating constant c_{12} shows correspondence with subscripts of y_1 and y_2 .

$$\therefore W(y_1, y_2) = c_{12} e^{-\int \frac{a_1(x)}{a_0(x)} dx} \quad (7)$$

The equation (7) is known as **Abel's identity** after the name of the Norwegian mathematician Niels Abel (1802–1829).

Let equation (5) possesses a third solution y_3 . Therefore, using Abel's identity for the pair (y_3, y_1) and (y_3, y_2) we have

$$W(y_3, y_1) = c_{31} e^{-\int P_1(x) dx} \text{ and } W(y_3, y_2) = c_{32} e^{-\int P_1(x) dx}$$

$$\text{i.e. } y_3 y_1' - y_1 y_3' = c_{31} e^{-\int P_1(x) dx}$$

$$\text{and } y_3 y_2' - y_2 y_3' = c_{32} e^{-\int P_1(x) dx}$$

$$\therefore y_3 = \frac{(y_1 c_{32} - y_2 c_{31}) e^{-\int P_1(x) dx}}{y_1 y_2' - y_2 y_1'} = \frac{(y_1 c_{32} - y_2 c_{31}) e^{-\int P_1(x) dx}}{c_{12} e^{-\int P_1(x) dx}} \quad \text{by (7)}$$

$$= \frac{c_{32}}{c_{12}} y_1 - \frac{c_{31}}{c_{12}} y_2 = c_1 y_1 + c_2 y_2 \quad (8)$$

$\therefore y_3$ is a linear combination of the two linearly independent solutions y_1 and y_2 i.e. y_3 is linearly dependent on y_1 and y_2 by definition of linear dependence.

Proceeding in this way, it can be shown that if y_1, y_2, y_3 are three solutions of $a_0(x) \frac{d^3 y}{dx^3} + a_1(x) \frac{d^2 y}{dx^2} + a_2(x) \frac{dy}{dx} + a_3(x) y = 0$ then there exists a fourth solution $y_4 = c_1 y_1 + c_2 y_2 + c_3 y_3$ which is a linear combination of the independent solutions y_1, y_2 and y_3 . Continuation of this procedure leads to $y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$ as a solution of (2) when $y_1, y_2, \dots, y_n$ are n linearly independent solutions of (2).

Theorem 3 : If the non-homogeneous equation (1) has a particular solution y_p and the homogeneous equation (2) has a general solution y_c , then the general solution of (1) is the sum of these two solutions i.e. $y = y_c + y_p$.

Proof : Since y_p is a solution of (1), therefore,

$$a_0(x) \frac{d^n y_p}{dx^n} + a_1(x) \frac{d^{n-1} y_p}{dx^{n-1}} + \dots + a_{n-1}(x) \frac{dy_p}{dx} + a_n(x) y_p = Q(x) \quad (9)$$

Now subtracting (9) from (1), we get,

$$a_0(x) \left[\frac{d^n y}{dx^n} - \frac{d^n y_p}{dx^n} \right] + a_1(x) \left[\frac{d^{n-1} y}{dx^{n-1}} - \frac{d^{n-1} y_p}{dx^{n-1}} \right] + \dots + a_n(x) [y - y_p] = 0$$

$$\text{or, } a_0(x) \frac{d^n}{dx^n} (y - y_p) + a_1(x) \frac{d^{n-1}}{dx^{n-1}} (y - y_p) + \dots + a_n(x) (y - y_p) = 0 \quad (10)$$

The equation (10) shows that $(y - y_p)$ is a solution of the homogeneous equation (2). But the latter has a solution y_c , by hypothesis. Therefore,

$$y_c = y - y_p \text{ or } y = y_c + y_p \quad (11)$$

$\therefore$ the theorem shows that the general solution of the non-homogeneous equation (1) consists of two parts – (i) y_c , called **complementary solution** or function, obtained from the associated homogeneous equation (2) with requisite number of arbitrary constants and (ii) y_p , called **particular integral**, obtained from the non-homogeneous equation (1) directly with no arbitrary constants.

21.2 FIRST DEGREE BUT HIGHER ORDER HOMOGENEOUS LINEAR ODE WITH CONSTANT COEFFICIENTS

The most general first degree but higher order homogeneous linear ODE with constant coefficients is

$$A_0 \frac{d^n y}{dx^n} + A_1 \frac{d^{n-1} y}{dx^{n-1}} + A_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + A_{n-1} \frac{dy}{dx} + A_n y = 0, A_0 \neq 0 \quad (1)$$

where $A_0, A_1, \dots, A_n$ are all constant. This type of ODE was known to Euler and Daniel Bernoulli in about 1739 and Euler in 1743 published his paper on it. The differential operator $\frac{d}{dx} \equiv D$, and for n th order $\frac{d^n}{dx^n} \equiv D^n$ can be used and with this symbol the equation (1) takes the form

$$(A_0 D^n + A_1 D^{n-1} + A_2 D^{n-2} + \dots + A_{n-1} D + A_n) y = 0 \quad (2)$$

Now the bracketed expression resembles an algebraic polynomial of degree n in the variable D . So, functional notation may be used for compact representation i.e. let

$$F(D) \equiv A_0 D^n + A_1 D^{n-1} + A_2 D^{n-2} + \dots + A_{n-1} D + A_n \quad (3)$$

so that equation (2) is equivalent to $F(D) y = 0$ (4)

Let $F(D)$ can be factorized into n linear factors as

$F(D) \equiv A_0 (D - b_1) (D - b_2) \dots (D - b_n)$. Therefore, equation (4) becomes $(D - b_1) (D - b_2) \dots (D - b_n) y = 0$. (5)

Since $A_0 \neq 0$. The solution of (5) are solution of (1) or (2). The equation (5) is equivalent to n linear first order first degree equations of the form $(D - b_n) y = 0$ whose general solution is $y = ce^{b_n x}$. This solution suggests $y = e^{mx}$ as a form of solution of the equation (1). So, n such linear independent solutions of (1) are expected and hence by Theorem 1, the general solution of (1) will be the linear combination of those independent solutions.

Therefore, to solve equation (1) or (2) $y = e^{mx}$ is substituted and the ODE (1) takes the form of an algebraic equation

$$(A_0 m^n + A_1 m^{n-1} + \dots + A_{n-1} m + A_n) e^{mx} = 0$$

or, $A_0 m^n + A_1 m^{n-1} + \dots + A_{n-1} m + A_n = 0$, $\because e^{mx} \neq 0$ (6)

This equation (6) is called the **auxiliary** or **characteristic** equation of equation (1). Each of the roots of (6) will give rise one solution of (1) and their linear combination will be the general solution of (1).

In the sense of Linear Algebra, each independent solution is called a **basis** of the **solution space** generated by them and the **dimension** of the solution space equals the order n of the ODE.

21.3 SECOND ORDER LINEAR HOMOGENEOUS ODE WITH CONSTANT COEFFICIENTS

This section deals with a particular case of 21.2. Let $n = 2$. So, equation (1) of 21.2 becomes

$$A_0 \frac{d^2 y}{dx^2} + A_1 \frac{dy}{dx} + A_2 y = 0 \quad (1)$$

Using substitution $y = e^{mx}$, the auxiliary equation (A.E.) arises as

$$A_0 m^2 + A_1 m + A_2 = 0, \because e^{mx} \neq 0 \quad (2)$$

This is a quadratic equation in m and it has two roots m_1, m_2 . They may be

(i) real distinct, if $A_1^2 > 4 A_0 A_2$

(ii) complex, if $A_1^2 < 4 A_0 A_2$

(iii) real equal, if $A_1^2 = 4 A_0 A_2$

With these three types of roots of A.E., the general solution of (1) simplifies into three forms as follows :

Case (i) Real distinct roots : Let the two real distinct roots of (2) be m_1 and m_2 . Then the two solutions of (1) corresponding to m_1 and m_2 are $y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$. Now the Wronskian,

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1' = (m_2 - m_1) e^{(m_1 + m_2)x} \neq 0,$$

$\therefore m_1 \neq m_2$ and $-\infty < x < \infty$. Hence the two solutions y_1 and y_2 of (1) are linearly independent. Therefore, by Theorem 1, the general solution of (1) is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}.$$

Case (ii) Complex roots : Let the two complex roots of (2) be $m_1 = a + ib$ and $m_2 = a - ib$, a and b real. Then the corresponding solutions are $y_1 = e^{(a+ib)x}$ and $y_2 = e^{(a-ib)x}$. Now the Wronskian,

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1' = -2ib e^{2ax} \neq 0 \text{ for } -\infty < x < \infty.$$

Hence the two solutions of (1) are linearly independent. Therefore, by Theorem 1, the general solution of (1) is

$$y = A e^{(a+ib)x} + B e^{(a-ib)x} = e^{ax} \{c_1 \cos bx + c_2 \sin bx\}$$

where $c_1 = A + B$ and $c_2 = i(A - B)$ and $e^{\pm ib} = \cos \theta \pm i \sin \theta$.

Case (iii) Real equal roots : Let the two roots be equal i.e. $m_1 = m_2$. This will hold true if the A.E. is $m^2 - 2m_1 m + m_1^2 = 0$ and the ODE is $\frac{d^2 y}{dx^2} - 2m_1 \frac{dy}{dx} + m_1^2 y = 0$. The solution of this equation cannot be written in the form (i), because Wronskian vanishes i.e. the corresponding solutions are not linearly independent as $y = c_1 e^{m_1 x} + c_2 e^{m_1 x} = (c_1 + c_2) e^{m_1 x} = c e^{m_1 x}$. Let us use variation of parameter. Let $y = c(x) e^{m_1 x}$ be the solution of $\frac{d^2 y}{dx^2} - 2m_1 \frac{dy}{dx} + m_1^2 y = 0$. On substitution the equation reduces to $\frac{d^2 c}{dx^2} c(x) = 0 \because e^{m_1 x} \neq 0$ or by integrating twice successively,

$$c(x) = c_1 + c_2 x, \therefore y = (c_1 + c_2 x) e^{m_1 x}$$

in which let $y_1 = e^{m_1 x}$ and $y_2 = x e^{m_1 x}$. Therefore, Wronskian, $W(y_1, y_2) = e^{2m_1 x} \neq 0$ for $-\infty < x < \infty$. So, these y_1 and y_2 are independent. Hence by theorem 1, the general solution of (1) when the roots of A.E. are equal, is $y = (c_1 + c_2 x) e^{m_1 x}$.

These results are summarised in the table below :

Cases	Nature of the roots of A.E., $A_0 m^2 + A_1 m + A_2 = 0$	Solution of the 2nd order ODE, $A_0 y'' + A_1 y' + A_2 y = 0$
(i)	Real distinct roots, m_1 and m_2	$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$, c_1 and c_2 arbitrary constants.
(ii)	Complex roots, $m_1 = a + ib, m_2 = a - ib$	$y = A e^{(a+ib)x} + B e^{(a-ib)x}$ $= e^{ax} \{c_1 \cos bx + c_2 \sin bx\}$, $c_1 = A + B, c_2 = i(A - iB)$
(iii)	Real equal roots, m_1, m_2	$y = (c_1 + c_2 x) e^{m_1 x}$

21.4 MORE THAN SECOND ORDER LINEAR HOMOGENEOUS ODE WITH CONSTANT COEFFICIENTS

The general equation is the equation (1) of 21.2. Solution procedures of this equation is the extension of the idea developed in 21.3 and the use of Theorem 1, 2. Let (a) p -roots $m_1, m_2, \dots, m_p$ of A.E. (6) in 21.2 are real distinct, (b) q -pairs of roots $\alpha_1 \pm i\beta_1, \alpha_2 \pm i\beta_2, \dots, \alpha_q \pm i\beta_q$ are complex, (c) r -roots m_1 are equal and (d) s -pairs equal complex roots $a \pm ib$. Therefore, $n = p + 2q + r + 2s$. Now the general solution of (1) of 21.2 consists of four parts - each part corresponds to an expression for each of the four categories of the roots mentioned namely

- (1) $c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_p e^{m_p x}$ corresponding to category (a)
- (2) $e^{\alpha_1 x} (D_1 \cos \beta_1 x + D_2 \sin \beta_1 x) + e^{\alpha_2 x} (D_3 \cos \beta_2 x + D_4 \sin \beta_2 x)$
+ + $e^{\alpha_q x} (D_{2q-1} \cos \beta_q x + D_{2q} \sin \beta_q x)$ corresponding to category (b)
- (3) $(E_1 + E_2 x + E_3 x^2 + \dots + E_r x^{r-1}) e^{m_1 x}$ corresponding to category (c)
- (4) Since imaginary roots occur in conjugate pair, so occurrence of at least two pairs of equal imaginary roots needs a fourth degree A.E. Let two pairs of imaginary roots be

$$m_1 = m_2 = a + ib \text{ and } m_3 = m_4 = a - ib.$$

Therefore, the solution of ODE is

$$y = (C_1 + C_2 x) e^{(a+ib)x} + (C_3 + C_4 x) e^{(a-ib)x}$$

$$\begin{aligned}
 &= e^{ax} \left[(C_1 + C_2 x) e^{ibx} + (C_3 + C_4 x) e^{-ibx} \right] \\
 &= e^{ax} \left[(A_1 + A_2 x) \cos bx + (A_3 + A_4 x) \sin bx \right] \quad (i)
 \end{aligned}$$

where $A_1 = C_1 + C_3$, $A_2 = C_2 + C_4$,

$$A_3 = i(C_1 - C_3), \quad A_4 = i(C_2 - C_4)$$

Proceeding in this way, category (d) can be easily extended to

$$e^{ax} (F_1 + F_2 x + \dots + F_s x^{s-1}) \cos bx + (F_{s+1} + F_{s+2} x + \dots + F_{2s} x^{s-1}) \sin bx.$$

Other equivalent forms of (i) are

$$y = e^{ax} [A_1 \cos bx + A_3 \sin bx] + x e^{ax} [A_2 \cos bx + A_4 \sin bx],$$

$$y = e^{ax} [(A_1 \cos bx + A_3 \sin bx) + x (A_2 \cos bx + A_4 \sin bx)],$$

$$y = B_1 e^{ax} \cos (bx + B_2) + B_3 x e^{ax} \cos (bx + B_2)$$

where $A_1 = B_1 \cos B_2$, $A_3 = -B_1 \sin B_2$,

$$A_2 = B_3 \cos B_4, \quad A_4 = -B_3 \sin B_4$$

The last one can be expressed in terms of sines instead of cosines easily.

$\therefore$ the required general solution of (1) in 21.2 is the sum of these four expressions i.e.

$$\begin{aligned}
 y = & (c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_p e^{m_p x}) + [(D_1 \cos \beta_1 x + D_2 \sin \beta_1 x) e^{\alpha_1 x} + \\
 & \dots + (D_{2q-1} \cos \beta_q x + D_{2q} \sin \beta_q x) e^{\alpha_q x} + (E_1 + E_2 x + E_3 x^2 + \\
 & \dots + E_r x^{r-1}) e^{m_x x} + [(F_1 + F_2 x + \dots + F_s x^{s-1}) \cos bx \\
 & + (F_{s+1} + F_{s+2} x + \dots + F_{2s} x^{s-1}) \sin bx] e^{ax}.
 \end{aligned}$$

Examples

1. Test the linear dependence or independence of the functions $y_1 = x$, $y_2 = x^2$, $y_3 = x^3$ when $x \neq 0$.

The Wronskian $W(y_1, y_2, y_3)$

$$= \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} = 2x^3 \neq 0 \text{ since } x \neq 0.$$

Since Wronskian does not vanish, so the given functions are linearly independent.

2. Prove that the general solution of $y'' + y = 0$ is $y = c_1 \sin x + c_2 \cos x$ and its particular solution is $y = \sin x + \frac{1}{2} \cos x$ when $y(0) = 1$, $y'(0) = \frac{1}{2}$.

If $y_1 = \sin x$ and $y_2 = \cos x$ are linearly independent and separately satisfy the given equation then by Theorem 1, 2, their linear combination is the general

solution. Now $y_1 = \sin x$ and $y_2 = \cos x$ are seen to satisfy $y'' + y = 0$ separately and the Wronskian

$$W(y_1, y_2) = (y_1 y_2' - y_1' y_2) = \sin x (-\sin x) - \cos x (\cos x) = -1 \neq 0.$$

So they are linearly independent as well as they satisfy the given equation. Hence by Theorem 1, 2, $y = c_1 \sin x + c_2 \cos x$ is the general solution of $y'' + y = 0$. Also for $y(0) = 1$ and $y'(0) = \frac{1}{2}$,

$$\text{we have } c_1 \sin 0 + c_2 \cos 0 = 1 \text{ and } c_1 \cos 0 - c_2 \sin 0 = \frac{1}{2}$$

giving $c_1 = \frac{1}{2}$ and $c_2 = 1$. Therefore $y = \sin x + \frac{1}{2} \cos x$ is a particular solution of $y'' + y = 0$.

3. Solve the ODE $\frac{d^2 y}{dx^2} + 5 \frac{dy}{dx} + 6y = 0$.

This is a second order linear homogeneous ODE with constant coefficients. Let $y = e^{mx}$ be a trial solution. So, $\frac{dy}{dx} = me^{mx}$ and $\frac{d^2 y}{dx^2} = m^2 e^{mx}$. Putting these in the given equation, it becomes $(m^2 + 5m + 6) e^{mx} = 0$. So, the auxiliary equation (A.E.) is

$$m^2 + 5m + 6 = 0 \quad \because e^{mx} \neq 0 \text{ for } -\infty < x < \infty$$

$$\text{or, } (m+2)(m+3) = 0 \quad \therefore m = -2, -3.$$

The roots are here real distinct. Hence the general solution of the given equation is $y = c_1 e^{-2x} + c_2 e^{-3x}$ where c_1 and c_2 are two arbitrary constants.

4. Solve the ODE $\frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} + 4y = 0$.

Let $y = e^{mx}$ be the trial solution. So, $\frac{dy}{dx} = me^{mx}$, $\frac{d^2 y}{dx^2} = m^2 e^{mx}$. As in

Example 1, the A.E. is $m^2 - 4m + 4 = 0$, $\because e^{mx} \neq 0$.

$\therefore$ the roots of the A.E. are $m = 2, 2$ i.e. the roots are equal. Hence the general solution of the given equation is $y = (c_1 + c_2 x) e^{2x}$, where c_1 and c_2 are two arbitrary constants.

5. Solve the ODE $\frac{d^2 y}{dx^2} - 6 \frac{dy}{dx} + 13y = 0$.

Let $y = e^{mx}$ be the trial solution. So, $\frac{dy}{dx} = me^{mx}$, $\frac{d^2 y}{dx^2} = m^2 e^{mx}$. As in Ex.

1, the A.E. is $m^2 - 6m + 13 = 0$.

$\therefore$ the roots of A.E. are $m = 3 \pm 2i$ i.e. the roots are complex. Therefore, the required general solution is

$$y = c_1 e^{(3+2i)x} + c_2 e^{(3-2i)x} = e^{3x} [c_1 e^{2ix} + c_2 e^{-2ix}]$$

$$= e^{3x} [c_1 (\cos 2x + i \sin 2x) + c_2 (\cos 2x - i \sin 2x)]$$

$= e^{3x} [A \cos 2x + B \sin 2x]$ where $A = c_1 + c_2$ and $B = i(c_1 - c_2)$ are two arbitrary constants.

6. Solve the ODE $\frac{d^7 y}{dx^7} - \frac{d^6 y}{dx^6} + 6 \frac{d^5 y}{dx^5} - 6 \frac{d^4 y}{dx^4} + 9 \frac{d^3 y}{dx^3} - 9 \frac{d^2 y}{dx^2} = 0$.

The equation is a 7th order homogeneous linear ODE with constant coefficients. Let $y = e^{mx}$ be a trial solution. So, the A.E. is

$$m^7 - m^6 + 6m^5 - 6m^4 + 9m^3 - 9m^2 = 0, \quad \because e^{mx} \neq 0$$

$$\text{or } m^2 (m - 1) (m^2 + 3)^2 = 0$$

$\therefore$ the roots of the A.E. are

0, 0, 1, $\pm i\sqrt{3}$, $\pm i\sqrt{3}$. Of which two are equal, one real distinct and two pairs of equal complex roots.

$\therefore$ the general solution of the given equation, by 21.4, is

$$y = c_1 e^x + (c_2 + c_3 x) e^{0x} + (c_4 + c_5 x) e^{i\sqrt{3}x} + (c_6 + c_7 x) e^{-i\sqrt{3}x}$$

$$= c_1 e^x + c_2 + c_3 x + (c_4 + c_5 x) (\cos \sqrt{3} x + i \sin \sqrt{3} x)$$

$$+ (c_6 + c_7 x) (\cos \sqrt{3} x - i \sin \sqrt{3} x)$$

$$= c_1 e^x + c_2 + c_3 x + (A + Bx) \cos \sqrt{3} x + (C + Dx) \sin \sqrt{3} x$$

where $c_1, c_2, c_3, A = c_4 + c_6, B = c_5 + c_7, C = i(c_4 - c_6)$ and $D = i(c_5 - c_7)$ are seven arbitrary constants.

Exercises 21 (1)

1. Test for linear independence or dependence of the following functions :

(a) $y_1 = x^2 + 2x, y_2 = x^3 + x, y_3 = 2x^3 - x^2$; (b) $1, \sin x, \cos x$;

(c) $e^{ix}, \sin x, \cos x$; (d) $e^{-x}, 1, e^x, \sinh x$.

2. Show that $y_1 = e^x$ and $y_2 = e^{-x}$ are two linearly independent solutions of $y'' - y = 0$. Also show $y_1 = \sinh x$ and $y_2 = \cosh x$ are two linearly independent solutions of the same equation. Now reason why the general solution is same?

3. Prove that the general solution of $x^2 y'' - 2x y' + 2y = 0$ is $y = c_1 x^2 + c_2 x, x \neq 0$ and the particular solution is $y = 2(2x - x^2)$ when $y(1) = 2$ and $y'(1) = 0$.

4. Show that the general solution of $y'' - 4y' + 4y = 0$ is $y = c_1 e^{2x} + c_2 x e^{2x}$ and find the particular solutions when

(i) $y(0) = y'(0) = 0$ and (ii) $y(0) = 1, y'(0) = 4$.

Solve the following ODEs :

5. $\frac{d^2 y}{dx^2} + (a+b) \frac{dy}{dx} + aby = 0$.

6. $\frac{d^2 y}{dx^2} - 6 \frac{dy}{dx} + 13y = 0$.

7. $\frac{d^2 s}{dt^2} + 4 \frac{ds}{dt} + 13s = 0$.

8. $y_2 + 2y_1 - 8y = 0$.

9. $y_2 - 7y_1 + 12y = 0$.

10. $(D^2 - n^2)y = 0$.

11. $(D+3)^2 y = 0$.

12. $\frac{d^2 y}{dx^2} - 3 \frac{dy}{dx} + 2y = 0$.

13. $\frac{d^2 y}{dx^2} + 4y = 0$.

14. $\frac{d^2 y}{dx^2} + 2\sqrt{3} \frac{dy}{dx} + 4y = 0$.

15. $2 \frac{d^2 y}{dx^2} + 3 \frac{dy}{dx} + 4y = 0$.

16. $y_2 + 3y_1 - 54y = 0$.

17. $y_2 + 2y_1 + 5y = 0$.

18. Find particular solution of the IVP (initial value problems) :

(i) $y_2 + 4y_1 + 3y = 0, y(0) = 1, y_1(0) = 0;$

(ii) $y_2 - 3y_1 + 2y = 0, y(0) = 1, y_1(0) = 0;$

(iii) $l \frac{d^2 \theta}{dt^2} + g\theta = 0$, given that $\theta = \alpha$ and $\frac{d\theta}{dt} = 0$ when $t = 0$;

(iv) $L \frac{d^2 Q}{dt^2} + R \frac{dQ}{dt} + \frac{Q}{C} = 0$ given that $Q = Q_0$ and $\frac{dQ}{dt} = 0$ when $t = 0$ and $CR^2 < 4L$.

19. Show that the general solution of $\frac{d^2 x}{dt^2} + k \frac{dx}{dt} + \mu x = 0$ is

$$x = e^{-\frac{1}{2}kt} (A \cos nt + B \sin nt) \text{ when } k^2 < 4\mu \text{ and } n^2 = \mu - \frac{1}{4}k^2$$

Solve the following ODEs :

20. $\frac{d^3 y}{dx^3} - y = 0$.

21. $\frac{d^4 y}{dx^4} + 4 \frac{d^3 y}{dx^3} + 8 \frac{d^2 y}{dx^2} + 8 \frac{dy}{dx} + 4y = 0$.

22. $\frac{d^3 y}{dx^3} + \frac{d^2 y}{dx^2} - \frac{dy}{dx} - y = 0$.

23. $\frac{d^3 y}{dx^3} + 3 \frac{d^2 y}{dx^2} + 3 \frac{dy}{dx} + y = 0$.

24. $\frac{d^4 y}{dx^4} - 2 \frac{d^3 y}{dx^3} + \frac{d^2 y}{dx^2} = 0$. 25. $4 \frac{d^5 y}{dx^5} - 3 \frac{d^3 y}{dx^3} - \frac{d^2 y}{dx^2} = 0$.
26. $(D^6 + 3D^4 + 3D^2 + 1)y = 0$. 27. $(D^4 - 2D^3 + 2D^2 - 2D + 1)y = 0$.
28. $(D^5 - D)y = 0$. 29. $\frac{d^6 y}{dx^6} - 64y = 0$.
30. $(D + 1)(D - 1)(D - 2)y = 0$. 31. $(D - 1)(D - 2)^2 y = 0$.
32. $(D^6 - a^6)y = 0$. 33. $(D^4 + 5D^2 + 6)y = 0$.

Answers

1. (a) Linearly dependent, (b) Linearly independent,
(c) Linearly dependent, (d) Linearly independent.
2. Because each of latter solution is a linear combination of the former solutions.
4. (i) $y = 0$. (ii) $y = (1 + 2x)e^{2x}$. 5. $y = c_1 e^{-ax} + c_2 e^{-bx}$.
6. $y = e^{3x}(A \cos 2x + B \sin 2x)$. 7. $s = e^{-2t}(A \cos 3t + B \sin 3t)$.
8. $y = c_1 e^{-4x} + c_2 e^{2x}$. 9. $y = c_1 e^{-3x} + c_2 e^{-4x}$. 10. $y = c_1 e^{\mu x} + c_2 e^{-\mu x}$.
11. $y = (c_1 + c_2 x)e^{-x}$. 12. $y = c_1 e^{-x} + c_2 e^{-2x}$. 13. $y = c_1 \cos 2x + c_2 \sin 2x$.
14. $y = e^{\sqrt{3}x}(A \cos x + B \sin x)$. 15. $y = e^{-3x/4} A \cos(x\sqrt{23}/4) + B \sin(x\sqrt{23}/4)$.
16. $y = c_1 e^{-9x} + c_2 e^{5x}$. 17. $y = e^x(A \cos 2x + B \sin 2x)$.
18. (i) $y = 2e^x - e^{-3x}$, (ii) $y = 2e^x - e^{2x}$, (iii) $\theta = \alpha \cos\left(\frac{\sqrt{g}}{l}t\right)$.
- (iv) $Q = Q_0 e^{-Rt/2L} \left[\cos\left(\frac{1}{2L} \sqrt{\frac{4L}{C} - R^2} t\right) + \left(R / \sqrt{\frac{4L}{C} - R^2} \right) \cdot \sin\left(\frac{1}{2L} \sqrt{\frac{4L}{C} - R^2} t\right) \right]$.
20. $y = c_1 e^x + e^{-\frac{1}{2}x} \left(c_2 \cos \frac{\sqrt{3}}{2} x + c_3 \sin \frac{\sqrt{3}}{2} x \right)$.
21. $y = e^{-\frac{1}{2}x} (c_1 + c_2 x) \cos \frac{\sqrt{3}x}{2} + (c_3 + c_4 x) \sin \frac{\sqrt{3}x}{2}$.
22. $y = c_1 e^x + (c_2 + c_3 x)e^{-x}$. 23. $y = (c_1 + c_2 x + c_3 x^2)e^{-x}$.
24. $y = c_1 + c_2 x + (c_3 + c_4 x)e^x$. 25. $y = c_1 + c_2 x + c_3 e^x + (c_4 + c_5 x)e^{-\frac{1}{2}x}$.
26. $y = (c_1 + c_2 x + c_3 x^2) \cos x + (c_4 + c_5 x + c_6 x^2) \sin x$.
27. $y = (c_1 + c_2 x)e^x + c_3 \cos x + c_4 \sin x$. 28. $y = c_1 + c_2 e^x + c_3 e^{-x} + c_4 \cos x + c_5 \sin x$.
29. $y = c_1 e^{2x} + c_2 e^{-2x} + e^{-x} \{ \cos(x\sqrt{3}) + c_4 \sin(x\sqrt{3}) \}$
 $+ e^x \{ c_5 \cos(x\sqrt{3}) + c_6 \sin(x\sqrt{3}) \}$
30. $y = c_1 e^{-x} + c_2 e^x + c_3 e^{2x}$. 31. $y = c_1 e^x + (c_2 + c_3 x)e^{2x}$.
32. $y = c_1 e^{ax} + c_2 e^{-ax} + e^{-ax/2} \{ c_3 \cos(ax\sqrt{3}/2) + c_4 \sin(ax\sqrt{3}/2) \}$
 $+ e^{ax/2} \{ c_5 \cos(ax\sqrt{3}/2) + c_6 \sin(ax\sqrt{3}/2) \}$.
33. $y = c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x + c_3 \sin \sqrt{3}x + c_4 \cos \sqrt{3}x$.

21.5 HIGHER ORDER LINEAR NON-HOMOGENEOUS ODE WITH CONSTANT COEFFICIENT

The most general n th order linear non-homogeneous ODE with constant coefficients is

$$A_0 \frac{d^n y}{dx^n} + A_1 \frac{d^{n-1} y}{dx^{n-1}} + A_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + A_{n-1} \frac{dy}{dx} + A_n y = Q(x), A_0 \neq 0, \quad (1)$$

where $A_0, A_1, \dots, A_n$ are constants and $Q(x)$ is a single valued continuous function. According to Th. 3 of 21.1, the solution of ODE (1) consists of two parts— y_c and y_p i.e. $y = y_c + y_p$. The complementary solution or function y_c is computed from the associated homogeneous equation (1) by the procedures developed in 21.3 and 21.4. Now remains the computation of y_p directly from (1). The computation of y_p requires the properties of the differential operators

$D^n \equiv \frac{d^n}{dx^n}$, ..., $D \equiv \frac{d}{dx}$ etc. So, these properties are first discussed in the following :

The symbols for differential operators obey the fundamental laws of algebra such as

- (i) Commutative law for addition : $(D^r + D^s) u = (D^s + D^r) u$
- (ii) Associative law for addition :

$$[D^r + (D^s + D^t)] u = [(D^r + D^s) + D^t] u$$

- (iii) Commutative law for multiplication : $D^r (D^s \cdot D^t) u = (D^r \cdot D^s + D^r \cdot D^t) u$ when the coefficients are variable and $(D^r \cdot D^s) u = (D^s \cdot D^r) u$ for constant coefficients.

- (iv) Distributive law : $D^r (D^s + D^t) u = (D^r D^s + D^r D^t) u$

- (v) Index law : $(D^r \cdot D^s) u = D^{r+s} u$.

Again if the operator

$$f(D) \equiv (A_0 D^n + A_1 D^{n-1} + A_2 D^{n-2} + \dots + A_{n-1} D + A_n)$$

Theorem 1 : $f(D) e^{ax} = e^{ax} f(a)$.

Theorem 2 : $f(D) \{e^{ax} v(x)\} = e^{ax} f(D+a) v(x)$.

Theorem 3 : $f(D^2) \cos(ax + \alpha) = f(-a^2) \cos(ax + \alpha)$

and $f(D^2) \sin(ax + \alpha) = f(-a^2) \sin(ax + \alpha)$.

Proof of Th. 1 : Since $De^{ax} = ae^{ax}$, $D^2 e^{ax} = ae^{ax}$, ..., $D^n e^{ax} = a^n e^{ax}$,

$$f(D) e^{ax} = [A_0 D^n + A_1 D^{n-1} + \dots + A_{n-1} D + A_n] e^{ax}$$

$$= [A_0 a^n + A_1 a^{n-1} + \dots + A_{n-1} a + A_n] = f(a) e^{ax}$$

Proof of Th. 2 : Since $D(e^{ax} v(x)) = e^{ax} D v(x) + a e^{ax} v(x)$

$$= e^{ax} (D + a) v(x),$$

$$\begin{aligned} D^2(e^{ax} v(x)) &= D[e^{ax} (D + a) v(x)] \\ &= e^{ax} D(D + a) v(x) + a e^{ax} (D + a) v(x) \\ &= e^{ax} [D^2 + 2aD + a^2] v(x) = e^{ax} (D + a)^2 v(x). \end{aligned}$$

Similarly, $D^n(e^{ax} v(x)) = e^{ax} (D + a)^n v(x)$. Therefore,

$$\begin{aligned} f(D) \{e^{ax} v(x)\} &= (A_0 D^n + A_1 D^{n-1} + \dots + A_{n-1} D + A_n) (e^{ax} v(x)) \\ &= e^{ax} [A_0 (D + a)^n + A_1 (D + a)^{n-1} + \dots + A_{n-1} (D + a) + A_n] v(x) \\ &= e^{ax} f(D + a) v(x). \end{aligned}$$

Proof of Th. 3 : Since $D^2 \cos(ax + \alpha) = -a^2 \cos(ax + \alpha)$,

$$D^4 \cos(ax + \alpha) = (-a^2)^2 \cos(ax + \alpha), \dots,$$

$$D^{2n} \cos(ax + \alpha)$$

$$= (-a^2)^n \cos(ax + \alpha)$$

$$\begin{aligned} \therefore f(D^2) \cos(ax + \alpha) &= [A_0 D^{2n} + A_1 D^{2n-2} + \dots \\ &\quad + A_{n-1} D^2 + A_n] \cos(ax + \alpha) \end{aligned}$$

$$= [A_0 (-a^2)^n + A_1 (-a^2)^{n-1} + \dots + A_{n-1} (-a^2) + A_n] \cos(ax + \alpha)$$

$$= f(-a^2) \cos(ax + \alpha)$$

$$\text{Similarly, } f(D^2) \sin(ax + \alpha) = f(-a^2) \sin(ax + \alpha)$$

INVERSE OPERATOR : Let $Dy = v$. $\therefore y = \frac{1}{D} v = D^{-1} v$, D is used

as an algebraic quantity. Again $v = Dy = DD^{-1} v$. This equality can only hold true if $DD^{-1} = 1$. Therefore D^{-1} or $\frac{1}{D}$ is inverse of the operator D and the

combined effect of D and its inverse, D^{-1} keeps the function unchanged.

Therefore, the inverse operator D^{-1} or $\frac{1}{D}$ performs the function of integration.

In the same way, it can be said that if $f(D)$ denotes an n th degree polynomial in D , then $\frac{1}{f(D)}$ or $[f(D)]^{-1}$ is an inverse operator. This is used in the computation of y_p .

Particular integral, y_p : Let the equation (1) be expressed as $f(D)y = Q(x)$, then the particular integral y_p is

$y_p = \frac{1}{f(D)} Q(x) = [f(D)]^{-1} Q(x)$. If $f(D) = D^n$ only then $y_p = \frac{1}{D^n} Q(x)$ which

can be computed by successive integration n times without adding any constant of integration as the requisite number constants comes in y_c . The computation of y_p for other forms of $f(D)$ and $Q(x)$ will be discussed below:

Methods for computing y_p :

Type I : If $Q(x) = X$, a polynomial in x

$$\text{then } y_p = \frac{1}{f(D)} X = [f(D)]^{-1} X.$$

This can be simplified by using binomial expansion for $[f(D)]^{-1}$ and multiplying term by term. The expansion can be made simple by using partial fraction, if possible.

Type II : If $Q(x) = e^{ax}$, ' a ' is a constant, then $y_p = \frac{e^{ax}}{f(a)}$, $f(a) \neq 0$.

$\therefore$ by theorem 1, $f(D) e^{ax} = f(a) e^{ax}$

$$\text{or, } \frac{e^{ax}}{f(a)} = \frac{1}{f(D)} e^{ax} \quad \because f(a) \neq 0.$$

$$\therefore y_p = \frac{1}{f(D)} e^{ax} = \frac{e^{ax}}{f(a)} \text{ if } f(a) \neq 0.$$

If $f(a) = 0$, Type II fails and use Type III with $v(x) = 1$ or use Type IV.

Type III : If $Q(x) = e^{ax} v(x)$, then $y_p = e^{ax} \frac{1}{f(D+a)} v(x)$.

To establish the result let X be a function of x . Then by Th. 2 $f(D)(e^{ax} X) = e^{ax} f(D+a) X$. Let

$$f(D+a) X = v(x) \text{ i.e. } C = \frac{1}{f(D+a)} v(x)$$

$$\therefore f(D) \left(e^{ax} \cdot \frac{1}{f(D+a)} v(x) \right) = e^{ax} f(D+a) v(x) = e^{ax} v(x)$$

Multiplying both sides by the inverse operator $\frac{1}{f(D)}$, we get,

$y_p = \frac{1}{f(D)} (e^{ax} \cdot v) = e^{ax} \frac{1}{f(D+a)} v(x)$. If $v(x)$ is a polynomial in x , use Type I or if $v(x)$ is a function like trigonometric sine or cosine use Type V.

Type IV : If Type II fails, then there must be factor like $(D-a)$ in $f(D)$ as $f(a) = 0$. So, let $f(D) = (D-a)^r \phi(D)$ when $\phi(a) \neq 0$. Therefore,

$$\begin{aligned} y_p &= \frac{1}{f(D)} e^{ax} = \frac{1}{(D-a)^r \phi(D)} e^{ax} = \frac{1}{(D-a)^r} \left\{ \frac{1}{\phi(D)} e^{ax} \right\} \\ &= \frac{1}{(D-a)^r} \cdot \frac{e^{ax}}{\phi(a)} \text{ by Type II} \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{\phi(a)} \cdot \frac{1}{(D-a)^r} e^{ax} = \frac{e^{ax}}{\phi(a)} \cdot \frac{1}{(D+a-a)^r} \cdot 1 \text{ by type III} \\
 &= \frac{e^{ax}}{\phi(a)} \cdot \frac{1}{D^r} \cdot 1 = \frac{x^r e^{ax}}{r! \phi(a)}.
 \end{aligned}$$

Type V : If $Q(x) = \sin(ax + \alpha)$ or $\cos(ax + \alpha)$, then there arise three cases to evaluate $y_p = \frac{1}{f(D)} \{\sin(ax + \alpha) \text{ or } \cos(ax + \alpha)\}$.

Case (a) : $y_p = \frac{1}{\psi(-a^2)} \{\sin(ax + \alpha) \text{ or } \cos(ax + \alpha)\}$ when

$f(D) \equiv \phi(D^2)$ i.e. $f(D)$ contains only even powers of D and $\phi(-a^2) \neq 0$. This result can be easily obtained by Th. 3 which states that $\phi(D^2) \sin(ax + \alpha) = \phi(-a^2) \sin(ax + \alpha)$. Multiplying this relation by the inverse operator $\frac{1}{(D)^2}$ and dividing by $\phi(-a^2) \neq 0$, we have the required result

$$y_p = \frac{1}{\phi(D)^2} \sin(ax + \alpha) = \frac{1}{\phi(-a^2)} \sin(ax + \alpha), \quad \phi(-a^2) \neq 0$$

Similarly,

$$y_p = \frac{1}{\phi(D)^2} \cos(ax + \alpha) = \frac{1}{\phi(-a^2)} \cos(ax + \alpha), \quad \phi(-a^2) \neq 0$$

Case (b) : $y_p = \frac{H(-D)}{G(-a^2)H(a^2)} \{\sin(ax + \alpha) \text{ or } \cos(ax + \alpha)\}$, where

$$G(-a^2)H(a^2) \neq 0$$

$$\text{and} \quad f(D) \equiv G(D)^2 H(D).$$

Here $f(D)$ is not an even powered polynomial in D as in case (a). Let $f(D) \equiv G(D^2)H(D)$ where G is even powered polynomial of D and H contains both even and odd powers of D . Therefore,

$$\begin{aligned}
 y_p &= \frac{1}{f(D)} \sin(ax + \alpha) = \frac{1}{G(D^2)H(D)} \sin(ax + \alpha) \\
 &= \frac{H(-D)}{G(D^2)H(-D)H(D)} \sin(ax + \alpha) \\
 &= \frac{H(-D)}{G(D^2)H(-D^2)} \sin(ax + \alpha) \\
 &= \frac{H(-D) \sin(ax + \alpha)}{G(-a^2)H(a^2)}, \text{ if } G(-a^2)H(a^2) \neq 0
 \end{aligned}$$

Now multiplying with each term of $H(-D)$, we have, y_p . Similarly, for cosine only, replace sine by cosine.

Case (c) : If case (a) and (b) fail, then Euler's equation $e^{i\theta} = \cos \theta + i \sin \theta$ is to be used to transfer the type V to type II. So, let

$$X = \frac{1}{f(D)} \cos(ax + \alpha) \text{ and } Y = \frac{1}{f(D)} \sin(ax + \alpha).$$

$\therefore X + iY = \frac{1}{f(D)} e^{i(ax + \alpha)}$. Now using the type II, equate the real or imaginary part according as $Q(x)$ involves cosine or sine term.

Cor. : If the operator be $(D^2 + a^2)$, then using case (c) above

$$y_p = \frac{-1}{D^2 + a^2} \sin(ax + \alpha) = \frac{x \cos(ax + \alpha)}{2a} = \frac{x}{2} \int \sin(ax + \alpha) dx$$

and
$$y_p = \frac{1}{D^2 + a^2} \cos(ax + \alpha) = \frac{x \sin(ax + \alpha)}{2a} = \frac{x}{2} \int \cos(ax + \alpha) dx.$$

Type VI : If $Q(x) = v(x) \{\sin(ax + \alpha) \text{ or } \cos(ax + \alpha)\}$, then $y_p = \frac{1}{f(D)} v(x) \{\sin(ax + \alpha) \text{ or } \cos(ax + \alpha)\}$. For its evaluation, if $v(x)$ involves exponential use type II first. Then a similar procedure as type V, otherwise also the same as type V. Let

$$X = \frac{1}{f(D)} v(x) \cos(ax + \alpha) \text{ and } Y = \frac{1}{f(D)} v(x) \sin(ax + \alpha)$$

$\therefore X + iY = \frac{1}{f(x)} v(x) e^{i(ax + \alpha)}$. Now using type III, equate for y_p the real or imaginary part according as $Q(x)$ involves cosine or sine term. This equation can be avoided by using exponential definitions of sine and cosine at the start and using type III. Note W.O. Ex. 5 Alter 2.

Type VII : If $Q(x) = xV(x)$, $V(x)$ is any function of x ,

then
$$y_p = \frac{1}{f(D)} xV(x)$$

$$= x \frac{1}{f(D)} xV(x) + \left\{ \frac{d}{dD} \frac{1}{f(D)} \right\} V(x) \quad \dots (1)$$

$$= x \frac{1}{f(D)} V(x) - f'(D) \left\{ \frac{1}{f(D)} \right\}^2 V(x) \quad \dots (1a)$$

This result can be established using Leibnitz's theorem as

$$\begin{aligned} D^n(xV(x)) &= xD^nV(x) + nD^{n-1}V(x) \\ &= xD^nV(x) + \left(\frac{d}{dD} D^n \right) V(x) \end{aligned}$$

Giving different values to n , we have,

$$f(D)xV(x) = xf(D)V(x) + f'(D)V(x) \quad \dots (2)$$

Now multiplying R.H.S. of (1) by $f(D)$, we obtain,

$$\begin{aligned}
 & f(D) \left[x \frac{1}{f(D)} V(x) + \left\{ \frac{d}{dD} \frac{1}{f(D)} \right\} V(x) \right] \\
 &= x f(D) \cdot \frac{1}{f(D)} V(x) + f'(D) \frac{1}{f(D)} V(x) \\
 &\quad - f(D) \left\{ f'(D) \frac{1}{\{f(D)\}^2} \right\} V(x) \\
 &\quad \text{by (2) and R.H.S. of (1a)} \\
 &= x V(x) + f' \frac{1}{f(D)} V(x) - f'(D) \left[f(D) \left\{ \frac{1}{f(D)} \left(\frac{1}{f(D)} V(x) \right) \right\} \right] \\
 &\quad \text{by re-arrangement} \\
 &= x V(x) + f'(D) \frac{1}{f(D)} V(x) - f'(D) \frac{1}{f(D)} V(x) \\
 &= x V(x) \\
 \therefore y_p &= \frac{1}{f(D)} x V(x) = x \frac{1}{f(D)} V(x) + \left\{ \frac{d}{dD} \frac{1}{f(D)} \right\} V(x) \text{ which is (1).}
 \end{aligned}$$

When x is not linear i.e. when $Q(x) = x^m V(x)$, then the above result can be extended to

$$\begin{aligned}
 y_p &= \frac{1}{f(D)} x^m V(x) = x^m \frac{1}{f(D)} V(x) + m x^{m-1} \left\{ \frac{d}{dD} \frac{1}{f(D)} \right\} V(x) \\
 &\quad + \frac{m(m-1)}{2!} x^{m-2} \left\{ \frac{d^2}{dD^2} \frac{1}{f(D)} \right\} V(x) + \dots
 \end{aligned}$$

Practically this is not a convenient procedure.

If $V(x) = \sin ax$ or $\cos ax$ and $m > 1$ then type VI is preferable. Note that in such case if $f(a) = 0$ then DO NOT use this procedure.

Type VIII : METHOD OF VARIATION OF PARAMETERS/ CONSTANTS

All the types I to VI discussed are meant for some particular cases. If $Q(x)$ involves some function other than these then those short methods will not be applicable for the computation of y_p . Here is a general method "variation of parameters or constants" applicable to the above cases and also to other cases. This method is due to French mathematician Lagrange. In this method the constants of the complementary solution are treated as function of the independent variable and so is the name of method "variation of parameters or constants" and assumed as general solution. For sake of convenience, $(n-1)$ simultaneous equations are constructed by equated to zero the sum of the first derivatives of these new constants arising in each successive differentiation of the dependent variable for a n th order ODE. Ultimately n

simultaneous equations resulted from the process, the n derivatives are then solved and integrated. The substitution of these into the assumed solution gives y_c with arbitrary constants and y_p with no arbitrary constants.

$$\text{Let } A_0 \frac{d^n y}{dx^n} + A_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + A_{n-1} \frac{dy}{dx} + A_n y = Q(x), A_0 \neq 0, \quad (1)$$

be the non-homogeneous equation. Let

$$y = c_1 y_1(x) + c_2 y_2(x) + \dots + c_n y_n(x) \quad (2)$$

be the solution of the associated homogeneous equation of (1).

By def., let $c_i = c_i(x)$, $i = 1, 2, \dots, n$. Therefore, let the general solution of (1) be

$$y = c_1(x) y_1(x) + c_2(x) y_2(x) + \dots + c_n(x) y_n(x) \quad (3)$$

Differentiating (3) w.r.t. x

$$y' = c_1(x) y_1'(x) + c_2(x) y_2'(x) + \dots + c_n(x) y_n'(x) + [c_1'(x) y_1(x) + c_2'(x) y_2(x) + \dots + c_n'(x) y_n(x)] \quad (4)$$

Now to avoid complexity in knowing $c_i(x)$, let

$$c_1'(x) y_1(x) + c_2'(x) y_2(x) + \dots + c_n'(x) y_n(x) = 0 \quad (a_1)$$

be a first condition for $c_i(x)$. Therefore, (4) becomes,

$$y' = c_1(x) y_1'(x) + c_2(x) y_2'(x) + \dots + c_n(x) y_n'(x) \quad (5)$$

Differentiating (5) w.r.t. x and setting to zero the sum of the derivatives of $c_i(x)$, we have,

$$y'' = c_1(x) y_1''(x) + c_2(x) y_2''(x) + \dots + c_n(x) y_n''(x) \quad (6)$$

$$\text{and } c_1'(x) y_1'(x) + c_2'(x) y_2'(x) + \dots + c_n'(x) y_n'(x) = 0 \quad (a_2)$$

Proceeding in this way, the $(n-1)$ th derivative of y and the $(n-1)$ th condition of $c_i(x)$, we have

$$y^{(n-1)} = c_1(x) y_1^{(n-1)}(x) + c_2(x) y_2^{(n-1)}(x) + \dots + c_n(x) y_n^{(n-1)}(x) \quad (n-4)$$

and

$$c_1'(x) y_1^{(n-2)}(x) + c_2'(x) y_2^{(n-2)}(x) + \dots + c_n'(x) y_n^{(n-2)}(x) = 0 \quad (a_{n-1})$$

Differentiating $(n-4)$ th equation once w.r.t. x ,

$$y^{(n)} = c_1(x) y_1^{(n)}(x) + c_2(x) y_2^{(n)}(x) + \dots + c_n(x) y_n^{(n)}(x) + c_1'(x) y_1^{(n-1)}(x) + c_2'(x) y_2^{(n-1)}(x) + \dots + c_n'(x) y_n^{(n-1)}(x) \quad (n-3)$$

Now using (3), (5), (6), ..., $(n-4)$, $(n-3)$ in (1), we have,

$$\begin{aligned} & c_1(x) [A_0 y_1^{(n)}(x) + A_1 y_1^{(n-1)}(x) + \dots + A_{n-1} y_1'(x) + y_1(x)] + \\ & c_2(x) [A_0 y_2^{(n)}(x) + A_1 y_2^{(n-1)}(x) + \dots + A_{n-1} y_2'(x) + y_2(x)] + \dots + \\ & c_n(x) [A_0 y_n^{(n)}(x) + A_1 y_n^{(n-1)}(x) + \dots + A_{n-1} y_n'(x) + y_n(x)] + \\ & [c_1'(x) y_1^{(n-1)}(x) + c_2'(x) y_2^{(n-1)}(x) + \dots + c_n'(x) y_n^{(n-1)}(x)] = Q(x) \end{aligned}$$

which reduces to the n th condition of $c_i(x)$ as

$$c_1'(x) y_1^{(n-1)}(x) + c_2'(x) y_2^{(n-1)}(x) + \dots + c_n'(x) y_n^{(n-1)}(x) = Q(x) \quad (a_n)$$

Since $y_1, y_2, \dots, y_n$ are the n independent solution of the associated homogeneous equation of (1) and so the coeffs. of $c_1(x), c_2(x), \dots, c_n(x)$ vanish. Now the n equations for n unknowns $c_i(x), i = 1, 2, \dots, n$ are

$$\sum_{i=1}^n c_i'(x) y_i(x) = 0, \sum_{i=1}^n c_i'(x) y_i'(x) = 0, \dots, \sum_{i=1}^n c_i'(x) y_i^{(n-2)}(x) = 0$$

and
$$\sum_{i=1}^n c_i'(x) y_i^{(n-1)}(x) = Q(x).$$

These n equations are solvable as their Wronskian $W(y_1, y_2, \dots, y_n) \neq 0$.

Solving these $c_i(x)$'s and using in (3) we have the required solution $y = y_c + y_p =$ sum of complementary solution and particular integral.

Examples

1. Solve the ODE $\frac{d^2 y}{dx^2} - 5 \frac{dy}{dx} + 6y = x^3 + x$ (1)

The associated homogeneous equation of the given non-homogeneous linear ODE with constant coefficient is

$$\frac{d^2 y}{dx^2} - 5 \frac{dy}{dx} + 6y = 0. \quad (2)$$

Let $y = e^{mx}$ be the trial solution of (2). Therefore, the A.E. is $m^2 - 5m + 6 = 0$ or $m = 2, 3$ which are real distinct. So, the complementary solution

$$y_c = c_1 e^{2x} + c_2 e^{3x} \quad (3)$$

Now, to find y_p , the particular integral, the differential operator D needs to be introduced. So, the equation (1) becomes, $(D^2 - 5D + 6)y = x^3 + x$

$$\therefore \text{by def., } y_p = \frac{1}{D^2 - 5D + 6} (x^3 + x)$$

Here, R.H.S. = $Q(x) = x^3 + x$, a polynomial in x of degree 3 and the operator $f(D) = D^2 - 5D + 6$ can be factorized. So, y_p is computed below by two ways of expansion of the inverse operator in question :

$$\text{1st way : } y_p = \frac{1}{6} \left(1 + \frac{D^2 - 5D}{6} \right)^{-1} (x^3 + x), \text{ taking out lowest degree}$$

$$\begin{aligned}
 D &= \frac{1}{6} \left[1 - \frac{D^2 - 5D}{6} + \left(\frac{D^2 - 5D}{6} \right)^2 - \left(\frac{D^2 - 5D}{6} \right)^3 + \dots \right] (x^3 + x) \\
 &= \frac{1}{6} \left[1 + \frac{5}{6}D + \frac{19}{36}D^2 + \frac{65}{216}D^3 + \dots \right] (x^3 + x) \\
 &= \frac{1}{6} \left[x^3 + \frac{5}{2}x^2 + \frac{25}{6}x + \frac{95}{36} \right]
 \end{aligned}$$

$$\begin{aligned}
 \text{2nd way: } y_p &= \frac{1}{(D-2)(D-3)} (x^3 + x) = \left(\frac{1}{D-3} - \frac{1}{D-2} \right) (x^3 + x) \\
 &= \left[-\frac{1}{3} \left(1 - \frac{D}{3} \right)^{-1} + \frac{1}{2} \left(1 - \frac{D}{2} \right)^{-1} \right] (x^3 + x) \\
 &= -\frac{1}{3} \left(1 + \frac{D}{3} + \frac{D^2}{9} + \frac{D^3}{27} + \dots \right) (x^3 + x) + \\
 &\quad \frac{1}{2} \left(1 + \frac{D}{2} + \frac{D^2}{4} + \frac{D^3}{8} + \dots \right) (x^3 + x) \\
 &= \frac{1}{6} \left[x^3 + \frac{5}{2}x^2 + \frac{25}{6}x + \frac{95}{36} \right]
 \end{aligned}$$

$\therefore$ the general solution of the given equation (1) is

$$y = y_c + y_p = c_1 e^{2x} + c_2 e^{3x} + \frac{1}{6} \left(x^3 + \frac{5}{2}x^2 + \frac{25}{6}x + \frac{95}{36} \right)$$

2. Solve the ODE $\frac{d^2 y}{dx^2} - 3 \frac{dy}{dx} + y = e^{2x}$.

The A.E. of the associated homogeneous equation is $m^2 - 3m + 1 = 0$

$$\therefore m = \frac{3 \pm \sqrt{5}}{2}, \text{ real distinct.}$$

$$\therefore y_c = c_1 e^{\frac{3+\sqrt{5}}{2}x} + c_2 e^{\frac{3-\sqrt{5}}{2}x}$$

$$\begin{aligned}
 \text{Now } y_p &= \frac{1}{D^2 - 3D + 1} e^{2x} = \frac{e^{2x}}{2^2 - 3 \cdot 2 + 1} \text{ by type II} \\
 &= -e^{2x}
 \end{aligned}$$

$\therefore$ the general solution is

$$y = y_c + y_p = c_1 e^{(3+\sqrt{5})x/2} + c_2 e^{(3-\sqrt{5})x/2} - e^{2x}$$

3. Solve the ODE $(D^2 - 3D + 2)y = e^{2x}$

The A.E. of the associated homogeneous equation is $m^2 - 3m + 2 = 0$.

$\therefore 3, 1$, are real distinct roots.

$$\therefore y_c = c_1 e^x + c_2 e^{3x}.$$

Now, $y_p = \frac{1}{D^2 - 3D + 2} e^{2x} = \frac{1}{(D-2)(D-1)} e^{2x}$, Type II is not applicable

$$\begin{aligned} &= \frac{1}{D-2} \cdot \left[\frac{1}{D-1} e^{2x} \right] = \frac{1}{D-2} \cdot \frac{1}{2-1} e^{2x}, \text{ by type II} \\ &= \frac{1}{D-2} e^{2x} = e^{2x} \frac{1}{D+2-2} \cdot 1 = e^{2x} \frac{1}{D} \cdot 1, \text{ by type III} \\ &= x e^{2x} \end{aligned}$$

$\therefore$ the required general solution is $y = c_1 e^x + c_2 e^{3x} + x e^{2x}$

Alter for y_p . Since II is not applicable. So, using III directly,

$$\begin{aligned} y_p &= e^{2x} \cdot \frac{1}{(D+2)^2 - 3(D+2) + 2} \cdot 1 \\ &= e^{2x} \cdot \frac{1}{D^2 + D} \cdot 1 = e^{2x} \cdot \frac{1}{D} (1+D)^{-1} \cdot 1 \\ &= e^{2x} \cdot \frac{1}{D} \cdot 1 = x e^{2x}. \end{aligned}$$

4. Solve the ODE $(D^3 - 3D - 2)y = 540x^3 e^{-x}$

Roots of the A.E. are $m = -1, -1, 2$

$$\therefore y_c = (c_1 + c_2 x) e^{-x} + c_3 e^{2x}$$

$$\begin{aligned} \text{Now, } y_p &= \frac{1}{D^3 - 3D - 2} 540x^3 e^{-x} = 540 e^{-x} \frac{1}{(D-1)^3 - 3(D-1) + 2} \cdot x^3 \\ &= 540 e^{-x} \frac{1}{D^3 - 3D^2} x^3 = -180 e^{-x} \frac{1}{D^2} \left(1 - \frac{D}{3}\right)^{-1} \cdot x^3 \\ &= -180 e^{-x} \frac{1}{D^2} \left(x^3 + x^2 + \frac{2}{3}x + \frac{2}{9}\right) \\ &= -180 e^{-x} \frac{1}{D} \left(\frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{3}x^2 + \frac{2}{9}x\right) \\ &= -e^{-x} (9x^5 + 15x^4 + 20x^3 + 20x^2) \\ \therefore y &= y_c + y_p = (c_1 + c_2 x) e^{-x} + c_3 e^{2x} \\ &\quad - e^{-x} (9x^5 + 15x^4 + 20x^3 + 20x^2). \end{aligned}$$

5. Find the particular integral of $(D^2 + 9)y = x \cos x$

Using type VIII,

$$\begin{aligned}
 y_p &= \frac{1}{D^2+9} x \cos x \\
 &= x \frac{1}{D^2+9} \cos x - 2D \left(\frac{1}{D^2+9} \right)^2 \cos x \\
 &= \frac{1}{8} x \cos x - 2D \cdot \frac{1}{D^4+18D^2+81} \cos x \\
 &= \frac{1}{8} x \cos x - 2D \cdot \frac{1}{(-1^2)^2+18(-1^2)+81} \cos x \\
 &= \frac{1}{8} x \cos x + \frac{1}{32} \sin x.
 \end{aligned}$$

Alter 1. (Using type VI)

$$\text{Let } X = \frac{1}{D^2+9} x \cos x, \quad Y = \frac{1}{D^2+9} x \sin x$$

$$\begin{aligned}
 \therefore X + iY &= \frac{1}{D^2+9} x e^{ix} \\
 &= e^{ix} \frac{1}{(D+i)^2+9} x \\
 &= \frac{1}{8} e^{ix} \left(x - \frac{i}{4} \right)
 \end{aligned}$$

$\therefore$ equating real parts

$$y_p = x = \frac{1}{8} x \cos x + \frac{1}{32} \sin x.$$

Alter 2.

$$\begin{aligned}
 y_p &= \frac{1}{D^2+9} x \cos x = \frac{1}{D^2+9} \frac{x}{2} (e^{ix} + e^{-ix}) \\
 &= \frac{1}{2} e^{ix} \frac{1}{(D+i)^2+9} x + \frac{1}{2} e^{-ix} \frac{1}{(D-i)^2+9} x \\
 &= \frac{1}{2} e^{ix} \cdot \frac{1}{8} \left(1 + \frac{2Di+D^2}{8} \right)^{-1} x + \frac{1}{2} e^{-ix} \cdot \frac{1}{8} \left(1 - \frac{2Di-D^2}{8} \right)^{-1} x \\
 &= \frac{1}{8} \left[\frac{1}{2} e^{ix} \left(x - \frac{i}{4} \right) + \frac{1}{2} e^{-ix} \left(x + \frac{i}{4} \right) \right] \\
 &= \frac{1}{8} \left[\frac{x}{2} (e^{ix} + e^{-ix}) - \frac{i}{4} \cdot \frac{1}{2} (e^{ix} - e^{-ix}) \right] \\
 &= \frac{1}{8} \left(x \cos x + \frac{1}{4} \sin x \right)
 \end{aligned}$$

$$= \frac{1}{8} x \cos x + \frac{1}{32} \sin x$$

N.B. Alter 2 requires no equating of real/imaginary parts.

6. Solve the ODE $(D^4 + 10D^2 + 9)y = 40 \sin 2x \cos x + \cos^2 x + e^x$.
The A.E. of the associated homogeneous equation is $m^4 + 10m^2 + 9 = 0$. Its roots are $m = \pm 3i, \pm i$. So, the complementary solution,

$$y_c = c_1 \cos 3x + c_2 \sin 3x + c_3 \cos x + c_4 \sin x$$

$$\begin{aligned} \text{Now, } y_p &= \frac{1}{D^4 + 10D^2 + 9} (40 \sin 2x \cos x + \cos^2 x + e^x) \\ &= \frac{1}{D^4 + 10D^2 + 9} \left[20 (\sin 3x + \sin x) + \frac{1}{2} (1 + \cos 2x) + e^x \right] \\ &= 20 \cdot \frac{1}{(D^2 + 1)} \cdot \frac{1}{(D^2 + 9)} \sin 3x + 20 \cdot \frac{1}{D^2 + 9} \cdot \frac{1}{D^2 + 1} \sin x + \\ &\quad \frac{1}{(D^2 + 9)(D^2 + 1)} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{(D^2 + 9)(D^2 + 1)} \cos 2x \\ &\quad + \frac{1}{(D^2 + 9)(D^2 + 1)} e^x \\ &= 20 \cdot \frac{1}{D^2 + 1} \cdot \frac{x}{2} \int \sin 3x dx + 20 \cdot \frac{1}{D^2 + 9} \cdot \frac{x}{2} \int \sin x dx + \frac{1}{2} \cdot \frac{1}{9} \\ &\quad - \frac{1}{30} \cos 2x + \frac{e^x}{20} \\ &= -\frac{10}{3} \cdot \frac{1}{D^2 + 1} x \cos 3x - 10 \cdot \frac{1}{D^2 + 9} x \cos x + \frac{1}{18} - \frac{\cos 2x}{30} + \frac{e^x}{20} \\ &= \frac{5}{12} x \cos 3x - \frac{5}{16} \sin 3x - \frac{5}{4} x \cos x - \frac{5}{16} \sin x - \frac{\cos 2x}{30} + \frac{e^x}{20} + \frac{1}{18} \end{aligned}$$

(Computing 1st and 2nd terms separately cf. W.O. Ex. 5)

$\therefore y = y_c + y_p$ is the general solution.

7. Find the particular integral of

$$\frac{d^2 y}{dx^2} - 6 \frac{dy}{dx} + 25y = 2e^{3x} \cos 4x + 8(1 - 2x)e^{3x} \sin 4x.$$

Using operator D ,

$$\begin{aligned} y_p &= \frac{1}{D^2 - 6D + 25} [2e^{3x} \cos 4x + 8e^{3x} (1 - 2x) \sin 4x] \\ &= \frac{1}{D^2 - 6D + 25} \cdot 2e^{3x} \cos 4x + \frac{1}{D^2 - 6D + 25} 8e^{3x} (1 - 2x) \sin 4x \end{aligned}$$

$$\begin{aligned}
&= 2e^{3x} \frac{1}{(D+3)^2 - 6(D+3) + 25} \cos 4x \\
&\quad + 8e^{3x} \frac{1}{(D+3)^2 - 6(D+3) + 25} (1-2x) \sin 4x \\
&= 2e^{3x} \frac{1}{D^2 + 16} \cos 4x + 8e^{3x} \cdot \frac{1}{D^2 + 16} (1-2x) \sin 4x \\
&= 2e^{3x} \cdot \frac{x}{2} \int \cos 4x \, dx + 8e^{3x} \cdot I_p \\
&= \frac{1}{4} x e^{3x} \sin 4x + 8e^{3x} I_p \quad (1)
\end{aligned}$$

Now to find I_p

$$\text{let } X = \frac{1}{D^2 + 16} (1-2x) \cos 4x \text{ and } Y = \frac{1}{D^2 + 16} (1-2x) \sin 4x$$

$$\text{or, } X + iY = \frac{1}{D^2 + 16} (1-2x) e^{4ix} = e^{4ix} \frac{1}{(D+4i)^2 + 16} (1-2x)$$

$$= e^{4ix} \frac{1}{D^2 + 8iD} (1-2x) = \frac{e^{4ix}}{8i} \frac{1}{D} \left(1 + \frac{D}{8i}\right)^{-1} (1-2x)$$

$$= -\frac{ie^{4ix}}{8} \cdot \frac{1}{D} \left(1 - \frac{D}{8i} + \frac{D^2}{(8i)^2} - \dots\right) (1-2x)$$

$$= -\frac{ie^{4ix}}{8} \cdot \frac{1}{D} \left(1 - 2x - \frac{1}{8i}(-2)\right)$$

$$= -\frac{1}{8} (-\sin 4x + i \cos 4x) \cdot \left[x - x^2 + \frac{1}{4i}x\right]$$

$$= -\frac{1}{8} (-\sin 4x + i \cos 4x) \cdot \left(x - x^2 - \frac{i}{4}x\right)$$

$$\therefore I_p = Y = -\frac{1}{32} x \sin 4x - \frac{1}{8} (x - x^2) \cos 4x$$

$$\therefore y_p = \frac{1}{4} x e^{3x} \sin 4x - \frac{1}{4} e^{3x} x \sin 4x - (x - x^2) e^{3x} \cos 4x$$

$$= x(x-1) e^{3x} \cos 4x.$$

8. Solve the ODE $(D^2 + a^2)y = \cot ax$.

Here $y_c = A \cos ax + B \sin ax$.

For y_p , variation of parameters is to be used. So, let the general solution be

$$y = A(x) \cos ax + B(x) \sin ax \quad (1)$$

Differentiating (1) w.r.t. x

$$y' = -A(x) a \sin ax + B(x) a \cos ax \quad (2)$$

after setting the first condition on $A(x)$ and $B(x)$ as

$$A'(x) \cos ax + B'(x) \sin ax = 0 \quad (3)$$

Differentiating (2) again w.r.t. x ,

$$y'' = -A(x) a^2 \cos ax - B(x) a^2 \sin ax - A'(x) a \sin ax + B'(x) a \cos ax \quad (4)$$

Using (1), (2) and (4) in the given equation, we have,

$$-a^2 [A(x) \cos ax + B(x) \sin ax] - A'(x) a \sin ax + B'(x) a \cos ax + a^2 [A(x) \sin ax + B(x) a \cos ax] = \cot ax$$

$$\text{or, } -A'(x) a \sin ax + B'(x) a \cos ax = \cot ax \quad (5)$$

Solving for $A'(x)$ and $B'(x)$ from (3) and (5), we have,

$$A'(x) = -\frac{1}{a} \cos ax \text{ and } B'(x) = \frac{1}{a} \frac{\cos^2 ax}{\sin ax}$$

Integrating each of them separately, we get,

$$A(x) = -\frac{\sin ax}{a^2} + A_1 \text{ and } B(x) = \frac{1}{a^2} \left(\log \tan \frac{ax}{2} + \cos ax \right) + A_2$$

Using $A(x)$ and $B(x)$ in (1), we have the general solution of the given equation as

$$\begin{aligned} y &= \left(-\frac{\sin ax}{a^2} + A_1 \right) \cos ax + \left[\frac{1}{a^2} \left(\log \tan \frac{ax}{2} + \cos ax \right) + A_2 \right] \sin ax \\ &= \underbrace{(A_1 \cos ax + A_2 \sin ax)}_{y_c} + \underbrace{\frac{1}{a^2} \sin ax \log \tan \frac{ax}{2}}_{y_p} \end{aligned}$$

9. Solve the ODE

$$\frac{d^2 y}{dx^2} - 6 \frac{dy}{dx} + 25y = 2e^{3x} \cos 4x + 8e^{3x} (1-2x) \sin 4x.$$

This has already been solved Ex. 7. Now to use variation of parameters, let the general solution be

$$y = e^{3x} [A(x) \cos 4x + B(x) \sin 4x] \quad (1)$$

Diff. (1) w.r.t. ' x ' and setting the first condition for $A(x)$ and $B(x)$, we get

$$y' = e^{3x} [-4A(x) \sin 4x + 4B(x) \cos 4x + 3A(x) \cos 4x + 3B(x) \sin 4x] \quad (2)$$

$$\text{and } e^{3x} [A'(x) \cos 4x + B'(x) \sin 4x] = 0$$

$$\text{i.e. } A'(x) \cos 4x + B'(x) \sin 4x = 0 \quad (3)$$

Diff. again (2) w.r.t. x

$$\begin{aligned}
 y'' &= e^{3x} [-16A(x) \cos 4x - 16B(x) \sin 4x - \\
 &\quad 12A(x) \sin 4x + 12B(x) \cos 4x] \\
 &+ 3e^{3x} [-4A(x) \sin 4x + 4B(x) \cos 4x + 3A'(x) \cos 4x \\
 &\quad + 3B'(x) \sin 4x] \\
 &+ e^{3x} [-4A'(x) \sin 4x + 4B'(x) \cos 4x + 3A''(x) \cos 4x \\
 &\quad + 3B''(x) \sin 4x] \\
 &= e^{3x} [-7A(x) \cos 4x - 7B(x) \sin 4x - 24A(x) \sin 4x \\
 &\quad + 24B(x) \cos 4x \\
 &\quad - 4A'(x) \sin 4x + 4B'(x) \cos 4x + 3A''(x) \cos 4x + 3B''(x) \sin 4x] \quad (4)
 \end{aligned}$$

Using (1), (2) and (4) in the given equation, we have

$$\begin{aligned}
 e^{3x} [-7A(x) \cos 4x - 7B(x) \sin 4x - 24A(x) \sin 4x + 24B(x) \cos 4x \\
 + A'(x) \{3 \cos 4x - 4 \sin 4x\} + B'(x) \{4 \cos 4x + 3 \sin 4x\} \\
 + 24A(x) \sin 4x - 24B(x) \cos 4x - 18A(x) \cos 4x - 18B(x) \sin 4x \\
 + 25A(x) \cos 4x + 25B(x) \sin 4x] = 2e^{3x} [\cos 4x + 4(1-2x) \sin 4x] \\
 \text{or, } A'(x) \{3 \cos 4x - 4x - 4 \sin 4x\} + B'(x) \{4 \cos 4x + 3 \sin 4x\} \\
 = 2 [\cos 4x + 4(1-2x) \sin 4x] \quad (5)
 \end{aligned}$$

Solving for $A'(x)$ and $B'(x)$ from (3) and (5), we get,

$$A'(x) = -\frac{1}{4} \sin 8x - (1-2x)(1 - \cos 8x)$$

$$\text{and, } B'(x) = \frac{1}{4} (1 + \cos 8x) + (1-2x) \sin 8x$$

Integrating each of them separately,

$$A(x) = -(x-x^2) + \frac{1}{8} (1-2x) \sin 8x + A_1$$

$$\text{and } B(x) = \frac{1}{4} x - \frac{1}{8} (1-2x) \cos 8x + B_1$$

$\therefore$ the required general solution is

$$y = e^{3x} [(x^2 - x) \cos 4x + A_1 \cos 4x + B_2 \sin 4x] \text{ where } B_1 + \frac{1}{8} = B_2.$$

Exercises 21 (2)

Solve the following ODEs :

$$1. \frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} + 3y = x^5 + x^2. \quad 2. (D^2 + D - 6)y = x + 1.$$

$$3. (D^2 + 4)y = x^2. \quad 4. \frac{d^2 y}{dx^2} - \frac{dy}{dx} - 2y = 44 - 76x - 48x^2,$$

5. $(D^2 + 6D + 25)y = 104e^{3x}$.
6. $(D^2 + 2rD + r^2 + s^2)y = e^{ax}$.
7. $\frac{d^2 y}{dx^2} - 9y = 54e^{3x}$.
8. $\frac{d^2 y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{2x}$.
9. $(D^2 + 2D + 2)y = xe^{-x}$.
10. $(D^2 - 5D + 6)y = 100 \sin 4x$.
11. $(D - 2)^2 y = x^2 e^{2x}$.
12. $(D^2 + 1)y = \cos^2 x$.
13. $(D^2 + 4)y = \sin 2x$.
14. $(D^2 + 1)y = \cos x$.
15. $(D^2 + D + 1)y = \cos 2x$.
16. $(D^2 - 1)y = xe^x \sin x$.
17. $(D^2 - 4D + 4)y = 8x^2 e^{2x} \sin 2x$.
18. $(D^2 + 2D + 2)y = 2e^{-x} \sin x$.
19. $(D^2 + a^2)y = \cot ax$.
20. $(D^2 + 1)y = \operatorname{cosec} x$.
21. $(D^2 + 4)y = 4 \tan 2x$.
22. $(D^2 - 1)y = \frac{2}{1 + e^x}$.
23. $(D^2 - 4D + 4)y = e^{2x} \sec^2 x$.
24. $(D^2 + a^2)y = \sec ax$.
25. $(D^2 + 1)y = 3x - 8 \cot x$.
26. $(D^2 - 8D + 25)y = 5 \cos x$, given that $y = 0$, $Dy = 0$ when $x = 0$.
27. $(D^2 - 2D - 3)y = 2x + 1$, given that $y = 1/3$, $Dy = -4/9$ when $x = 0$.
28. $(D^5 - D^4 + 2D^3 - 2D^2 + D - 1)y = \cos x$.
29. $(D^4 + D^2 + 1)y = ax^2 + be^{-x} \sin 2x$.
30. $(D^5 - D)y = 12e^x + 8 \sin x - 2x$.
31. $(D^2 - 1)(D^3 - 1)y = xe^{ax} + 2 \cos x$.
32. $(D^3 + 1)y = x^3 + e^{-x} + \cos 2x$.
33. $(D^2 - 1)y = \cosh x \cos x$. [Hints : for y_p use exponential definition of hyperbolic functions. Simplify the y_p back to hyperbolic functions.]

Answers

1. $y = c_1 e^x + c_2 e^{3x} + \frac{1}{3}x^5 + \frac{20}{9}x^4 + \frac{260}{27}x^3 + \frac{809}{27}x^2 + \frac{4912}{81}x + \frac{13794}{243}$.
2. $y = c_1 e^{2x} + c_2 e^{-3x} - \frac{1}{30}x - \frac{71}{180}$.
3. $y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{4}x^2 - \frac{1}{8}$.
4. $y = Ae^{-x} + Be^{2x} + 24x^2 + 14x - 5$.
5. $y = e^{-3x} (A \cos 4x + B \sin 4x) + 2e^{3x}$.
6. $y = e^{-rx} (A \cos sx + B \sin sx) + e^{ax} / \{(a+r)^2 + s^2\}$.
7. $y = (A + 9x)e^{3x} + Be^{-3x}$.
8. $y = c_1 e^x + c_2 e^2 + xe^{2x}$.
9. $y = e^{-x} (A \sin x + B \cos x) + xe^{-x}$.
10. $y = Ae^{2x} + Be^{3x} + 4 \cos 4x - 2 \sin 4x$.
11. $y = (c_1 + c_2 x)e^{2x} + x^4 e^{2x} / 12$.
12. $y = c_1 \cos x + c_2 \sin x + \frac{1}{2} - \frac{1}{6} \cos 2x$.

$$13. y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{2}x \cos 2x. \quad 14. y = c_1 \cos x + c_2 \sin x + \frac{1}{2}x \sin x.$$

$$15. y = e^{-\frac{1}{2}x} \left(c_1 \cos (x\sqrt{3}/2) + c_2 \sin (x\sqrt{3}/2) \right) + \frac{2}{13} \sin 2x - \frac{1}{13} \cos 2x.$$

$$16. y = c_1 e^x + c_2 e^{-x} + e^x/25 + [(14-5x) \sin x - 2(5x+1) \cos x].$$

$$17. y = e^{2x} (c_1 + c_2 x + 3 \sin 2x - 2x^2 \sin 2x - 4 \cos 2x).$$

$$18. y = (A \sin x + (B-x) \cos x) e^{-x}.$$

$$19. y = A \cos ax + B \sin ax + \frac{1}{a^2} (\sin ax) \log \tan \frac{ax}{2}.$$

$$20. y = (A-x) \cos x + (B + \log \sin x) \sin x.$$

$$21. y = \left[A - \log \tan \left(\frac{\pi}{4} + x \right) \right] \cos 2x + B \sin 2x.$$

$$22. y = \{A - e^{-x} + \log(1 + e^{-x})\} e^x + \{B - \log(1 + e^x)\} e^{-x}.$$

$$23. y = (A + Bx + \frac{1}{4} \log \sec 2x) e^{2x}.$$

$$24. y = A \cos ax + B \sin ax + \frac{x}{a} \sin ax + \frac{1}{a^2} \log(\cos ax) \cdot \cos ax.$$

$$25. y = \left\{ -\frac{3}{4}x(2x - \sin 2x) + \frac{3}{8}(2x^2 + \cos 2x) + 11 \cos x + A \right\} \cos x \\ + \{3x \sin x + 11 \cos x + B\} \sin x.$$

$$26. y = e^{4x} (18 \cos 6x - 13 \sin 6x)/96 + (\sin x - 3 \cos x)/16.$$

$$27. y = \frac{8}{9} e^{-x} - \frac{4}{3}x - \frac{5}{9}.$$

$$28. y = Ae^x + (B + Cx) \cos x + (D + Ex) \sin x + \frac{1}{16} x^2 (\cos x - \sin x).$$

$$29. y = e^{x/2} (A \cos (x\sqrt{3}/2) + B \sin (x\sqrt{3}/2)) \\ + e^{-x/2} (C \cos (x\sqrt{3}/2) + D \sin (x\sqrt{3}/2)) - \frac{be^{-x}}{481} (20 \cos 2x + 9 \sin 2x).$$

$$30. y = A + Bx + Ce^{-x} + (D \cos x + E \sin x) + 2x \sin x + x^2.$$

$$31. y = (A + Bx) e^x + Ce^{-x} + e^{-x/2} (D \cos (x\sqrt{3}/2) + E \sin (x\sqrt{3}/2)) + \frac{e^x}{72} (2x^3 - 9x^2).$$

$$32. y = Ae^{-x} + e^{x/2} (B \cos (x\sqrt{3}/2) + C \sin (x\sqrt{3}/2)) + \frac{1}{65} (\cos 2x - 8 \sin 2x).$$

$$33. y = c_1 e^x + c_2 e^{-x} + \frac{1}{5} (2 \sinh x \sin x - \cosh x \cos x).$$

21.6 APPLICATIONS

ODE with constant coefficients is used in the mathematical modelling of various real life problems and its solution is used in the study of those problems. As "Mathematics is the science of killing many birds with one stone". Thus one *ODE* may be used in the study of planetary and satellite motion, mechanical vibration, electric flow through series circuit, economic system etc. So, knowledge of one system may be easily transferred to another system with necessary changes of interpretation. Thus a prototype mechanical vibrational problem is considered in the following section.

20.6.1 ODE FOR MECHANICAL VIBRATION AND ELECTRIC CIRCUIT

Let a mass m be hanging at the end of an elastic spring in vacuum and the spring is stretched to s unit. Let the spring obey's Hooke's law : force is proportional to displacement. So, force $F = mg$, g is the acceleration due to gravity, is proportional to s i.e. $F = ks$, the constant of proportionality k is called the modulus of elasticity of the spring. Let the mass be set in vertical motion from its equilibrium position and its downward displacement, y be regarded positive. So, y displacement of mass causes a force of $[-k(y + s)]$ on the spring and the weight of the mass is mg . Using Newton's second law of motion,

$$-k(y + s) + mg = m \frac{d^2 y}{dt^2} \text{ or } m \frac{d^2 y}{dt^2} = -ky \quad \because F = mg = ks$$

$$\text{or,} \quad my'' + ky = 0 \quad (1)$$

This is the second order linear homogeneous ODE with constant coefficient for the displacement y in time t and is called the ODE for harmonic oscillator. If an external force $F(t)$ is applied to the mass, the equation (1) becomes $my'' + ky = F(t)$ (2)

Again for any physical problem friction or damping is natural and damping is proportional to velocity, so the equation (2) takes form

$$my'' + ry' + ky = F(t) \quad (3)$$

when damping is counted, r is a positive constant of proportionality. The equation (3) is called ODE for damped and forced harmonic oscillator.

The analogy of equation (3) with electric flow of current in series circuit can be easily seen. The ODE for R - L circuit in 19.3.3 (3) is $L \frac{di}{dt} + Ri = E(t)$ and that for R - C circuit is $Ri + \frac{1}{C} \int i dt = E(t)$, $i = \frac{dQ}{dt}$ or $R \frac{dQ}{dt} + \frac{Q}{C} = E(t)$. But in R - L - C circuit (fig. 158) the potential difference

or voltage drop in inductance L is $L \frac{di}{dt}$. Now using Kirchhoff's law,

$$L \frac{di}{dt} + Ri + \frac{1}{C} \int i dt = E(t) \quad (4)$$

$\therefore i = \frac{dQ}{dt}$. $\therefore$ (4) becomes

$$L \frac{d^2 Q}{dt^2} + R \frac{dQ}{dt} + \frac{Q}{C} = E(t) \quad (5)$$

Now, analogy between equation (3) and (5) is tabulated below :

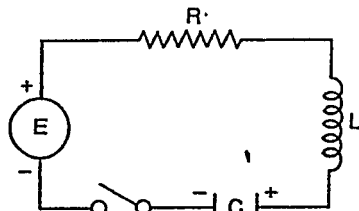


Fig. 158

Mechanical quantities		Electric quantities
displacement, y	$\leftrightarrow$	charge, Q
mass, m	$\leftrightarrow$	inductance, L
friction, r	$\leftrightarrow$	resistance, R
modulus of elasticity, k	$\leftrightarrow$	reciprocal of capacitance, $\frac{1}{C}$
external force, $F(t)$	$\leftrightarrow$	e.m.f., $E(t)$
velocity $v = y'$	$\leftrightarrow$	current $i = \frac{dQ}{dt}$

21.6.2 SIMPLE HARMONIC MOTION (S.H.M.)

The solution and the properties for equation (3) of 21.6.1 will be discussed under different circumstances.

(i) **Free undamped vibration** : Since the system is free, so R.H.S. of (3) or (5) equals to zero and it being undamped, so $r=0$ in (3) and $R=0$ in (5). Hence the equation (3) and (5) reduces to $my'' + ky = 0$ according to equation (3). Its general solution is

$$y = c_1 \cos(\sqrt{k/m})t + c_2 \sin(\sqrt{k/m})t = c_1 \cos \omega t + c_2 \sin \omega t \quad (6)$$

when $\omega = \sqrt{k/m}$ (7)

Since in equation (6) both $\cos \omega t$ and $\sin \omega t$ are periodic functions with common period $(2\pi/\omega)$. Therefore, the equation of motions governed by equation (1) or by (6) are *simple harmonic*. i.e. **A body executes simple harmonic motion (S.H.M.) if it is attracted to a fixed point by a force proportional to the distance of the body from the fixed point.** Therefore, free undamped or frictionless motion is simple harmonic. Examples are the motion of helical spring and simple pendulum, sinking of a floating body from equilibrium etc.

In equation (7) $\omega = \sqrt{k/m}$ is called the **circular frequency** and

$$f_n = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{k}{m}} \quad (8)$$

is called the **natural frequency** or simply **frequency** of the system and

$$T = \frac{1}{f_n} = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{k}} \quad (9)$$

is called **period** of the system (fig. 159).

The solution (6) can be further simplified. Let $c_1 = A \cos \delta$ and $c_2 = A \sin \delta$ where A and δ are constants.

$$\therefore A = \sqrt{c_1^2 + c_2^2} \text{ and } \delta = \tan^{-1} \left(\frac{c_2}{c_1} \right)$$

So, the solution (6) takes the form

$$y = A \cos \delta \cos \omega t + A \sin \delta \sin \omega t \\ = A \cos (\omega t - \delta) \quad (10)$$

Here A is called the amplitude (fig. 159) and δ is called the phase of the system (fig. 159).

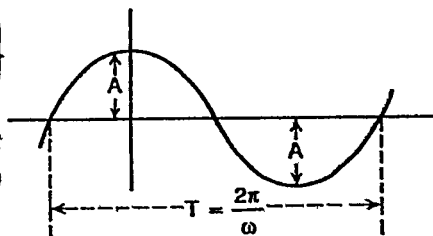


Fig. 159

(ii) **Forced undamped vibrations** : In this case, in equation (3), $r = 0$ and $F(t) \neq 0$ and in equation (5) $R = 0$, $E(t) \neq 0$. Therefore, equation (3) reduces to

$$my'' + ky = F(t) \quad (11)$$

In terms of computer technology, the $F(t)$ is called the **input** and the y is called the **output**. The general solution of (11) is of the form $y = y_c + y_p$. This solution will be in S.H. if y_p is periodic as y_c is already seen in (i) as S.M. If y remains finite with time t , then the motion is **stable**, otherwise **unstable**. Normally, the frequency of $F(t)$ and y_c are unequal for stable vibration and equal for unstable case. In the latter case y_c is called the **undamped resonance frequency**.

(iii) **Free damped vibration** : In this case, in equation (3) $r \neq 0$, $F(t) = 0$ and in equation (5) $r \neq 0$, $E(t) = 0$. Therefore, equation (3) becomes,

$$my'' + ry' + ky = 0 \quad (12)$$

In this case, the motion is not S.H. because oscillation decreases with time. So, the equation is called damped harmonic oscillation equation. The A.E. equation of (12) is,

$$m\lambda^2 + r\lambda + k = 0 \quad \therefore \lambda = \frac{-r \pm \sqrt{r^2 - 4km}}{2m} \quad (13)$$

$$= \frac{-r \pm \sqrt{\Delta}}{2m}, \quad \Delta = r^2 - 4km \quad (13a)$$

There arise three cases : (a) $\Delta > 0$, (b) $\Delta = 0$, (c) $\Delta < 0$

Case (a) $\Delta > 0$, **Over damped case** : When $\Delta > 0$ then λ has two real distinct values $\delta = -a \pm b$ where $a = \frac{r}{2m}$ and $b = \frac{\sqrt{r^2 - 4km}}{2m}$.

$\therefore$ the solution of (12) is

$$y = c_1 e^{-(a-b)t} + c_2 e^{-(a+b)t} \quad (14)$$

This solution (14) is **not periodic**, since exponential functions are not periodic, and as t increases y dies out. Again, $c_1 \neq 0$, $c_2 \neq 0$ and c_1, c_2 are of

same sign, y will never be 0 but for c_1, c_2 of opposite sign $y = 0$ only once. As the damping force overpowers the restoring force and hinders in oscillation, the motion of this type is called the *over damped motion*.

Case (b) $\Delta = 0$, Critically damped motion : Here roots of the A.E. are equal $\lambda = -\frac{r}{2m}, -\frac{r}{2m}$. Therefore the general solution of (12) is

$$y = (c_1 + c_2 t) e^{-(r/2m)t} \quad (15)$$

This is also similar to case (a) and $y \rightarrow 0$ as $t \rightarrow \infty$. Also damping force is equal to restoring force and hinders oscillation, so this type of motion is called **critically damped motion**.

Case (c) $\Delta < 0$ Damped Oscillatory motion : When $\Delta < 0$ i.e. $r^2 < 4km$, the roots of the A.E. are imaginary i.e.

$$\lambda = -\alpha \pm i\beta \text{ where } \alpha = \frac{r}{2m} \text{ and}$$

$$\beta = \frac{\sqrt{4km - r^2}}{2m} = \sqrt{\frac{k}{m} - \left(\frac{r}{2m}\right)^2} = \sqrt{\omega^2 - \alpha^2} \quad (16)$$

$\therefore$ the general solution of (12) is

$$y = e^{-\alpha t} (c_1 \cos \beta t + c_2 \sin \beta t) \quad (17)$$

$$= A e^{-\alpha t} \cos (\beta t - \delta) \quad (18)$$

Here y is periodically negative and oscillates with period $2\pi/\sqrt{\omega^2 - \alpha^2}$. But as $e^{-\alpha t} \rightarrow 0$ as $t \rightarrow \infty$, the amplitude $A e^{-\alpha t}$ gradually decreases and vanishes. So, $A e^{-\alpha t}$ is called **damping factor**. This type of motion is **damped oscillatory motion**.

In all the above three cases $e^{-\alpha t} \rightarrow 0$ as $t \rightarrow \infty$ i.e. as t increases, the motion gradually dies. So, these motions are not simple harmonic. These are called **transient motion**.

(iv) Forced damped oscillations / vibrations : Here the support of the spring is itself vibrating in mechanical system and in electric system $e.m.f.$ is a function of t . If damping (or resistance) and external force together act upon to create motion then that motion is called **forced damped vibration** i.e. $r \neq 0$, $F(t) \neq 0$ and the equation (3) remains unchanged. Its general solution is

$$y = y_c + y_p \quad (19)$$

$\therefore$ the solution of (12), $y_c \rightarrow 0$ as $t \rightarrow \infty$. So, the solution of (19) becomes $y = y_p$ as $t \rightarrow \infty$ and increases indefinitely with the increase of time. Such a motion is called **steady state motion** or **vibration**. Its period

$$= \frac{2\pi}{\text{Frequency of } F(t)} \text{ and Frequency} = \frac{\text{frequency of } F(t)}{2\pi}.$$

Again, with the presence of friction or damping, the frequency of $F(t)$ and that of the mechanical body are unequal, then the amplitude of steady vibration or motion is maximum when $2m^2$ (frequency of $F(t)$)² = $2mk - r^2$. In this situation $F(t)$ creates resonance with the mechanical body.

A body destroys when its self frequency as well as its internal friction are minimum in comparison with the frequency of the external force.

In engineering education resonance plays an important role. In radiating advantages of electric resonance are taken. Car, aeroplane, ship, bridge's etc. construction requires mechanical vibrational resonance.

21.6.2 BENDING OF BEAMS

A beam may be simple and cantilever. A simple beam is one which rests on two supports at the two ends while a cantilever beam rests on one support at one end.

Let a simple beam be considered such that

(i) it is longer than its width and thickness; (ii) its cross-section is uniform and (iii) C.G. of each cross-section is collinear. The C.G. of a rectangular cross-sectional simple beam of length L is the C.G. of a geometrical rectangle. Let the axis of the simple beam be the line joining $(0, 0)$ and $(L, 0)$. Let the length of each fibre be L ft. and each be parallel to the axis of the beam. This simple beam will create a sag when a load is applied to it. For this sag, the fibres of one side will be compressed and that on the other side will be stretched and between these two stages there exists a neutral surface of the original length L . The curve of any particular fiber in the neutral surface is known as the elastic curve or deflection curve of the beam. Let this curve joins $(0, 0)$ and $(L, 0)$.

In fig. 160 Δx is a small part of bent beam. $g - g'$ is its neutral surface and its radius of curvature is R . Let the vertical planes AB and CD intersect at P and since $BP = DP$ and $gP = g'P$; therefore the centre of curvature of Δx lies at P .

From Euclidian geometry central angles of the same circular part are proportional to their radii. Therefore, if a fibre ST at a distance Z from the neutral surface bends from its original Δx by e , then the relation between Δx and its radius of curvature R is

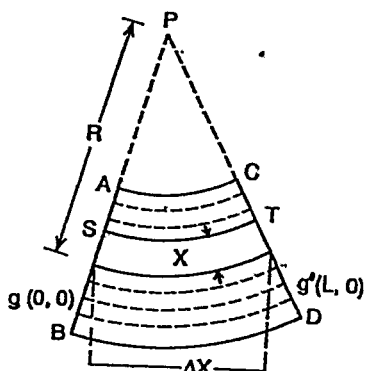


Fig. 160

$$\frac{\Delta x}{\Delta x - e} = \frac{R}{R - Z} \quad \text{or} \quad \frac{Z}{R} = \frac{e}{\Delta x}$$

Again by Hooke's law, strain (per unit stretched) is proportional to stress *f* i.e.

$$f = E \frac{e}{\Delta x}, \quad E \text{ Young's modulus}$$

or,
$$f = E \frac{Z}{R}.$$

If the cross-sectional area of *ST* on the surface *AB* or *CD* be *dA*, then the applied force on the fibre *ST* is

$$dF = f \, dA = E \cdot \frac{Z}{R} \cdot dA$$

$$\therefore \text{integrating, total force is } F = \int E \cdot \frac{Z}{R} \, dA = \frac{E}{R} \int Z \, dA$$

As the beam carries transverse load and remains static equilibrium, so tensile and compressive forces throughout the whole cross-section remains equilibrium i.e. total force $F = 0$. So,

$$F = \frac{E}{R} \int Z \, dA = 0 \quad \therefore \int Z \, dA = 0 \quad \because \frac{E}{R} \neq 0$$

i.e. the moment of an area about an axis from which *Z* is measured is zero. This is only true for the lines of the centroidal axes. Clearly neutral axis crosses the centroidal axis.

Again, when the force acts perpendicularly to a cross-section (say *AB*), then the moment due to a differential force at a distance of *Z* units from the neutral axis is

$$dM = Z \, dF = Z \cdot E \cdot \frac{Z}{R} \cdot dA = \frac{E}{R} Z^2 \, dA$$

$$\therefore M = \frac{E}{R} \int Z^2 \, dA = \frac{E}{R} I, \quad \because I = \int Z^2 \, dA = \text{moment of inertia.}$$

But radius of curvature,

$$R = \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{3/2} \bigg/ \left(\frac{d^2y}{dx^2} \right) = 1 \bigg/ \left(\frac{d^2y}{dx^2} \right)$$

$\therefore \frac{dy}{dx}$ is small for small deflection of the beam and $\left(\frac{dy}{dx} \right)^2$ is neglected.

Therefore,

$$M = EI \frac{d^2y}{dx^2} \quad \text{or} \quad \frac{d^2y}{dx^2} = \frac{M}{EI}$$

This is the required *ODE* for the **bending moment** of a simple beam. In this equation x is the horizontal length of the beam, y vertical deflection, M bending moment, E Young's modulus and I is the moment of inertia.

For a **cantilever beam**, Moment = (sum of the algebraic moments due to the applied forces at each point of the free end of the beam) and the forces are equal to the sum of the vertical shears at those points right to the points under consideration. Let $P_1, P_2, P_3, \dots, P_n$ be vertical forces act at $x_1, x_2, \dots, x_n$ right to x and $x + \Delta x$ (fig. 161). Then the moment at x and $x + \Delta x$ are

$$M_x = P_1(x - x_1) + P_2(x - x_2) + \dots + P_n(x - x_n)$$

$$\text{and } M_{x+\Delta x} = P_1(x + \Delta x - x_1) + P_2(x + \Delta x - x_2) + \dots + P_n(x + \Delta x - x_n)$$

$$\therefore \Delta M = M_{x+\Delta x} - M_x$$

$$= (P_1 + P_2 + \dots + P_n) \Delta x$$

$$= (\text{Vertical deflection, } V) \Delta x$$

$$\begin{aligned} \text{or, } \frac{\Delta M}{\Delta x} &= V \quad \text{or} \quad \lim_{\Delta x \rightarrow 0} \frac{\Delta M}{\Delta x} = V \quad \text{or} \quad V = \frac{dM}{dx} = EI \frac{d}{dx} \left(\frac{d^2 y}{dx^2} \right) \\ &= EI \frac{d^3 y}{dx^3} \end{aligned}$$

This is the *ODE* for **vertical deflection** of a cantilever beam.

21.6.3 FLOW OF HEAT

In this section flow of heat through a rod and a fin will be considered.

(A) **Through a rod** : Let the rod satisfies the following conditions :

(i) flow of heat is steady i.e. temperature does not change at any point of the rod with time, (ii) the rod is homogeneous and its thermal conductivity is constant, (iii) surrounding temperature is uniform, (iv) heat transfer is constant, (v) no temperature gradient towards the end of the rod. Length of the rod should be enough compare to diameter so that temperature gradient remains active along the length only, (vi) negligible amount of heat radiates from the end.

Let A = cross-section of rod, ft^2

b = length of the rod, ft

h = coefficient of heat conduction, $Btu/\text{ft}^2 - \text{hr} - ^\circ F$

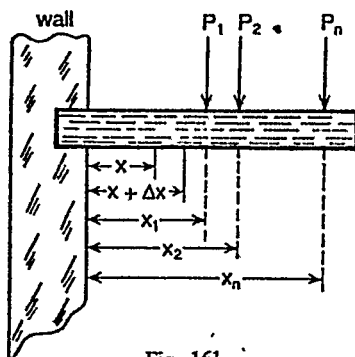


Fig. 161

d = diameter, ft

k = thermal conductivity, $Btu/ft-hr-^{\circ}F$

q = heat flow, Btu/hr

s = external heat flow surface, ft^2

T = temperature, $^{\circ}F$

T_s = surrounding temperature, $^{\circ}F$

x = length co-ordinate, ft (fig. 162).

Now, from Fourier's law of heat conduction, heat flow in the

length x is $q_x = -KA \frac{dT}{dx} \Big|_x$ and heat

flow out of the length $x + \Delta x$ is

$$q_{x+\Delta x} = -KA \frac{dT}{dx} \Big|_{x+\Delta x}$$

Also from Newton's law of surrounding surface (Δs) heat transmission $q_s = h \Delta s (T - T_s)$. But (heat loss) = (heat gain),

$$\therefore q_x = q_{x+\Delta x} + q_s$$

$$\text{or, } -KA \frac{dT}{dx} \Big|_x = -KA \frac{dT}{dx} \Big|_{x+\Delta x} + h \Delta s (T - T_s)$$

$$\text{or, } K \frac{\pi}{4} d^2 \left[\frac{dT}{dx} \Big|_{x+\Delta x} - \frac{dT}{dx} \Big|_x \right] = h d \pi (T - T_s) \Delta x$$

$$\therefore A = \frac{\pi d^2}{4} \text{ and } \Delta s = \pi d \Delta x$$

$$\text{or, } \frac{\frac{dT}{dx} \Big|_{x+\Delta x} - \frac{dT}{dx} \Big|_x}{\Delta x} = \frac{4h}{Kd} (T - T_s)$$

$$\text{or, } \lim_{\Delta x \rightarrow 0} \frac{\frac{dT}{dx} \Big|_{x+\Delta x} - \frac{dT}{dx} \Big|_x}{\Delta x} = \frac{4h}{Kd} (T - T_s)$$

$$\text{or, } \frac{d^2 T}{dx^2} = \frac{4h}{Kd} (T - T_s).$$

This is a non-homogeneous linear second order ODE with

I.C. : $T = T_0$ at $x = 0$ and B.C. $\frac{dT}{dx} = 0$ at $x = b$.

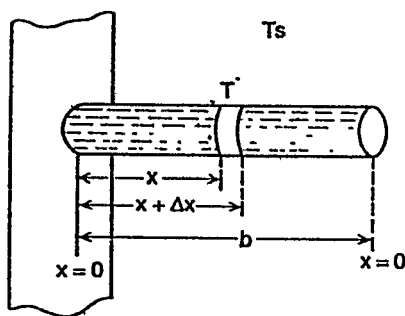


Fig. 162

(B) Through a rectangular fin : Let the origin of reference for x -coordinate be the base of the fin and positive direction of x -axis be towards the end of the fin. Here $A = \delta L$, $\Delta s = 2(L + \delta) \Delta x = 2L \Delta x$, $\therefore L \gg \delta, b \gg \delta$. Difference of temperature between Δx and its surroundings, $\theta = T - T_s$.

$\therefore d\theta = dT$ (Symbols remain as above) (fig. 163).

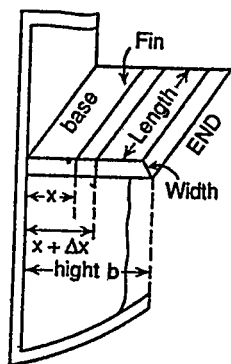


Fig. 163

$$\text{As in (A), } q_x = -KA \frac{dT}{dx} \Big|_x = -K\delta L \frac{d\theta}{dx} \Big|_x$$

$$q_{x+\Delta x} = -KA \frac{dT}{dx} \Big|_{x+\Delta x} = -K\delta L \frac{d\theta}{dx} \Big|_{x+\Delta x}$$

$\therefore$ heat loss = heat gain,

$$\therefore q_x = q_{x+\Delta x} + q_s$$

$$\text{or, } -K\delta L \frac{d\theta}{dx} \Big|_x = -K\delta L \frac{d\theta}{dx} \Big|_{x+\Delta x} + 2Lh\Delta x \cdot \theta$$

$$\text{or, } \frac{\frac{d\theta}{dx} \Big|_{x+\Delta x} - \frac{d\theta}{dx} \Big|_x}{\Delta x} = \frac{2h}{K\delta} \theta$$

or taking limit when $\Delta x \rightarrow 0$, we have,

$$\frac{d^2 \theta}{dx^2} - m^2 \theta = 0 \text{ where } m^2 = \frac{2h}{K\delta}$$

If radiation is not negligible, then radiation depends on $\Delta s = P \cdot \Delta x$ where P is a parameter of the fin. Therefore heat equilibrium is

$$-K\delta L \frac{d\theta}{dx} \Big|_x = -K\delta L \frac{d\theta}{dx} \Big|_{x+\Delta x} + hP\Delta x \cdot \theta$$

$\therefore$ taking limit when $\Delta x \rightarrow 0$, the required ODE becomes

$$\frac{d^2 \theta}{dx^2} - m^2 \theta = 0 \text{ where } m^2 = \frac{hP}{K\delta L} = \frac{hP}{KA}$$

with I.C. : $\theta = \theta_0 = T_0 - T_s$ at $x = 0$ and B.C. : $\frac{d\theta}{dx} = 0$ at $x = b$.

Examples

1. A body of weight 16 lbs is moving along a horizontal line and is attracted towards the origin by a force proportional to its distance from the origin. The force of attraction is 9 lbs at $x = -2$. It is also acted upon by an external applied force of $\sin 3t$. If $x = 2$ and $v = 0$ at $t = 0$ then (a) find the equation of motion and (b) also find the resonant frequency of the system.

Here force of attraction = 9 lbs and applied force = $\sin 3t$. So, the ODE of system is $x'' + 9x = \sin 3t$. Its general solution i.e. general equation of motion is

$$x = c_1 \cos 3t + c_2 \sin 3t - \frac{1}{6} t \cos 3t \quad (1)$$

But I.C. : $x = 2$ and $v = 0$ at $t = 0$. Therefore,

$$2 = c_1 \text{ and } x' = v = 0 = 3c_2 - \frac{1}{6} \text{ i.e. } c_1 = 2, c_2 = \frac{1}{18}$$

$\therefore$ the required equation of motion is

$$x = 2 \cos 3t + \frac{1}{18} \sin 3t - \frac{1}{6} t \cos 3t.$$

Second part 1 : $x = \left(2 - \frac{1}{6} t\right) \cos 3t + \frac{1}{18} \sin 3t$

$$= r \cos(3t - \theta) \text{ where } r = \sqrt{\left(2 - \frac{1}{6} t\right)^2 + \left(\frac{1}{18}\right)^2},$$

$$\omega = 3 \text{ and } \theta = \tan^{-1} \left\{ \frac{1}{3(12 - t)} \right\}.$$

$\therefore$ the max. displacement $r = \sqrt{\left(2 - \frac{1}{6} t\right)^2 + \frac{1}{18^2}}$ which is a function of t and r increases as t increases. So, the motion is unstable. Its undamped resonant frequency = $\omega = 3$ rad/sec. or $\frac{3}{2\pi}$ cycle/sec.

2. A capacitor of $\frac{2}{1010}$ farad, a resistance of 1 ohm and an inductance of $(1/20)$ henry are connected in series. At $t = 0$, $i = 0$, charge of the capacitor is 1 coulomb. Find the charge and current in time $t = .01$ sec.

Here $E(t) = 0$, $C = \frac{2}{1010}$, $L = \frac{1}{20}$ and $R = 1$.

I.C. $i = 0$ at $t = 0$ and $Q = 1$ at $t = 0$. B.C. $\bar{i} = ?$, $\bar{Q} = ?$ at $t = .01$ sec.

From equation (5) of 21.6.1, we have

$$L \frac{d^2 Q}{dt^2} + R \frac{dQ}{dt} + \frac{Q}{C} = E(t)$$

$$\text{or, } \frac{1}{20} \frac{d^2 Q}{dt^2} + 1 \cdot \frac{dQ}{dt} + 505 Q = 0$$

$$\text{or, } \frac{d^2 Q}{dt^2} + 20 \frac{dQ}{dt} + 10,100 Q = 0$$

Its general solution is

$$Q = e^{-10t} (c_1 \cos 100t + c_2 \sin 100t) \quad (1)$$

$$\begin{aligned} \therefore i = \frac{dQ}{dt} &= -10 e^{-10t} (c_1 \cos 100t + c_2 \sin 100t) \\ &\quad + e^{-10t} (-100 c_1 \sin 100t + 100 c_2 \cos 100t) \end{aligned} \quad (2)$$

Using I.C. in (1) and (2) we get $c_1 = 1$ and $c_2 = 1$

$\therefore$ charge Q at $t = 0.01$ sec, is

$$Q = e^{-1} (\cos 1 + 1 \cos 1) = 0.538 \text{ coulomb}$$

and current flow is

$$\begin{aligned} i &= -10 e^{-1} (\cos 1 + 1 \sin 1) + e^{-1} (-100 \sin 1 + 100 \cos 1) \\ &= -101 e^{-1} \sin 1 = -76.9 \text{ amperes} \end{aligned}$$

3. A 16 lbs weight stretches a spring by 1 ft. It is immersed in a medium whose coefficient of friction is 4. External force of $10 \sin 2t$ is applied in its equilibrium position. (a) Find its equation of motion, (b) Determine its transient and steady speed, (c) magnification. (d) Also find the amplitude, period and frequency of steady speed.

$$\text{Here } W = mg = 16 \text{ lbs. } \therefore m = \frac{1}{2} \text{ slug, } s = 1 \text{ ft.}$$

$$L(t) = 10 \sin 2t, r = 4$$

$$\text{I.C.: } x = 0, x' = 0 \text{ at } t = 0.$$

From equation (3) of 21.6.1, we have,

$$\frac{1}{2} x'' + 4x' + 16x = 10 \sin 2t$$

$$\text{or, } x'' + 8x' + 32x = 20 \sin 2t$$

Its general solution i.e. general equation of motion is

$$x = x_c + x_p = e^{-4t} (c_1 \cos 4t + c_2 \sin 4t) + \frac{1}{13} (7 \sin 2t - 4 \cos 2t)$$

Using I.C., we get $c_1 = (4/13)$ and $c_2 = (1/26)$.

$\therefore$ (a) The required equation of motion is

$$x = \frac{e^{-4t}}{26} (8 \cos 4t + \sin 4t) + \frac{1}{13} (7 \sin 2t - 4 \cos 2t)$$

$$(b) \text{ Transient speed} = \frac{e^{-4t}}{26} (8 \cos 4t + \sin 4t)$$

$$\text{and steady speed} = \frac{1}{13} (7 \sin 2t - 4 \cos t)$$

$$(c) \text{ Magnification} = \frac{\text{Amplitude of } x_p}{(\text{Coeff. of applied force})/(\text{Coeff. of } x \text{ in ODE})}$$

$$= \frac{\sqrt{(7/13)^2 + (4/13)^2}}{(20)/32} = (8/\sqrt{65})$$

(d) $\omega = 2$, so amplitude, period and frequency are respectively

$$(\sqrt{65}/13), \frac{2\pi}{2} = \pi \text{ sec and } \frac{1}{\pi} \text{ cycle/sec.}$$

4. A simple beam $2L$ ft in length rests on two supports at its ends. The beam sags due to putting an uniform load of q lbs/ft on it. The weight of beam is negligible in comparison to the applied load. (a) Find the elastic curve and max. deflection of the beam. (b) Calculate (a) if q is replaced by W lb/ft at the middle of the beam.

Let origin at O be the left of the simple beam. Let $P(x, y)$ be a point on the elastic curve. Let upward thrust at O be qL and at the middle point of OP the downward thrust be qx (fig. 164).

$\therefore$ moment at P is $M =$
(moment at P due to upward thrust qL)
– (moment at the mid point of OP
due to downward thrust)

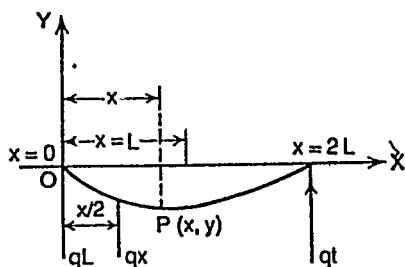


Fig. 164

$$= qLx - qx \left(\frac{1}{2} x \right) = qLx - \frac{1}{2} qx^2.$$

$\therefore$ ODE for elastic beam is $\frac{d^2 y}{dx^2} = \frac{M}{EI}$ becomes

$$EI \frac{d^2 y}{dx^2} = qLx - \frac{1}{2} qx^2$$

$$\text{Integrating w.r.t. } x, EI \frac{dy}{dx} = \frac{1}{2} qLx^2 - \frac{1}{6} qx^3 + c_1$$

$$\text{But at the middle point } x = L, \frac{dy}{dx} = 0. \therefore c_1 = -\frac{1}{3} qL^3.$$

$$\therefore EI \frac{dy}{dx} = \frac{1}{2} qLx^2 - \frac{1}{6} qx^3 - \frac{1}{3} qL^3$$

Integrating again,

$$\therefore EIy = \frac{1}{6} qLx^3 - \frac{1}{24} qx^4 - \frac{1}{3} qL^3x + c_2$$

Again at $O, x=0, y=0. \therefore c_2=0$

$$\therefore y = \frac{q}{24EI} (4Lx^3 - x^4 - 8L^3x)$$

Now, the deflection of beam at a distance x from O is $(-y)$ and the max. deflection is at the middle. Therefore, max. deflection is

$$(-y)_{\max} = -\frac{q}{24EI} (4L^3 - L^4 - 8L^3L) = \frac{5qL^4}{24EI}.$$

(b) When the load W is at the middle of the beam, then there arise two cases $-P(x, y)$ on the (i) left or (ii) right of the middle.

Case (i) : (Fig. 165) $P(x, y)$ lies on the left of the middle point of the beam. Therefore, moment at P is

$$M = \left(\text{moment of } \frac{W}{2} \text{ at } 0 \right) = \frac{Wx}{2}$$

$\therefore$ the ODE for elastic beam is $EI \frac{d^2y}{dx^2} = \frac{W}{2}x$. Its general solution

$$\text{is } EIy = \frac{W}{12}x^3 + c_1x + c_2.$$

$$\therefore \text{ at } x=0, y=0 \text{ and at } x=L, \frac{dy}{dx}=0. \therefore c_1 = -\frac{W}{4}L^2 \text{ and } c_2=0.$$

Therefore, the required equation of the elastic curve is

$$y = \frac{1}{12EI} Wx(x^2 - 3L^2), 0 < x < L.$$

$$\therefore \text{ the max. deflection is at } x=L \text{ and it is } (-y)_{\max} = \frac{1}{6EI} WL^3$$

Case (ii) : (Fig. 166) $P(x, y)$ lies on the right of the middle point of the beam. Now moment at P is

$$\begin{aligned} M &= (\text{moment at } O \text{ due to } W/2) - (\text{moment at } (L, 0) \text{ due to } W) \\ &= \frac{W}{2}x - W(x-L) \end{aligned}$$

$\therefore$ the ODE for elastic beam

is

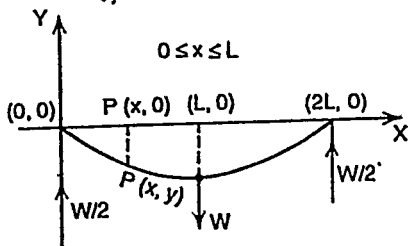


Fig. 165

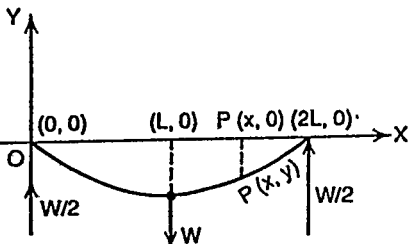


Fig. 166

$$EI \frac{d^2 y}{dx^2} = \frac{W}{2}x - W(x-L)$$

$$\text{Integrating, } EI \frac{dy}{dx} = \frac{W}{4}x^2 - W\left(\frac{1}{2}x^2 - Lx\right) + c_1$$

$$\therefore \text{ at } x=L, \frac{dy}{dx}=0. \therefore c_1 = -\frac{3}{4}WL^2$$

$$\therefore EI \frac{dy}{dx} = \frac{W}{4}x^2 - W\left(\frac{x^2}{2} - Lx\right) - \frac{3}{4}WL^2$$

Integrating again,

$$EIy = \frac{W}{12}x^3 - W\left(\frac{x^3}{6} - \frac{L}{2}x^2\right) - \frac{3}{4}WL^2x + c_2$$

$$\therefore \text{ at } x=2L, y=0. \therefore c_2 = \frac{1}{6}WL^3$$

$$\therefore EIy = \frac{Wx}{12}(x^2 - 3L^2) + \frac{W}{6}(L-x)^3$$

$\therefore$ the required equation of the elastic curve is

$$y = \frac{Wx}{12EI}(x^2 - 3L^2) + \frac{W}{6EI}(L-x)^3, \quad L \leq x \leq 2L$$

$$\therefore \text{ max. deflection is at } x=L \text{ and it is } (-y)_{\max} = \frac{WL^3}{6EI}.$$

$\therefore$ the max. deflection of the beam = $\frac{WL^3}{6EI}$ wherever irrespective of the position of the load on the right or left of the middle.

Exercises 21 (3)

1. Solve the ODE, $x'' + 8x' + 36x = 24 \cos 6t$ and discuss the behaviour of the solution as $t \rightarrow \infty$.
2. An iron ball weighs 64 lbs. It hangs at the end of a spring. The spring elongates due to the weight of the ball. From the equilibrium position of the ball, it is pulled 6" below and then released. Find the equation of motion of the ball. Find the position of the ball after t secs. (Assume no friction).
3. A helical springs elongates 6" due to the weight of 12 lbs attached to its end. It is pulled down to 4" from its equilibrium position and is then released. Find the equation of motion of the weight. Find also period, frequency and amplitude of motion. With what velocity it passes its position of equilibrium?

[Hint : I.C. $y = 4''$, $y' = 0$ when $t = 0$ and at equilibrium $x = 0$]

4. A helical spring elongates 4" when a 5 lbs weight is attached to it. It is released from its equilibrium position after pulling it down to 6" more. Determine its equation of motion, period, frequency and amplitude. (Assume there is no friction).
5. A mass is moving along the x -axis. It is attracted towards the origin by a force proportional to its distance from the origin. Find its equation of motion when it starts from rest at $x = x_0$. Also find its equation of motion when it starts at $x = x_0$ with a velocity $v = v_0$. In both cases find amplitude and period of motion.
6. The length of a simple pendulum is l and the weight of its bob is $W = mg$. If its oscillating angle is very small, show that it performs simple harmonic motion. Find amplitude and period of oscillation.
7. The ODE of a physical system is $y'' + 4y = \cos 2t$. Find its equation of motion. Is it stable or unstable? What is its undamped resonant frequency?
8. A spring is elongated by 6" by the weight of 16 lbs. Determine its equation of motion when an external force, $f(t) = 10 \sin 2t$ is applied to this system. Find the maximum displacement of the weight. Is it stable or unstable? What frequency of the external force creates resonance?
9. The arms of galvanometer swings at an angle θ such that $\ddot{\theta} + k\dot{\theta} + h\theta = 0$. What condition of h and k arms oscillates? Express θ as a function of t .
10. A series circuit consists of $L = 1$, $R = 1,000$ and $C = 6.25 \times 10^{-6}$. The series has no previous charge or current flow. Suddenly the series is connected with a constant voltage $E = 24$. Find the peak current flowing through the circuit.

[Hints : $\frac{d^2 Q}{dt^2} + 1000 \frac{dQ}{dt} + \frac{Q}{6.25 \times 10^{-6}} = 24$]

Answers

1. $x = e^{-4t} (c_1 \cos 2\sqrt{5} t + c_2 \sin 2\sqrt{5} t) + \frac{1}{2} \sin 6t$, at $t \rightarrow \infty$, $x \rightarrow \frac{1}{2} \sin 6t$, periodic oscillation with period $\frac{\pi}{3}$, damped harmonic.
2. $x'' + 8x = 0$, $x = \frac{1}{2} \cos 2\sqrt{2} t$.
3. $y = \frac{1}{3} \cos 8t$; $\frac{\pi}{4}$ sec; 8 rad/sec or $\frac{4}{\pi}$ C.P.S.; $\frac{1}{3}$ ft, $\pm \frac{8}{3}$ ft/sec.
4. $y = \frac{1}{2} \cos 4\sqrt{6} t$, $\frac{\pi}{2\sqrt{3}}$ secs., $\frac{2\sqrt{3}}{\pi}$ cycle/sec., $\frac{1}{2}$ ft.

5. $x = x_0 \cos Kt$, $x_0, \frac{2\pi}{K}$; $x = \frac{v_0}{K} \sin Kt + x_0 \cos Kt$, $\frac{\sqrt{v_0^2 + K^2 x^2}}{K}$, $\frac{2\pi}{K}$, where K is const. of proportionality.
6. $\sqrt{c_1^2 + c_2^2}$, $2\pi\sqrt{l/g}$. 7. $y = c \cos(2t + \delta) + \frac{1}{4}t \sin 2t$; stable; 2 rad./unit time.
8. $y = c \cos(8t + \delta) + \frac{1}{3}t \sin 2t$; stable; 8 rad/unit time.
9. $k^2 < 4h$; $\theta = e^{-kt/2} \left\{ c_1 e^{(t\sqrt{k^2 - 4h})/2} + c_2 e^{-(t\sqrt{k^2 - 4h})/2} \right\}$.
10. $i_{\max} = .019$ amp.

ORDER REDUCTION OF ODE AND EULER'S ODE

22.1 REDUCTION OF ORDER

When a method of solution is not apparent for some *ODEs*, one may look for lowering down the order so that previous known methods may be used. With this view in mind the following special types are discussed to expedite the solution procedure by reduction of order.

I. $\frac{d^n y}{dx^n} = f(x)$: This is an easiest type of n th order linear non-homogeneous *ODE*. This can be solved by successive integration n times, because the R.H.S. = $f(x)$ is a function of the independent variable, x only. So, first integration gives

$$\frac{d^{n-1} y}{dx^{n-1}} = \int f(x) dx + c_1$$

Second time integration gives,

$$\frac{d^{n-2} y}{dx^{n-2}} = \iint f(x) (dx)^2 + c_1 x + c_2$$

Integrating in this way, the general solution is

$$y = \iiint \dots \int f(x) (dx)^n + A_1 x^{n-1} + A_2 x^{n-2} + \dots + A_{n-1} x + A_n$$

where $A_r = \frac{c_r}{(n-r)!}$. Particular solution can be obtained by using particular conditions starting from the first integral.

II. $\frac{d^n y}{dx^n} = f(y)$: This equation is not so simple as type I although both are of n th order first degree. The linearity of the equation depends on the linearity of $f(y)$ and homogeneity too. The equation is homogeneous or non-homogeneous according as $f(y)$ contains or does not contain a constant term. It can be easily solved for **three** special cases – (i) $f(y) = \text{const.}$, using type I, (ii) $n = 1$, using separation of variables, (iii) $n = 2$, multiply both sides by $2 \frac{dy}{dx}$. In the last case, equation becomes

$$2 \frac{dy}{dx} \cdot \frac{d^2y}{dx^2} = 2 f(y) \cdot \frac{dy}{dx}$$

or,
$$\frac{d}{dx} \left(\frac{dy}{dx} \right)^2 = 2 f(y) \frac{dy}{dx}$$

Integrating,
$$\left(\frac{dy}{dx} \right)^2 = 2 \int f(y) \frac{dy}{dx} dx + c = 2 \int f(y) dy + c$$

$$= \phi(y) + c$$

Taking square root and separating variables

$$\frac{dy}{\sqrt{\phi(y) + c_1}} = \pm dx \text{ or } \int \frac{dy}{\sqrt{\phi(y) + c_1}} = \pm x + c_2$$

is the general solution of *ODE* $\frac{d^2y}{dx^2} = f(y)$ which occurs frequently in dynamics particularly in motion under a force directed to a fixed point and magnitude depends on the distance from the fixed point.

III. ODE with two derivatives only : The *ODE* containing two derivatives can be solved by **reduction of order** if the orders of the two derivatives differ by one or two.

a. Orders differ by one : The general equation of this type is

$$\frac{d^n y}{dx^n} = f \left(\frac{d^{n-1} y}{dx^{n-1}} \right)$$

Let $\frac{d^{n-1} y}{dx^{n-1}} = Y \therefore \frac{d^n y}{dx^n} = \frac{dY}{dx}$

$\therefore$ the given equation takes the form $\frac{dY}{dx} = f(Y)$ which is a first order and first degree *ODE*. The order reduces by $(n-1)$. By separation of variables,

$$\psi(Y) = \int \frac{dY}{f(Y)} = x + A$$

Solving for Y , let it be $Y = \phi(x + A)$ or $\frac{d^{n-1} y}{dx^{n-1}} = \phi(x + A)$ which is of type I and by $(n-1)$ successive integration this can be solved.

b. Orders differ by two : The general equation of this type is

$$\frac{d^n y}{dx^n} = f \left(\frac{d^{n-2} y}{dx^{n-2}} \right)$$

Let $\frac{d^{n-2} y}{dx^{n-2}} = u.$

$\therefore \frac{d^n y}{dx^n} = \frac{d^n u}{dx^2}$. So the given equation becomes a second order ODE as

$$\frac{d^2 u}{dx^2} = f(u). \text{ This is of type II.}$$

Its solution is $\int \frac{du}{\sqrt{\{c_1 + 2 \int f(u) du\}}} = x + c_2$. After performing this

integration and replacing $u = \frac{d^{n-2} y}{dx^{n-2}}$, the equation will be of the type I which requires $(n-2)$ successive integrations to solve completely.

IV. Absence of dependent or independent variable : The explicit absence of either dependent or independent variable helps to reduce the order of ODE by the substitution for the lowest order derivative present in the equation.

a. Independent variable, x is absent explicitly : The general equation of second order is $F\left(y, \frac{dy}{dx}, \frac{d^2 y}{dx^2}\right) = 0$. This equation is normally of variable coefficient and non-linear. The substitution $\frac{dy}{dx} = p$ reduces the given second to a first order equation. Thus $\frac{d^2 y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy} \cdot \frac{dy}{dx} = p \frac{dp}{dy}$ and the given equation becomes $F\left(y, p, p \frac{dp}{dy}\right) = 0$. This reduced equation can be solved by the methods of chapter 19 and replacing p by $\frac{dy}{dx}$, it needs a single integration to get the general solution with two arbitrary constants.

This procedure is applicable to n th order ODE in which x is explicitly absent. Thus, the general equation is $F\left(y, \frac{dy}{dx}, \frac{d^2 y}{dx^2}, \dots, \frac{d^n y}{dx^n}\right) = 0$. As above let $\frac{dy}{dx} = p$,

$$\begin{aligned} \frac{d^2 y}{dx^2} &= p \frac{dp}{dy}, \quad \frac{d^3 y}{dx^3} = \frac{d}{dx} \left(p \frac{dp}{dy} \right) = \frac{dp}{dx} \frac{dp}{dy} + p \frac{d}{dy} \left(\frac{dp}{dy} \right) \frac{dy}{dx} \\ &= p \left(\frac{dp}{dy} \right)^2 + p^2 \frac{d^2 p}{dy^2} = p \left[\left(\frac{dp}{dy} \right)^2 + p \frac{d^2 p}{dy^2} \right] \dots \text{etc.} \end{aligned}$$

Therefore, the

given equation takes the form in p and y as $\phi\left(y, p, \frac{dp}{dy}, \dots, \frac{d^{n-1} p}{dy^{n-1}}\right) = 0$. The solution p requires a single integration to get the general solution with requisite number of arbitrary constants.

b. Dependent variable, y is absent explicitly : The general second order ODE is $F\left(x, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0$. This equation is also of variable coefficients and non-linear. The substitution $\frac{dy}{dx} = p, \frac{d^2y}{dx^2} = \frac{dp}{dx}$ reduces the given equation to a first order equation as $F\left(x, p, \frac{dp}{dx}\right) = 0$ which can be solved by the methods of chapter 19 and replacement of p by $\frac{dy}{dx}$ and integration w.r.t. x gives the general solution with two arbitrary constants.

This is again applicable to $F\left(x, \frac{d^k y}{dx^k}, \frac{d^{k+1} y}{dx^{k+1}}, \dots, \frac{d^n y}{dx^n}\right) = 0$ in which y and first $(k-1)$ derivatives are absent. Substituting for the lowest order derivative namely $\frac{d^k y}{dx^k} = v, \frac{d^{k+1} y}{dx^{k+1}} = \frac{dv}{dx}, \dots, \frac{d^n y}{dx^n} = \frac{d^{n-k} v}{dx^{n-k}}$, the given equation reduces to

$$F\left(x, v, \frac{dv}{dx}, \dots, \frac{d^{n-k} v}{dx^{n-k}}\right) = 0$$

in which the order reduces by k . Solving v , and then by k successive integration as in type I gives the general solution with requisite number of arbitrary constants.

(V) Reduction of homogeneous ODE of higher order : Higher order homogeneous ODE may be tested either from definition of homogeneity in 18.3.1 (iii) or by calculating dimensions as follows : let the dimension of x and y be unity i.e. 1. Hence the dimensions of derivatives y_1, y_2, y_3 etc. are 0, -1, -2 etc. On this basis if the dimension of each term be same, the ODE is homogeneous.

(a) Change of independent variable : A non-linear equation with variable coefficients can be reduced to non-linear with constant coeffs. i.e. the coefficients will be free of x explicitly by the substitution $x = e^t$ or $t = \log x$. Because

$$y_1 = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{1}{x} \frac{dy}{dt} \text{ or } xy_1 = \frac{dy}{dt} = D_t y \text{ (say)}$$

$$\text{Similarly, } y_2 = \frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{1}{x} \frac{dy}{dt} \right) = -\frac{1}{x^2} \frac{dy}{dt} + \frac{1}{x^2} \frac{d^2 y}{dt^2}$$

$$\text{or, } x^2 y_2 = \frac{d^2 y}{dt^2} - \frac{dy}{dt} = D_t(D_t - 1)y$$

Proceeding in this way,

$$x^n y_n = D_t(D_t - 1)(D_t - 2) \dots (D_t - n + 1)y$$

which shows that terms with variable coefficients like $x^n y_n$ are expressed in another expression free of the new independent variable, t . But the non-linearity due to y , its derivative will remain and the equation will be reduced to type IV (a) i.e. the general form $F(x, y, xy_1, x^2 y_2, \dots, x^n y_n) = 0$ will be reduced to $\phi\left(y, \frac{dy}{dt}, \dots, \frac{d^n y}{dt^n}\right) = 0$.

(b) **Change of both variables** : Both variables can be changed simultaneously or one after another. The dependent variable is expressed as a product of independent variable itself and an arbitrary function of the independent variable i.e. $y = v(x)x$ (say) as in 19.4 and the independent variable is changed to a new variable by $x = e^t$ or $t = \log x$ as in (a) above.

$$\text{since } y = v(x)x \therefore y_1 = v(x) + xv_1(x), xy_2 = 2v_1x + x^2v_2,$$

$$x^2y_3 = 3v_2x^2 + x^3v_3, \dots, x^{n-1}y_n = n v_{n-1}x^{n-1} + x^n v_n$$

dropping the argument x in the derivatives of v w.r.t. x .

$$\text{Also } x = e^t, v_1(x) = \frac{dv}{dt} e^t \text{ or } xv_1(x) = \frac{dv}{dt} = D_t v, \text{ similarly}$$

$$x^2 v_2(x) = \frac{d^2 v}{dt^2} - \frac{dv}{dt} = D_t(D_t - 1)v, \dots,$$

$$x^n v_n(x) = D_t(D_t - 1) \dots (D_t - n + 1)v$$

$$\therefore y_1 = v + \frac{dv}{dt} = (D_t + 1)v, xy_2 = \frac{d^2 v}{dt^2} + \frac{dv}{dt} = D_t(D_t + 1)v,$$

$$x^2 y_3 = D_t(D_t^2 - 1)v, \dots$$

$$x^{n-1} y_n = D_t(D_t^2 - 1)(D_t - 2) \dots (D_t - n + 2)v, n > 2.$$

Using all these in a general homogeneous equation

$$f\left(\frac{y}{x}, y_1, xy_2, x^2 y_3, \dots, x^{n-1} y_n\right) = 0, \text{ it reduces to}$$

$$f(v, (D_t + 1)v, D_t(D_t + 1)v, D_t(D_t^2 - 1)v, \dots,$$

$$D_t(D_t^2 - 1)(D_t - 2) \dots (D_t - n + 2)v, n > 2) = 0$$

which belongs to type IV (a) again.

Homogeneous ODE of order greater than two may give rise to simultaneous first order and first degree ODEs which are beyond the scope of this book. So, only problems of second order are included in this book.

(VI) **Reduction by variation of parameters** : Earlier in 19.7 and 21.5 (vii) this method was used to find the solution of a non-homogeneous equation when the solution of the associated homogeneous equation is known. Here this method will be used to reduce the order of a homogeneous or a non-homogeneous equation by unity when a solution of the homogeneous or

the associated homogeneous equation can be known by trial or given. Now a second order ODE is considered for the purpose. Let

$$y'' + a(x)y' + b(x)y = c(x) \quad (1)$$

be a non-homogeneous ODE. Let $y_1(x)$ be a solution of the associated homogeneous equation

$$y'' + a(x)y' + b(x)y = 0 \quad (2)$$

Let $y_2(x)$ be assumed another independent solution of (2). Hence

$\frac{y_2(x)}{y_1(x)} = v(x)$ must be a non-constant, otherwise $y_1(x)$ and $y_2(x)$ will not be linearly independent. Therefore,

$y_2(x) = y_1(x) v(x)$. Differentiating twice successively,

$$y_2'(x) = y_1'(x) v(x) + y_1(x) v'(x)$$

and $y_2''(x) = y_1''(x) v(x) + 2 y_1'(x) v'(x) + y_1(x) v''(x)$

Using y_2, y_2' and y_2'' in (1), it becomes,

$$v(x) [y_1''(x) + a(x)y_1'(x) + b(x)y_1(x)] + v'(x) [2 y_1'(x) + a(x)y_1(x)] + y_1(x) v''(x) = c(x) \quad (3)$$

In this equation coefficient of $v(x)$ is identically zero as $y_1(x)$ is a solution of (2). So equation (3) becomes

$$y_1(x) v''(x) + v'(x) [2 y_1'(x) + a(x)y_1(x)] = c(x) \quad (4)$$

which is type IV (b) as $v(x)$ is explicitly absent. Let $v'(x) = p(x)$.

$v''(x) = p'(x)$. So, (4) becomes

$$y_1(x) p'(x) + [2 y_1'(x) + a(x)y_1(x)] p(x) = c(x) \quad (5)$$

which is a first order linear equation in $p(x)$. If (1) is homogeneous i.e. $c(x) = 0$, then (5) can be solved by separation of variables. Thus variation of parameter can reduce the order of a n th order ODE by unity giving rise to an $(n-1)$ th order ODE.

(VII). Reduction of order by factorization of operators : If the operators of an equation can be factorized into linear and quadratic factors of D , then the introduction of auxiliary variables reduces the order of the original equation. Let a second order equation be factorized as

$$(PD + Q)(RD + S)y = f(x) \quad (1)$$

where P, Q, R and S are function of x, y . The substitution $(RD + S)y = u$ (2)

reduces (1) to $(PD + Q)u = f(x)$ (3)

where both (1) and (2) are first order equations. Solving u from (3) and using it in (2) gives the solution of (1). So, factorization of operators reduces a higher order ODE into a number of first and second order ODE according as the factors in D are linear and quadratic.

The question may arise about how to factorize the operators? The operators with constant coefficients can be factorized easily as in ordinary algebra. But the operators with variable coefficients are not so easy as these operators are not always commutative i.e. $D(xD)y = Dy + xD^2y$ whereas $(xD)Dy = xD^2y$; so, $D(xD)y \neq (xD)Dy$. Two forms of factorization of variable coefficient operators are

(i) $[MD^2 + (MC + N)D + CN]y = (MD + N)(D + C)y$ where M and N are variables, C is a constant.

$$(ii) PR \frac{d^2y}{dx^2} + \left(P \frac{dR}{dx} + PS + QR \right) \frac{dy}{dx} + \left(P \frac{dS}{dx} + QS \right) y$$

$$= (PD + Q)(RD + S)y \text{ where } P, Q, R \text{ and } S \text{ are variables}$$

which can be used for second order operators with variable coefficients.

22.2 EULER'S DIFFERENTIAL EQUATIONS

An n th order differential equation with variable coefficients of the form

$$A_0 x^n \frac{d^n y}{dx^n} + A_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + A_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \dots + A_{n-1} x \frac{dy}{dx} + A_n y = f(x), A_0 \neq 0$$

is called Euler's or Cauchy-Euler's ODE. Here each term on the left side of the equation is homogeneous of the same degree and $A_0, A_1, \dots, A_n$ are constants. So, if $f(x) = 0$, the equation becomes homogeneous and is then called Euler's homogeneous ODE. This equation can easily be transferred to one with constant coefficient by the substitution $x = e^t$ or $\log x$ so that

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot \frac{1}{x} \therefore x \frac{dy}{dx} = \frac{dy}{dt} = D_t y$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dt} \cdot \frac{1}{x} \right) = -\frac{1}{x^2} \frac{dy}{dt} + \frac{1}{x^2} \frac{d^2 y}{dt^2}$$

$$\therefore x^2 \frac{d^2 y}{dx^2} = \frac{d^2 y}{dt^2} - \frac{dy}{dt} = D_t(D_t - 1)y$$

Proceeding in this way,

$$x^n \frac{d^n y}{dx^n} = D_t(D_t - 1) \dots (D_t - n + 1)y$$

$\therefore$ with these substitutions, Euler's equation with constant coefficient becomes

$$F(D)y \equiv (B_0 D_t^n + B_1 D_t^{n-1} + \dots + B_{n-1} D_t + B_n)y = f(e^t).$$

This can be solved by 21.5.

It may be mentioned that the variable coeffs. x 's in Euler's equation may

be replaced by $(ax+b)$ or even by (ax^2+bx+c) . In such case substitution will be $ax+b=e^t$ and $ax^2+bx+c=e^t$ and correspondingly the new derivatives are to be determined.

22.3 APPLICATIONS

(A) **Geometric applications :** Problems like tangent, normal and radius of curvature of a curve produce *ODE* of higher order. From differential calculus,

$$\text{Radius of curvature, } R = \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{3/2} \bigg/ \left(\frac{d^2y}{dx^2} \right)$$

$$\text{Length of tangent} = \frac{y}{y_1} \sqrt{1 + y_1^2}$$

$$\text{Length of normal} = y \sqrt{1 + y_1^2}$$

Remember that normal and radius of curvature are in same or in opposite directions if the sign of y and $\frac{d^2y}{dx^2}$ are opposite or same respectively.

(B) **The catenary :** The curve created by the sag of a suspended cord or cable at two points is called a catenary.

Let a homogeneous flexible cord or cable is suspended by its own weight at two fixed points. Let its weight from its lowest point A to any point P be W lbs, the tension at P along the tangent be T lbs, the tangent at P be inclined to horizon at an acute angle θ and the horizontal tension at A be H (fig. 167). Now the horizontal and the vertical components of tension in the portion AP are

$$T \sin \theta = W \text{ and } T \cos \theta = H$$

$$\therefore \tan \theta = W/H$$

Position of origin with respect to the cord will be decided latter on. Under this condition let the upper side of y -axis and the right side of x -axis be positive.

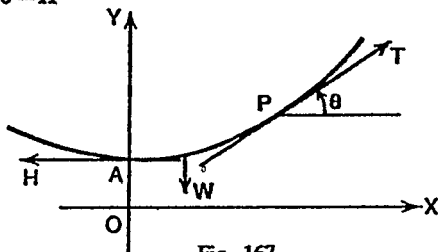


Fig. 167

So, $\frac{dy}{dx} = \tan \theta$. Let ω the weight per unit length and s be the length of AP .

So, $W = \omega s$. Hence

$$\frac{dy}{dx} = \frac{W}{H} = \frac{\omega s}{H} \quad (1)$$

This is the ODE for the cord. But the equation contains three variables x , y and s . Now to eliminate s from (1), let us differentiate (1) w.r.t. ' x '. Therefore

$$\frac{d^2 y}{dx^2} = \frac{\omega}{H} \frac{ds}{dx} = \frac{\omega}{H} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \quad (2)$$

To solve (2) let $p = \frac{dy}{dx}$, $\frac{dp}{dx} = \frac{d^2 y}{dx^2}$. So, (2) becomes

$$\frac{dp}{dx} = \frac{\omega}{H} \sqrt{1 + p^2} \quad \text{or} \quad \frac{dp}{\sqrt{1 + p^2}} = \frac{\omega}{H} dx$$

$$\text{Integrating, } \sinh^{-1} p = \frac{\omega}{H} x + c_1 \quad (3)$$

But when y -axis crosses A , $p = \frac{dy}{dx} = 0$ when $x = 0$.

$\therefore$ from (3), $c_1 = 0$. So, (3) becomes

$$\sinh^{-1} p = \frac{\omega}{H} x \quad \text{or} \quad p = \frac{dy}{dx} = \sinh \frac{\omega}{H} x$$

$$\text{Integrating, } y = \frac{H}{\omega} \cosh \frac{\omega}{H} x + c_2 \quad (4)$$

Let the position of x -axis below A be $\frac{H}{\omega}$. Hence $y = \frac{H}{\omega}$ when $x = 0$.

Therefore, from (4), $c_2 = 0$. Let $a = H/\omega$. So (4) becomes

$$y = a \cosh (x/a) \quad (5)$$

This is the standard equation of a catenary. Therefore, if the origin lies ' a ' ft below A , then the co-ordinate of any point P is $P(x, y)$. Now, length of the arc AP ,

$$s = \int_0^x \sqrt{1 + (y')^2} dx = \int_0^x \sqrt{1 + \sinh^2 \frac{x}{a}} dx = a \sinh \frac{x}{a} \quad (6)$$

Let the points of suspension lie on the same horizontal line i.e. they are at the same height from x -axis. Let d be the sag, distance between the suspension points be L and the length of the cord be s (fig. 168). Therefore from (6),

$$s = 2a \sinh \frac{L}{2a}$$

$$\text{and } d = y \Big|_{x=L/2} - a = a \left(\cosh \frac{L}{2a} - 1 \right) \quad \text{by (5)}$$

$$\text{Tension at any point, } T = H \sec \theta = H \sqrt{1 + (y')^2}$$

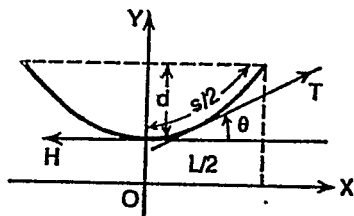


Fig. 168

$$= H \cosh \frac{x}{a} = \omega y$$

(C) **Heat flow in a fin** : Let the fin be of the shape of a parabola (fig. 169) instead of rectangular in 20.6.3 (B) but the conditions remain same. Here the equation of the parabolic profile is

$$y = \frac{\delta_0}{2} \left(\frac{x}{b} \right)^2$$

$\therefore$ area of cross-section of unit length, $A(x) = 2y = \delta_0 \left(\frac{x}{b} \right)^2$

$\therefore$ difference of heat flow in a differential dx in heat gain and loss is

$$q = \frac{d}{dx} \left[k A(x) \frac{dT}{dx} \right] dx.$$

Heat loss due to conduction,

$$q_s = 2h(T - T_s) dx.$$

$$\therefore k \frac{d}{dx} \left[A(x) \frac{dT}{dx} \right] dx = 2h(T - T_s) dx,$$

$$\therefore \theta = T - T_s \quad \therefore d\theta = dT$$

$$\therefore k A(x) \frac{d^2 \theta}{dx^2} + k \frac{d}{dx} A(x) \cdot \frac{d\theta}{dx} = 2h\theta$$

or,

$$x^2 \frac{d^2 \theta}{dx^2} + 2x \frac{d\theta}{dx} - m^2 b^2 \theta = 0,$$

$$\therefore m^2 = \frac{2h}{k \delta_0}, A(x) = \delta_0 \left(\frac{x}{b} \right)^2$$

This is a Euler's homogeneous equation. So, using 22.2 the general solution is

$$\theta = c_1 x^{(-1 + \sqrt{1 + 4m^2 b^2})/2} + c_2 x^{(-1 - \sqrt{1 + 4m^2 b^2})/2} \quad (1)$$

But excess heat distribution will be infinite at $x=0$ by (1). This is impossible, so $c_2 = 0$. Therefore, (1) becomes

$$\theta = c_1 x^{(-1 + \sqrt{1 + 4m^2 b^2})/2} \quad (2)$$

$$\text{Again } \theta = \theta_0 \text{ when } x = b \quad \therefore \theta_0 = c_1 b^{(-1 + \sqrt{1 + 4m^2 b^2})/2}$$

$$\therefore \theta = \theta_0 \left(\frac{x}{b} \right)^p \text{ where } p = \left(-\frac{1}{2} + \frac{1}{2} \sqrt{1 + 4m^2 b^2} \right)$$

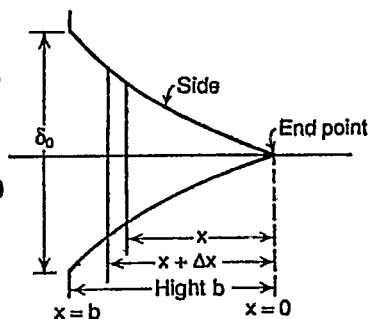


Fig. 169

Examples

1. Solve the ODE $\frac{d^3 y}{dx^3} = x^5$.

This is of type I of 22.1. Integrating 3 times successively, we have,

$$\frac{d^2 y}{dx^2} = \frac{1}{6} x^6 + c_1$$

or, $\frac{dy}{dx} = \frac{1}{42} x^7 + c_1 x + c_2$

or, $y = \frac{1}{336} x^8 + c_1' x^2 + c_2 x + c_3, \quad c_1' = \frac{c_2}{2}$

2. Solve the ODE $\frac{d^2 y}{dx^2} + 4y^2 = 0$ given $\frac{dy}{dx} = 1$ when $y = 0$ and $y = 0$ when $x = 0$.

This is of type II of 22.1. So, multiplying by $2 \frac{dy}{dx}$,

$$2 \frac{dy}{dx} \frac{d^2 y}{dx^2} = -8y^2 \frac{dy}{dx} \quad \text{or} \quad \frac{d}{dx} \left(\frac{dy}{dx} \right)^2 = -8y^2 \frac{dy}{dx}$$

or, $\int \frac{d}{dx} \left(\frac{dy}{dx} \right)^2 dx = -8 \int y^2 \frac{dy}{dx} \cdot dx + c_1$

or, $\left(\frac{dy}{dx} \right)^2 = -\frac{8}{3} y^3 + c_1$

$\therefore \frac{dy}{dx} = 1$ when $y = 0 \therefore c_1 = 1$

$\therefore \left(\frac{dy}{dx} \right)^2 = -\frac{8}{3} y^3 + 1 \quad \text{or} \quad \frac{dy}{dx} = \pm \sqrt{1 - \frac{8}{3} y^3}$

or, separating variables, $\int \frac{dy}{\sqrt{1 - \frac{8}{3} y^3}} = \pm \int dx + c_2$

or, $\int \left(1 - \frac{8}{3} y^3 \right)^{-1/2} dy = \pm x + c_2$

or, $\int \left[1 + \frac{4}{3} y^3 + \frac{8}{9} y^6 + \dots \infty \right] dy = \pm x + c_2$

or, $y + \frac{1}{3} y^4 + \frac{8}{63} y^7 + \dots \infty = \pm x + c_2$

$\therefore y=0$ when $x=0 \therefore c_2=0$

$\therefore$ the required solution is

$$y + \frac{1}{3}y^4 + \frac{8}{63}y^7 + \dots \infty = \pm x.$$

3. Solve the ODE $a^3 \frac{d^3 y}{dx^3} \frac{d^2 y}{dx^2} = \sqrt{1 + c^2 \left(\frac{d^2 y}{dx^2} \right)^2}$

This is of type III (a). The difference of the orders is 1. Let

$$\frac{d^2 y}{dx^2} = Y \text{ or } \frac{d^3 y}{dx^3} = \frac{dY}{dx}.$$

$\therefore$ the given equation becomes

$$a^3 Y \frac{dY}{dx} = \sqrt{1 + c^2 Y^2} \text{ or } \int \frac{a^3 Y dY}{\sqrt{1 + c^2 Y^2}} = \int dx + A$$

or, $\frac{a^3}{2c^2} \cdot 2 \sqrt{1 + c^2 Y^2} = x + A \text{ or } 1 + c^2 Y^2 = \left[\frac{c^2}{a^3} (x + A) \right]^2$

or, $Y = \pm \frac{1}{c} \sqrt{\left\{ \frac{c^2}{a^3} (x + A) \right\}^2 - 1}$

or, $\frac{d^2 y}{dx^2} = \pm \frac{1}{c} \sqrt{\frac{c^2}{a^3} (x + A)^2 - 1}$, which is of type I.

Integrating,

$$\frac{dy}{dx} = \pm \frac{1}{c} \frac{a^3}{c^2} \left[\frac{c^2}{2a^3} (x + A) \sqrt{\left[\frac{c^2}{a^3} (x + A) \right]^2 - 1} - \frac{1}{2} \log \left\{ \frac{c^2}{a^3} (x + A) + \sqrt{\left[\frac{c^2}{a^3} (x + A) \right]^2 - 1} \right\} \right] + A_1$$

or, $y = \pm \left(\frac{a}{c} \right)^3 \left[\frac{1}{2} \cdot \frac{a^3}{c^2} \cdot \frac{1}{3} \left\{ \left[\frac{c^2}{a^3} (x + A) \right]^2 - 1 \right\}^{3/2} - \frac{1}{2} \cdot \frac{c^2}{a^3} (x + A) \log \left\{ \frac{c^2}{a^3} (x + A) + \sqrt{\left[\frac{c^2}{a^3} (x + A) \right]^2 - 1} \right\} \right] + A_1$

$$\begin{aligned}
 & -\frac{1}{4} \sqrt{\left[\frac{c^2}{a^3} (x+A) \right]^2 - 1} + A_1 x + A_2 \\
 \therefore y = & \pm \left(\frac{a}{c} \right)^3 \left[\frac{a^3}{6c^2} \left\{ \frac{c^4}{a^6} (x+A)^2 - 1 \right\}^{3/2} \right. \\
 & \left. - \frac{c^2}{2a^3} (x+A) \cdot \log \left\{ \frac{c^2}{a^3} (x+A) + \sqrt{\frac{c^4}{a^6} (x+A)^2 - 1} \right\} \right. \\
 & \left. - \frac{1}{4} \sqrt{\frac{c^4}{a^6} (x+A)^2 - 1} \right] + A_1 x + A_2
 \end{aligned}$$

4. Solve the ODE $a^2 \frac{d^4 y}{dx^4} = \frac{d^2 y}{dx^2}$.

This is an example of type III (b) of 22.1. Here the difference of orders

is 2. So, let $\frac{d^2 y}{dx^2} = u$

$\therefore \frac{d^4 y}{dx^4} = \frac{d^2 u}{dx^2}$. Therefore, the given equation becomes,

$a^2 \frac{d^2 u}{dx^2} = u$ which is of type I of 22.1 or of 21.3.

$\therefore$ its general solution is $u = c_1 e^{x/a} + c_2 e^{-x/a}$

i.e. $\frac{d^2 y}{dx^2} = c_1 e^{x/a} + c_2 e^{-x/a}$ which is of type I.

$\therefore$ integrating twice successively, the required general solution is

$$y = A e^{x/a} + B e^{-x/a} + Cx + D.$$

5. Solve the ODE $y \frac{d^2 y}{dx^2} + \left\{ \left(\frac{dy}{dx} \right)^2 + a^2 \left(\frac{d^2 y}{dx^2} \right)^2 \right\}^{1/2} = \left(\frac{dy}{dx} \right)^2$.

This is an example of type IV (b) of 22.1. So, let $\frac{dy}{dx} = p$.

$\therefore \frac{d^2 y}{dx^2} = \frac{dp}{dx} = p \frac{dp}{dy}$. Therefore, the given equation becomes,

$$yp \frac{dp}{dy} + \left\{ p^2 + a^2 p^2 \left(\frac{dp}{dy} \right)^2 \right\}^{1/2} = p^2$$

or, $y \frac{dp}{dy} + \sqrt{1 + a^2 \left(\frac{dp}{dy} \right)^2} = p$ which is a first order second degree equation of chapter 20. Let $\frac{dp}{dy} = q$. Then $p = yq + \sqrt{1 + a^2 q^2}$ which is Clairaut's equation with p and y as dependent and independent variables respectively. Its general solution is $p = Ay + \sqrt{1 + a^2 A^2}$, A is an arbitrary constant. So,

$$\frac{dy}{dx} = Ay + \sqrt{1 + a^2 A^2}$$

Integrating it, the required general solution is,

$$Ay = e^{A(x+B)} - \sqrt{1 + a^2 A^2}, B \text{ is an arbitrary const.}$$

$$6. \text{ Solve the ODE } 1 + \left(\frac{dy}{dx} \right)^2 + x \frac{dy}{dx} \frac{d^2 y}{dx^2} = a^2 \frac{d^2 y}{dx^2} \sqrt{1 + \left(\frac{dy}{dx} \right)^2}.$$

This is an example of type IV (a). So, let $\frac{dy}{dx} = p$.

$\therefore \frac{d^2 y}{dx^2} = \frac{dp}{dx}$. Therefore, the given equation becomes,

$$1 + p^2 + xp \cdot \frac{dp}{dx} = a^2 \frac{dp}{dx} \sqrt{1 + p^2}.$$

This is a first order first degree ODE of chapter 18 in p and x . This can be re-arranged as,

$$\frac{dx}{dp} + \frac{p}{1 + p^2} x = a^2 \cdot \frac{1}{\sqrt{1 + p^2}} \text{ which is linear in } x.$$

Its general solution is $x \sqrt{1 + p^2} = a^2 p + A$, A is an arbitrary constant. This is a first order higher degree ODE of 20.2 (ii). So, solution of its reduced

equation is $y = \frac{Ap - a^2}{\sqrt{1 + p^2}} - A \log(p + \sqrt{1 + p^2}) + B$.

$\therefore$ the required solution is the parametric solution

$$\begin{cases} y = \frac{Ap - a^2}{\sqrt{1 + p^2}} - A \log(p + \sqrt{p^2 + 1}) + B \\ x = \frac{a^2 p + A}{\sqrt{1 + p^2}} \end{cases}$$

where p is a parameter.

$$7. \text{ Solve the ODE } 2x^2 y y_2 + y^2 = x^2 y_1^2. \quad (1)$$

This is a homogeneous equation of type V; because dimension of each term are $(2 + 1 - 1)$, 2 , $(2 + 0^2) = 2$, 2 , 2 . The equation can be written as

$$2 y y_2 + (y/x)^2 = y_1^2 \quad (2)$$

which belongs to type V (b) i.e. Both variables are to be changed. Let $y = vx$ first. So, $y_1 = v + xv_1$, $y_2 = xv_2 + 2v_1$. Therefore, the equation (2) becomes

$$2 vx (xv_2 + 2v_1) + v^2 = (v + xv_1)^2$$

or,

$$2x^2 v v_2 + 2xv v_1 - x^2 v_1^2 = 0 \quad (3)$$

which still remain homogeneous. Now putting $x = e^t$,

$$\text{so, } xv_1 = \frac{dv}{dt}, x^2 v_2 = \frac{d^2 v}{dt^2} - \frac{dv}{dt}. \text{ Therefore, equation (3) becomes}$$

$$2v \left(\frac{d^2 v}{dt^2} - \frac{dv}{dt} \right) + 2v \frac{dv}{dt} - \left(\frac{dv}{dt} \right)^2 = 0$$

or,

$$2v \frac{d^2 v}{dt^2} - \left(\frac{dv}{dt} \right)^2 = 0 \quad (4)$$

In (4) the independent variable t is explicitly absent. So, it is type IV (a).

$$\text{Let } \frac{dv}{dt} = p. \text{ So } \frac{d^2 v}{dt^2} = p \frac{dp}{dv}.$$

$\therefore$ equation (4) becomes,

$$2vp \frac{dp}{dv} - p^2 = 0 \text{ or } 2v \frac{dp}{dv} - p = 0 \quad (5)$$

which is a first order first degree equation. By separation of variables and integration, the general solution of (5) is $p = \sqrt{v} c$ or $\frac{dv}{dt} = \sqrt{v} c$ or integrating

$$2\sqrt{v} = ct + a \text{ or } 2\sqrt{y/x} = c \log x + a$$

or, $y = (a \log x + b)^2$ is the required general solution of (1).

Alter. Simultaneous substitution of $y = v(x)x$ and $x = e^t$ can be used and V (b) equation (2) can be written as $2v \left(\frac{d^2 v}{dt^2} + \frac{dv}{dt} \right) + v^2 = \left(v + \frac{dv}{dt} \right)^2$ which simplifies to (4). Then proceed as before.

8. Find a particular solution of the associated homogeneous equation of the non-homogeneous ODE

$$x^2 \frac{d^2 y}{dx^2} + x^3 \frac{dy}{dx} - 2(1+x^2)y = x$$

and hence find its general solution by reduction of order.

By inspection it is seen that $y = x^2$ is a particular solution of the associated homogeneous ODE of the given ODE. Let the general solution be

$$y = v(x) \cdot x^2 \quad (1)$$

$$\therefore \frac{dy}{dx} = v'(x) \cdot x^2 + v(x) \cdot 2x$$

$$\frac{d^2y}{dx^2} = v''(x) \cdot x^2 + 4x \cdot v'(x) + 2v(x)$$

$\therefore$ the given equation becomes,

$$x^2 [v''(x) \cdot x^2 + 4x \cdot v'(x) + 2v(x)] + x^3 [v'(x) \cdot x^2 + v(x) \cdot 2x] - 2(1+x^2)v(x) \cdot x^2 = x$$

$$\text{or, } x^3 v''(x) + x^2(x^2 + 4)v'(x) = 1 \quad (2)$$

The equation (2) contains no $v(x)$ explicitly. So, it is of type IV (b). Therefore, let $v'(x) = p(x) \therefore v''(x) = p'(x)$

$$\therefore \text{ equation (2) reduces to } p'(x) + \frac{x^2 + 4}{x} p(x) = \frac{1}{x^3} \quad (3)$$

which is a linear equation in $p(x)$. Therefore, the general solution of (3) is

$$p(x) \cdot x^4 e^{x^2/2} = \int x e^{x^2/2} dx + c_1$$

$$\text{or, } p(x) \cdot x^4 e^{x^2/2} = e^{x^2/2} + c_1$$

$$\text{or, } v'(x) = \frac{1}{x^4} + \frac{c_1}{x^4} e^{-x^2/2}$$

$$\text{Integrating, } v(x) = -\frac{1}{3x^3} + c_1 \int \frac{e^{-x^2/2}}{x^4} dx + c_2$$

$\therefore$ from (1) the required general solution is

$$y = -\frac{1}{3x} + c_1 x^2 \int \frac{e^{-x^2/2}}{x^4} dx + c_2 x^2.$$

9. Factorize the operators to reduce the order of the ODE

$$x^2 D^2 y + (5x - x^2) Dy + (3 - 2x)y = 4e^x$$

Let the operators

$$\begin{aligned} x^2 D^2 y + (5x - x^2) Dy + (3 - 2x)y &\equiv (xD + M)(xD + N)y \\ &= x^2 D^2 y + x(M + N + 1) Dy + (MN + xN')y \end{aligned}$$

$$\therefore \text{ by equating coeffs. of like operators, } x(M + N + 1) = 5x - x^2 \quad (i)$$

$$\text{and } MN + xN' = 3 - 2x \quad (ii)$$

From (i) $M = 4 - x - N$.

$$\therefore (ii) \text{ becomes } (4 - x - N)N + xN' = 3 - 2x \quad (iii)$$

Let $N = ax + b$

$$\therefore \text{ from (iii) } (4-x)(ax+b) - (a^2x^2 + 2abx + b^2) + xa = 3 - 2x$$

or, by equating coeffs. of like powers of x ,

$$-a - a^2 = 0, 5a - b - 2ab = -2, 4b - b^2 = 3$$

which give $a = -1, b = 3$. $\therefore N = -x + 3$ and $M = 1$

$\therefore$ from given equation

$$(xD + 1)(xD - x + 3)y = 4e^x \quad (\text{iv})$$

$$\text{Let } (xD - x + 3)y = u. \therefore (\text{iv}) \text{ becomes } (xD + 1)u = 4e^x \quad (\text{v})$$

Both (iv) and (v) are reduced first order first degree equations of the given equation. Now from (v) $u = \frac{4}{x}e^x + \frac{c_1}{x}$.

$$\therefore \text{ from (iv), } x^3 e^{-x} \cdot y = 2x^2 - c_1(x+1)e^{-x} + c_2$$

$\therefore$ the required general solution is

$$y = \frac{2e^x}{x} + \frac{c_2}{x^3}e^x - c_1 \frac{(x+1)}{x^3}$$

$$\text{or, } x^3 y = (2x^2 + c_2)e^x - c_1(x+1).$$

$$10. \text{ Solve } (a+bx)^2 \frac{d^2 y}{dx^2} + (a+bx)b \frac{dy}{dx} + b^2 y = 4 \cos \log(a+bx).$$

This is an equation of type 22.2. Let $a+bx = e^t$,

$$\therefore t = \log(a+bx). \text{ Now } \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dx} \cdot \frac{b}{a+bx}$$

$$\begin{aligned} \text{or, } (a+bx) \frac{dy}{dx} &= b \frac{dy}{dt}. \text{ Again } \frac{d^2 y}{dx^2} = \frac{d}{dx} \left[\frac{b}{a+bx} \frac{dy}{dt} \right] \\ &= \frac{d^2 y}{dt^2} \cdot \frac{b^2}{(a+bx)^2} - \frac{b^2}{(a+bx)^2} \cdot \frac{dy}{dt} \end{aligned}$$

$$\text{or, } (a+bx)^2 \frac{d^2 y}{dx^2} = b^2 \left(\frac{d^2 y}{dt^2} - \frac{dy}{dt} \right)$$

$\therefore$ the given equation becomes,

$$b^2 \left(\frac{d^2 y}{dt^2} - \frac{dy}{dt} \right) + b^2 \frac{dy}{dt} + b^2 y = 4 \cos t$$

$$\text{or, } b^2 \left(\frac{d^2 y}{dt^2} + y \right) = 4 \cos t \quad \text{or} \quad \frac{d^2 y}{dt^2} + y = \frac{4}{b^2} \cos t.$$

which is a second order linear ODE.

$$\therefore y_c = c_1 \cos t + c_2 \sin t = c_1 \cos \log(a+bx) + c_2 \sin \log(a+bx)$$

and $y_p = \frac{1}{D_t^2 + 1} \frac{4}{b^2} \cos t = \frac{2}{b^2} t \sin t = \frac{2}{b^2} \log(a + bx) \cdot \sin(a + bx)$

$\therefore$ the required general solution is

$$y = y_c + y_p = c_1 \cos \log(a + bx) + c_2 \log \sin(a + bx) + \frac{2}{b^2} \log(a + bx) \cdot \sin(a + bx).$$

Exercises 22

Solve the following ODEs : —

- $\frac{d^2 y}{dx^2} = \log x$, if $y = 2$, $\frac{dy}{dx} = 0$ when $x = 1$.
- $\frac{d^3 y}{dx^3} = \sqrt{1 - x^2}$, if $y = \pi$, $\frac{dy}{dx} = \frac{\pi}{4}$ and $\frac{d^2 y}{dx^2} = -\frac{\pi}{4}$ when $x = -1$.
- $\frac{d^3 y}{dx^3} = e^x \sin x$, if $y = \frac{3}{4}$, $\frac{dy}{dx} = \frac{1}{2}$ and $\frac{d^2 y}{dx^2} = \frac{3}{2}$ when $x = 0$.
- $\frac{d^2 y}{dx^2} + \alpha^2 y = 0$.
- $\frac{d^2 y}{dx^2} = 2y^3$, if $y = 1$, $\frac{dy}{dx} = 1$ when $x = 0$.
- $\alpha \frac{d^3 y}{dx^3} = \frac{d^2 y}{dx^2}$.
- $a \frac{d^2 y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$.
- $(y_n)^2 = 4y_{n-1}$.
- $(1 + y_1^2)^{3/2} = Ky_2$.
- $x^2 \frac{d^2 y}{dx^2} = 1 - x \frac{dy}{dx}$.
- $(1 + x^2) \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + ax = 0$.
- $a^2 \frac{d^2 y}{dx^2} + 2x \left\{ 1 + \left(\frac{dy}{dx}\right)^2 \right\}^{3/2}$
- $y \frac{d^2 y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 1 = 0$ if $y = 2$ and $\frac{dy}{dx} = 0$ when $x = 0$.
- $\left[1 + \left(\frac{dy}{dx}\right)^2 \right]^{3/2} = 2y \frac{d^2 y}{dx^2} \cdot \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$
- $2(2a - y) \frac{d^2 y}{dx^2} = 1 + \left(\frac{dy}{dx}\right)^2$.
- (i) $xy y_2 + x y_1^2 = 3y y_1$, (ii) $(x^2 + y^2)(y - x y_1) + x^2 y^2 y_2 = 0$,
(iii) $x^2 y_2 - x y_1 + y = 0$, (iv) $x y_2 + x y_1^2 - y_1 = 0$.

17. (i) $x^2 y_2 + x y_1 - 9y = 0$ given that $y = x^3$ is a solution.

(ii) $x y_2 (x \cos x - 2 \sin x) + (x^2 + 2) y_1 \sin x - 2y (x \sin x + \cos x) = 0$,
given that $y = x^2$ is a solution.

18. Factorize the operator and solve by reducing the order :

(i) $[xD^2 + (2x-1)D - 2]y = 4x^2$,

(ii) $[x^2 D + (2x^2 - x)D - 2x]y = 0$,

(iii) $[x^3 D^2 + (x^3 - x^2)D - x^2]y = 2x^4$.

Solve the ODEs : —

19. (i) $\sec^2 y \frac{d^2 y}{dx^2} + 2 \frac{\sin y}{\cos^3 y} \left(\frac{dy}{dx} \right)^2 + \tan y = x$,

(ii) $x \frac{d^2 y}{dx^2} + \frac{dy}{dx} + x = 0$,

(iii) $y(1 - \log y) y_2 + (1 + \log y) y_1^2 = 0$,

(iv) $y y_2 - y_1^2 = y^2 \log y$.

20. (i) $u = r \frac{d}{dr} \left(r \frac{du}{dr} \right) + ar^3$,

(ii) $r \frac{d^2 y}{dr^2} + \frac{dy}{dr} - \frac{y}{r} = -ar^2$.

21. $(x^2 D^2 + 4x D + 2)y = \log x$.

22. $(x^2 D^2 - 2x D + 2)y = x \log x$.

23. $x^2 \frac{d^2 y}{dx^2} - 4x \frac{dy}{dx} + 6y = 0$.

24. $x^2 y_2 - 2x y_1 + 2y = 4x^3$.

25. $(1+2x)^2 \frac{d^2 y}{dx^2} - 6(1+2x) \frac{dy}{dx} + 16y = 8(1+2x)^2$.

26. $(2x-3)^2 y'' + (2x-3)y' + y = \sec \log(2x-3)$.

27. $(1+x)^2 \frac{d^2 y}{dx^2} + (1+x) \frac{dy}{dx} + y = 4 \cos \log(1+x)$.

28. $x^2 y_2 - 3x y_1 + 4y = 0$ given that $y = 1$ and $\frac{dy}{dx} = 1$ when $x = 1$.

29. $x^2 y_2 - 3x y_1 + 3y = 0$ given that $y = 0$ and $y_1 = -2$ when $x = 1$.

30. $x^3 y_3 + 3x^2 y_2 + x y_1 = 24x^2$.

31. $(x^3 D^3 + x D - 1)y = 2x \log x$.

32. $x^3 y''' + x y' - y = x \log x$.

33. $x^3 D^3 y + 3x^2 D^2 y + 4x Dy = \sin(\sqrt{3} \log x)$.
34. $x^3 \frac{d^3 y}{dx^3} + 3x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + 8y = 65 \cos(\log x)$.
35. $(x^3 D^3 + 3x^2 D^2 + xD)y = x^3$.
36. $x^4 \frac{d^4 y}{dx^4} + 2x^3 \frac{d^3 y}{dx^3} + x^2 \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + y = \log x$.
37. $(x^4 D^3 + 2x^3 D^2 - x^2 D + x)y = 1$.
38. A cable is fastened with two pillars 100 ft apart at the same height. Its sag is 25 ft. (a) Find the length of the cable and (b) if the weight per foot is $w = 10$ lbs, then find the tension at the lowest point of the cable.
39. A cord is tight to two pillars 30 ft apart.
- If the length of the cord is 40 ft, find its sag.
 - If the sag is 10 ft, find the length of the cord.
 - If weight per foot is 1.5 lbs, find the tension of (b) at the point of suspension and at the lowest point.

Answers

1. $4y = x^2(2 \log x - 3) + 4x + 5$.
2. $y = -\frac{1}{24}x(1-x^2)^{3/2} + \frac{5}{6}x\sqrt{1-x^2} + \frac{4x^2+1}{16}\sin^{-1}x + \frac{37\pi}{32}$.
3. $4y + e^x(\cos x + \sin x) + x = 1$.
4. $y = c \sin(c_1 x + \alpha)$.
5. $(1-x)y = 1$.
6. $y = Ae^{x/\alpha} + Bx + C$.
7. $y = a \cosh\left(\frac{x+c_1}{a}\right) + c_2$.
8. $y = \frac{2x^{n+1}}{(n+1)!} + 2a \frac{x^n}{n!} + a^2 \frac{x^{n-1}}{(n-1)!} + bx^{n-2} + cx^{n-2} + \dots + hx + k$.
9. $(x-a)^2 + (y-b)^2 = k^2$.
10. $y = \frac{1}{2}(\log x)^2 + c_1 \log x + c_2$.
11. $y + ax = c_1 \sinh^{-1}x + c_2$.
12. $y = B + \int \frac{x^2 + A}{\sqrt{a^4 - (x^2 + A)^2}} dx$.
13. $x^2 + y^2 = 4$.
14. $(x-c)^2 = 4cy - 4c^2$.
15. $\int \sqrt{\frac{2a-y}{A-2a+y}} dy = x + B$.
16. (i) $y^2 + b = ax^4$, (ii) $\log x = ay/x + a^2 \log(y-ax) - a^2 \log x + b$,
(iii) $y = x(a \log x + b)$, (iv) $y = a + \log(x^2 + b)$.
17. (i) $y = ax^3 + bx^{-3}$, (ii) $y = ax^2 + b \sin x$.

17. (i) $y = ax^3 + bx^{-3}$, (ii) $y = ax^2 + b \sin x$.

18. (i) $y = 2x^2 - 2x + 1 + c_1(2x-1) + c_2 e^{-2x}$,

(ii) $y = c_1(2x-1) + c_2 e^{-2x}$,

(iii) $y = c_2 e^{-x} + c_1 x - c_1 + 2x^2 - 4x + 4$.

19. (i) $\tan y = c_1 \cos x + c_2 \sin x + x$, (ii) $y = c \log x - \frac{x^2}{4} + c_1$,

(iii) $\log y = 1 - \frac{1}{cx + c_1}$,

(iv) $\log y = c_1 e^x + c_2 e^{-x}$.

20. (i) $u = c_1 r + \frac{c_2}{r} - \frac{ar^3}{8}$,

(ii) $y = c_1 r + \frac{c_2}{r} - \frac{ar^3}{8}$.

21. $y = \frac{c_1}{x^2} + \frac{c_2}{x} + \frac{1}{2} \log x - \frac{3}{4}$.

22. $y = c_1 x^2 + c_2 x - x \left\{ \frac{1}{2} (\log x)^2 + \log x \right\}$.

23. $y = c_1 x^2 + c_2 x^3$.

24. $y = Ax + Bx^2 + 2x^3$.

25. $y = (1+2x)^2 \{ \log(1+2x) + A \log(1+2x) + B \}$.

26. $y = A \cos \log(2x-3) + \{ B + \log(2x-3) \} \sin \log(2x-3) +$

$\{ \log \cos \log(2x-3) \} \cdot \cos \log(2x-3)$.

27. $y = A \cos \{ \log(1+x) - \alpha \} + 2 \{ \log(1+x) \} \sin \log(1+x)$.

28. $y = (1 - \log x) x^2$.

29. $y = x - x^3$.

30. $y = c_1 + c_2 \log x + c_3 (\log x)^2 + 3x^2$.

31. $y = \{ A + B \log x + C (\log x)^2 \} x + \frac{1}{12} x (\log x)^4$.

32. $y = \{ A + B \log x + C (\log x) \} x + \frac{1}{24} x (\log x)^4$.

33. $y = A + B \sin(\sqrt{3} \log x) + C \cos(\sqrt{3} \log x) - \frac{1}{6} (\log x) \sin(\sqrt{3} \log x)$.

34. $y = 8 \cos(\log x) - \sin(\log x) + Ax^{-2} + Bx \cos(\sqrt{3} \log x - \alpha)$.

35. $y = A + B \log x + C (\log x)^2 + \frac{1}{27} x^3$.

36. $y = 4 + \log x + Ax + Bx \log x + Cx (\log x)^2 + Dx (\log x)^3$.

37. $y = (A + B \log x) x + C/x + (\log x)/4x$.

38. (a) 115 ft, (b) 5.37 lbs.

MISCELLANEOUS EXERCISES

A GUIDE TO GRE (MATH.) SUBJECT TEST

1. What is the mean value of $\cos \theta$ over $0 \leq \theta \leq \frac{\pi}{2}$?

[Hints: By sec. 11.5 $\left(\int_0^{\pi/2} \cos \theta \, d\theta \right) / \left(\frac{\pi}{2} - 0 \right) = \frac{2}{\pi}$]

2. Evaluate $\int_0^2 x^2 f(x) \, dx$ when $f(x) = \begin{cases} x, & x < 1 \\ x-1, & x \geq 1 \end{cases}$.

[Hints: $\int_0^2 x^2 f(x) \, dx = \int_0^1 x^2 \cdot x \, dx + \int_1^2 x^2 (x-1) \, dx = 5/3$]

3. Evaluate $\int_{-1}^1 |2x-1| \, dx$.

[Hints: $y = |2x-1| = \begin{cases} 2x-1, & x \geq 1/2 \\ -(2x-1), & x < 1/2 \end{cases}$. Its graph by sec. 11.4 is

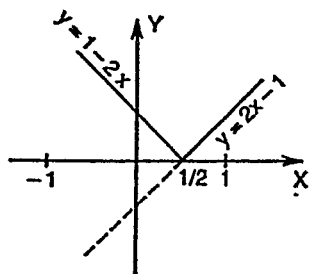


Fig. M-1

$\int_{-1}^1 |2x-1| \, dx = \int_{-1}^{1/2} (1-2x) \, dx + \int_{1/2}^1 (2x-1) \, dx = 2 \frac{1}{2}$

4. Evaluate $\int_a^b f(x) g(x) \, dx$ if $\frac{d}{dx} g(x) = f(x)$.
5. Let f be a continuous function such that for all $c > 0$,
- $f(c) > 0$ and
 - the region bounded by the line $x = c$, by the co-ordinate axes, and by the curve $y = f(x)$ has area $c e^c$. For all $x > 0$, $f(x) =$

- (A) e^x (B) xe^x
 (C) $xe^x - e^x$ (D) $xe^x + e^x$
 (E) $x^2 e^x - xe^x$

Choose the correct one for $f(x)$.

[Hints : Area = $\int_0^{e^c} f(x) dx = ce^c$ (given). Use sec. 8.8 because c is a parameter in the definite integral. Diff. w.r.t. ' c ', then $f(c) = e^c (c+1)$. Hence $f(x) = e^x (x+1) = xe^x + e^x$].

6. Which of the following is an equation of the curve through origin and intersect at right angles all the curves satisfying the differential equation $\frac{dy}{dx} = x+1$?

- (A) $e^{-y} = x+1$ (B) $e^y + x - 1 = 0$
 (C) $e^y = x+1$ (D) $2y = x^2 + 2x$
 (E) $y(x^2 + 2x) = -2$

[Hints: Slope of the required curve must be $\left(\frac{-1}{x+1}\right)$ i.e. $\frac{dy}{dx} = \frac{-1}{x+1}$. Solve ODE and use initial condition $y(0) = 0$].

7. Let f be a continuous function of x (f not identically zero) and let s and t be non-zero numbers. If $I = t \int_0^{s/t} f(s+tx) dx$, then the value of I

- (A) varies with x
 (B) depends on the ratio (s/t)
 (C) depends on s , but is independent of t and x
 (D) depends on t , but is independent of s and x
 (E) is constant for s , t and x

Choose the correct value.

[Hints : Let $s+tx = u$. Then $I = \int_s^{2s} f(u) du$. So I depends on s only].

8. If $f(x) = \begin{cases} \frac{x}{|x|} & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$, then $\int_{-1}^3 f(x) dx =$

- (A) $-e$ (B) $-\log 2$
 (C) 1 (D) 2
 (E) e

Choose the correct value.

9. Let f be continuous in $[-1, 1]$ and P be the partition of $[-1, 1]$. Let $L_f(P)$ and $U_f(P)$ denote the lower and upper sum of f . Let

(i) $L_f(P) = 3$ and $U_f(P) = 2$

(ii) $L_f(P) = 3, U_f(P) = 6$ and $\int_{-1}^1 f(x) dx = 2$

(iii) $L_f(P) = 3, U_f(P) = 6$ and $\int_{-1}^1 f(x) dx = 10$

Then which of the following is true ?

(A) (i)

(B) (ii)

(C) (iii)

(D) (i) and (ii)

(E) None

[Hints : $L_f(P) \leq \int_{-1}^1 f(x) dx \leq U_f(P)$ must be true for integrability and $L_f(P) \leq U_f(P)$. So, none is acceptable].

10. Given that $\int_1^4 f(x) dx = 5$, $\int_3^4 f(x) dx = 7$, $\int_1^8 f(x) dx = 11$, find each of the following :

(A) $\int_4^8 f(x) dx$

(B) $\int_4^3 f(x) dx$

(C) $\int_1^3 f(x) dx$

(D) $\int_3^8 f(x) dx$

(E) $\int_8^4 f(x) dx$

(F) $\int_4^4 f(x) dx$

[Hints : Use 8.7]

11. (i) Compare $\frac{d}{dx} \left[\int_a^x f(t) dt \right]$ to $\int_a^x \frac{d}{dt} [f(t)] dt$

[Hints : Take $f(t) = F'(t)$]

- (ii) Compare $\frac{d}{dx} \left[\int f(x) dx \right]$ to $\int \frac{d}{dx} [f(x)] dx$.

12. Find f from the information given :

(i) $f''(x) = x^2 - x, f'(1) = 0, f(1) = 2$.

(ii) $f''(x) = \cos x, f'(0) = 1, f(0) = 2$.

13. Solve the following equation for A , a constant :

$$\int_a^b [f(x) - A] dx = 0$$

14. Determine whether the ascertion is true or false.

(a) (the average of $f + g$ on $[a, b]$) = (the average of f on $[a, b]$) + (the average of g on $[a, b]$),

- (b) (the average of αf on $[a, b]$, α is a constant) = α (the average of f on $[a, b]$).
- (c) (the average of $f g$ on $[a, b]$) = (the average of f on $[a, b]$) (the average of g on $[a, b]$).

15. Find a formula for the area of each region shown below :

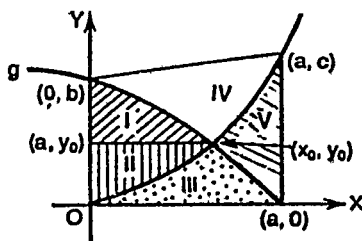


Fig. M-2

16. Show that

(i) $\int_{a+c}^{b+c} f(x-c) dx = \int_a^b f(x) dx$ if $c \neq 0$

(ii) $\frac{1}{c} \int_{ac}^{bc} f\left(\frac{x}{c}\right) dx = \int_a^b f(x) dx$

Assume that f is everywhere continuous.

17. Sketch the region bounded by the curves. Represent the area as an integral in terms of x and y .

(i) $y = |x|$, $3y - x = 8$

(ii) $x = |y|$, $x = 2$

18. Let T be a solid with volume

$$V = \iiint_T dx dy dz = \int_0^3 \int_x^{6-x} \int_0^{2x} dz dy dx.$$

Sketch T and fill in the blanks.

(a) $V = \int \dots \int \dots \int \dots dy dx dz$

(b) $V = \int \dots \int \dots \int \dots dy dz dx$

(c) $V = \int \dots \int \dots \int \dots dx dy dz + \int \dots \int \dots \int \dots dx dy dz$

19. Are the areas bounded by $y = g(x)$, $y = f(x)$, $x = a$, $x = b$ and $y = g(x) + c$, $y = f(x) + c$, $x = a$, $x = b$ identical ?

20. If $\int_a^c f(x) dx = \int_a^b f(x) dx - \int_b^c f(x) dx$, then $f(x)$ must intersect x -axis.

Sketch and explain.

21. If $f(x) = x$ then for any partition (subintervals) P on $[a, b]$

$$L_f(P) < \frac{b^2 - a^2}{2} < U_f(P)$$

22. If $\frac{d}{dx} \sqrt{\frac{x+1}{x-1}} = -\frac{1}{(x-1)\sqrt{x^2-1}}$ then find $\int_2^3 \frac{dx}{(x-1)\sqrt{x^2-1}}$.

23. The area of a solid is the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Find the volume of the solid given that the cross-sections perpendicular to x -axis are equilateral triangle.

[Hints : Area of an equilateral Δ is $\frac{1}{4}\sqrt{3}s^2$, where s is the length of a side of Δ . Since Δ is $\perp$ to x -axis. So, one median is $\perp$ to x -axis and a side should be $\parallel$ to y -axis. Let the length PQ of a side $= 2y = \frac{2b}{a}\sqrt{a^2 - x^2}$.

So, area $\Delta = A(x) = \sqrt{3} \frac{b^2}{a^2} (a^2 - x^2)$.

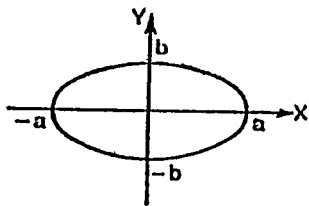


Fig. M-3

$\therefore$ by parallel cross-section method (not discussed in the book), the volume of the solid

$$= \int_{-a}^a A(x) dx = \frac{\sqrt{3} b^2}{a^2} \int_{-a}^a (a^2 - x^2) dx = \frac{4\sqrt{3}}{3} ab^2 \text{ units.}]$$

24. A function F is defined by the following indefinite integral :

$$F(x) = \int_1^x \frac{e^t}{t} dt \text{ if } x > 0.$$

Prove that $\int_1^x \left\{ \frac{e^t}{t+a} \right\} dt = e^{-a} [F(x+a) - F(1+a)]$.

Also express $\int_1^x \frac{e^{at}}{t} dt$, $\int_1^x \frac{e^t}{t^2} dt$, $\int_1^x e^{1/t} dt$

in terms of F .

25. Select the best answer of $\int_0^1 \int_0^x xy \, dy \, dx$ from (A) 0, (B) $\frac{1}{8}$, (C) $\frac{1}{3}$, (D) 1 and (E) 3.
26. Select the best answer of $\int_0^\pi e^{\sin^2 x} e^{\cos^3 x} \, dx$ from (A) π , (B) $e\pi$, (C) e^π , (D) $e^{\sin^2 \pi}$, (E) $e^\pi - 1$.
27. Evaluate $\int_0^1 \left(\int_0^{\sin y} \frac{1}{\sqrt{1-x^2}} \, dx \right) dy$ and match the best answer with (A) $\frac{1}{3}$, (B) $\frac{1}{2}$, (C) $\frac{\pi}{4}$, (D) 4, (E) $\frac{\pi}{3}$.
28. Let $y=f(x)$ be a solution of the differential equation $x \, dy + (y - xe^x) \, dx = 0$ such that $y=0$ when $x=1$. What is the value of $f(2)$?

Answers

1. $\frac{2}{\pi}$ 2. $5/3$ 3. $2\frac{1}{2}$ 4. $\frac{1}{2}[g(b)]^2 - [g(a)]^2$ 5. $xe^x + e^x$
6. (A) 7. (C) 8. (D) 9. (E)
10. (a) 6, (b) -7, (c) -2, (d) 13, (e) -6, (f) 0 11. (i) $f(x)$ and $f(x) - f(a)$ (ii) same.
12. (i) $\frac{1}{12}(x^4 - 2x^3 + 2x + 23)$ (ii) $x - \cos x + 3$
13. $A = \frac{\int_a^b f(x)}{b-a}$ 14. (a) True, (b) True, (c) False
15. $I = \int_0^{x_0} [g(x) - y_0] \, dx$ $II = \int_0^{x_0} [y_0 - f(x)] \, dx$ $III = \int_0^{x_0} f(x) \, dx + \int_{x_0}^a g(x) \, dx$
- $IV = \int_0^{x_0} \left[\left(\frac{c-b}{a}x + b \right) - g(x) \right] \, dx + \int_{x_0}^a \left[\left(\frac{c-b}{a}x + b \right) - f(x) \right] \, dx$
- $V = \int_{x_0}^a [f(x) - g(x)] \, dx$
17. (i) (a) $A = \int_{-2}^0 \left[\frac{8+x}{8} - (-x) \right] \, dx + \int_0^4 \left(\frac{8+x}{3} - x \right) \, dx$
- (b) $A = \int_0^2 [y - (-y)] \, dy + \int_2^4 [y - (3y-8)] \, dy$

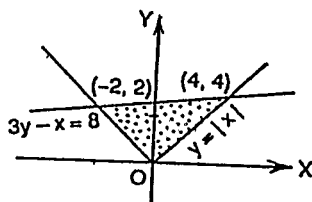


Fig. M-4

(ii) (a) $A = 2 \int_a^2 x \, dx$

(b) $A = 2 \int_0^2 y \, dy$

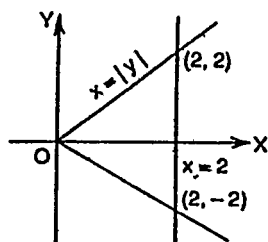


Fig. M-5

18. (a) $V = \int_0^6 \int_{z/2}^3 \int_x^{6-x} dy \, dx \, dz$

(b) $V = \int_0^3 \int_0^{2x} \int_x^{6-x} dy \, dz \, dx$

(c) $V = \int_0^6 \int_{z/2}^3 \int_{z/2}^y dx \, dy \, dz + \int_0^6 \int_3^{(12-z)/2} \int_{z/2}^{6-y} dx \, dy \, dz$

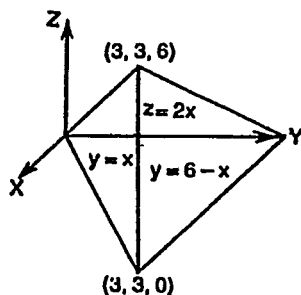


Fig. M-6

22. $(-\sqrt{2} + \sqrt{3})$

24. $F(ax) - F(a); F(x) - \frac{e^x}{x}; xe^{1/x} - e - F\left(\frac{1}{x}\right).$

25. (B).

26. (B).

27. (B).

28. $e^2/2.$

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